An inverse problem of identifying the coefficient of semilinear parabolic equation

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Abstract

In this paper, a variational iteration method (VIM), which is a well-known method for solving nonlinear equations, has been employed to solve an inverse parabolic partial differential equation. Inverse problems in partial differential equations can be used to model many real problems in engineering and other physical sciences. The VIM is to construct correction functional using general Lagrange multipliers identified optimally via the variational theory.This method provides a sequence of function which converges to the exact solution of the problem. This technique does not require any discretization, linearization or small perturbations and therefore reduces the numerical computations a lot. Numerical examples are examined to show the efficiency of the technique.

Introduction

Parabolic systems appear naturally in a number of physical and engineering settings, in particular in hydrology, material sciences, heat transfer, combustion systems, medical imaging and transport problems. Usually the function that characterizes a certain property of the system is unknown, and the interest is to identify the unknown function based on some time dependent measurements, which leads to an inverse problem for a parabolic system. A number of investigators have considered such problems for various applications using different methods [1], the reader can refer to [2-6].

In this paper, we solve an inverse semilinear parabolic problem using the VIM. The method is capable of reducing the size of calculations and handles both linear and non-linear equations, homogeneous or inhomogeneous, in a direct manner. The method gives the solution in the form of rapidly convergent successive approximations that may give the exact solution if such a solution exists. For concrete problems where exact

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solution is not obtainable, it was found that a few numbers of approximations can be used for numerical purposes. The VIM is a powerful tool to searching for approximate solutions of nonlinear equation without requirement of linearization or perturbation. This method, which was first proposed by He [7, 8] in 1998, has been proved by many authors to be a powerful mathematical tool for various kinds of non-linear problems [9-12].

The rest of this paper is organized as follows: In Section 2, we introduce an inverse semilinear parabolic problem and transform it into a direct linear parabolic problem. In Section 3, the variational iteration method is reviewed. In Section 4, application of the VIM is presented to solve the discussed inverse problem. In Section 5, several numerical examples are presented to confirm the accuracy and efficiency of the new method and finally a conclusion is presented in Section 6.

Statement of the problem

Consider the semilinear parabolic equation:

$$u_t(x,t) = u_{xx}(x,t) + p(t)u(x,t) + f(x,t); \qquad 0 < x < 1, \quad 0 < t < T, \tag{1}$$

with unknown coefficient $p(t)$ in a domain $Q_T = \{(x,t): 0 < x < 1, 0 < t < T\}$. Impose the initial and boundary conditions:

$$u(x,0) = u_0(x); \qquad 0 \le x \le 1, \tag{2}$$

$$u_x(0,t) = g_0(t); \qquad 0 \le t \le T, \tag{3}$$

$$Bu(1,t) = g_1(t); \qquad 0 \le t \le T, \tag{4}$$

and subject to an extra measurement:

$$\int_0^{s(t)} u(x,t)dx = E(t); \qquad 0 < s(t) < 1, 0 \le t \le T, \tag{5}$$

where $T > 0$ is final time, B is boundary operator (i.e. $B = \frac{\partial^i}{\partial x^i}; i = 0\, or\, 1$) and f, u_0, g_0, g_1, s and $E \neq 0$ are known functions.

The existence and uniqueness to some kind of these inverse problems are discussed in [13, 14, 20]. Certain types of physical problems can be modeled by (1)–(5). For example, if u represents a temperature distribution, then (1)–(5) can be interpreted as the control problem with source parameter. We want to identify the control function $p(t)$ that will yield a desired energy prescribed in a portion of the spatial domain. The applications of these inverse problems and some other similar parameter identification problems are discussed in [15, 20].

In order to solve the above problem by using VIM, we require transforming the problem with only one unknown function as follows [16]:

$$r(t) = \exp\{-\int_0^t p(s)ds\}, \qquad w(x,t) = u(x,t)r(t). \tag{6}$$

Thus, we have:

$$u(x,t) = \frac{w(x.t)}{r(t)}, \qquad p(t) = \frac{-r'(t)}{r(t)}. \tag{7}$$

We reduce the original inverse problem (1)-(5) to the following auxiliary direct problem:

$$w_t(x,t) = w_{xx}(x,t) + r(t)f(x,t); \qquad 0 < x < 1, \quad 0 < t < T, \tag{8}$$

$$w(x,0) = u_0(x); \qquad 0 \le x \le 1, \tag{9}$$

$$w_x(0,t) = r(t)g_0(t); \qquad 0 \le t \le T, \tag{10}$$

$$Bw(1,t) = r(t)g_1(t); \qquad 0 \le t \le T, \tag{11}$$

subject to:

$$r(t) = \frac{\int_0^{s(t)} w(x,t)dx}{E(t)}; \qquad 0 < s(t) < 1, 0 \le t \le T. \tag{12}$$

It is easy to show that the original inverse problem (1)-(5) is equivalent to the auxiliary direct problem (8)-(12).

Variational iteration method

To illustrate the basic idea of the method, we consider the following general non-linear differential equation:

$$Lu(t) + Nu(t) = f(t), \tag{13}$$

where L and N are linear and nonlinear operators, respectively and f is source or sink term. We can construct a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t,\tau)\{Lu_n(\tau) + N\tilde{u}_n(\tau) - f(\tau)\}d\tau; \quad n \ge 0, \tag{14}$$

where λ is general Lagrange multiplier [17], which can be identified optimally via the variational theory [7-9, 18]. The subscript n denotes the nth order approximation and $\tilde{u}_n$ is the restricted variation so that its variation is zero which means $\delta\tilde{u}_n = 0$. By this method, it is firstly required to determine the Lagrange multiplier λ that will be identified optimally via integration by part. Assuming $u_0(t)$ is the solution of $Lu=0$, the successive approximation $u_{n+1}(t)$; $n \geq 0$, of the solution $u(x,t)$ will be readily obtained upon using the determined Lagrange multiplier and any selective function $u_0(t)$. Consequently, the solution is given by $u=\lim_{n\to\infty} u_n$, because we will rewrite Equation (14) in the operator form as follows:

$$u_{n+1}(t)=A[u_n],$$

where the operator A takes the following form:

$$A[u(t)]=u(t)+\int_0^t \lambda(t,\tau)\{Lu(\tau)+Nu(\tau)-f(\tau)\}d\tau.$$

Theorem Let $(X, \|.\|)$ be a Banach space and $A: X \to X$ be a nonlinear mapping and suppose that:

$$\|A[u]-A[\tilde{u}]\| \leq \gamma \|u-\tilde{u}\|, \quad u, \tilde{u} \in X,$$

for some constant γ. Then, A has a unique fixed point. Furthermore, the sequence (14) using VIM with an arbitrary choice of $u_0 \in X$, converges to the fixed point of A and

$$\|u_n-u_m\| \leq \|u_1-u_0\| \sum_{j=m-1}^{n-2} \gamma^j.$$

Proof: A complete proof is given by Tatari and Dehghan [19].

According to the above theorem, a sufficient condition for the convergence of the variational iteration method is strictly contraction of A. Furthermore, sequence (14) converges to the fixed point of A, which is also the solution of the equation (13). Also, the rate of convergence depends on γ.

For variational iteration method, the key is the identification of the Lagrangian multiplier. For linear problems, its exact solution can be obtained by only one iteration step due to the fact that, no non-linear exists so the Lagrange multiplier can be exactly identified. For nonlinear problems, the Lagrangian multiplier is difficult to be identified exactly. To overcome the difficulty, we apply restricted variations to nonlinear terms. Due to the approximate identification of the Lagrangian multiplier, the approximate solutions converge to their exact solutions relatively slowly. It should be specially pointed out that the more accurate the identification of the multiplier, the faster the approximations converge to their exact solutions.

Application

In this section, the variational iteration method is used for solving the problem (8)-(12). If the VIM is applied to equation (8), the correction functional is derived in the first place:

$$w_{n+1}(x,t)=w_n(x,t)+\int_0^t \lambda(t,\tau)\{w_{n\tau}(x,\tau)-\tilde{w}_{nxx}(x,\tau)-r(t)f(x,\tau)\}d\tau; \quad n\geq 0.$$

Making the above correction functional stationary, we have:

$$\delta w_{n+1}(x,t)=\delta w_n(x,t)+\delta\int_0^t \lambda(t,\tau)\{w_{n\tau}(x,\tau)-\tilde{w}_{nxx}(x,\tau)-r(t)f(x,\tau)\}d\tau; \quad n\geq 0,$$

and it follows that:

$$\delta w_{n+1}(x,t)=\delta w_n(x,t)+\delta\int_0^t \lambda(t,\tau)\{w_{n\tau}(x,\tau)\}d\tau; \quad n\geq 0.$$

Note that $\delta w_n(x,0)=0$, $\delta\tilde{w}_n(x,)=0$ and $\delta f(x,t)=0$. Thus, its stationary condition can be obtained as follows:

$$\begin{cases}\lambda'(t,\tau)=0,\\ 1+\lambda(t,\tau)|_{\tau=t}=0.\end{cases}$$

Therefore $\lambda(t,\tau)=-1$. And the following iteration formula can be obtained:

$$w_{n+1}(x,t)=w_n(x,t)-\int_0^t \{w_{n\tau}(x,\tau)-w_{nxx}(x,\tau)-r(\tau)f(x,\tau)\}d\tau; \quad n\geq 0. \qquad (15)$$

For sufficiently large values of n we can consider u_n as an approximation of the exact solution. According to Adomian's decomposition method in t-direction which is equivalent to the VIM in t- direction [19], we choose its initial approximate solution $w_0(x,t)=w(x,0)$. Having $w(x,t)$ determined, then $u(x,t)$ and $p(t)$ can be computed by using equation (7).

Test examples

In this section the theoretical considerations introduced in the previous sections will be illustrated with some examples. These tests are chosen such that their analytical solutions are known. But the method developed in this research can be applied to more complicated problems. The numerical implementation is carried out in microsoft Maple13.

Example 1: This example is solved in [20] by using the finite difference scheme. Solve the following inverse problem:

$$u_t(x,t)=u_{xx}(x,t)+p(t)u(x,t)+(\pi^2+2t)e^2\cos(\pi x)+2e^t xt; \qquad 0<x<1, 0<t<1,$$

$$u(x,0)=x+\cos(\pi x); \qquad 0\leq x\leq 1,$$

$$u_x(0,t)=e^t; \qquad 0\le t\le 1,$$
$$u(1,t)=0; \qquad 0\le t\le 1,$$
$$\int_0^{\frac{1+t}{2}} u(x,t)dx = e^t\{\frac{1}{\pi}\sin(\frac{\pi(1+t)}{2})+\frac{(1+t)^2}{8}\}; \qquad 0\le t\le 1.$$

The true solution is $u(x,t)=e^t(x+\cos(\pi x))$ while $p(t)=1-2t$. We can select $w(x,0)=x+\cos(\pi x);$ by using the given initial value and from equations (15). According to (15), one can obtain the successive approximations $w_n(x,t)$ of $w(x,t)$ as follow: $w_1(x,t)=(x+\cos(\pi x))+t^2(x+\cos(\pi x)),$

$$w_2(x,t)=(x+\cos(\pi x))+t^2(x+\cos(\pi x))+\frac{t^4}{2}(x+\cos(\pi x)),$$
$$w_3(x,t)=(x+\cos(\pi x))+t^2(x+\cos(\pi x))+\frac{t^4}{2}(x+\cos(\pi x))+\frac{t^6}{6}(x+\cos(\pi x)).$$

And the rest of the components of iteration formula (15) are obtained using the Maple Package. Now from (12), we can obtain the successive approximations $r_n(t)$ of $r(t)$ as:

$$r_n(t)=\frac{\int_0^{s(t)} w_n(x,t)dx}{E(t)}.$$

Finally, using (7), we can obtain the successive approximations $u_n(x,t)$ of $u(x,t)$ and $p_n(t)$ of $p(t)$ as following:

$$u_n(x,t)=\frac{w_n(x.t)}{r_n(t)}, \qquad p_n(t)=\frac{-r_n'(t)}{r_n(t)}.$$

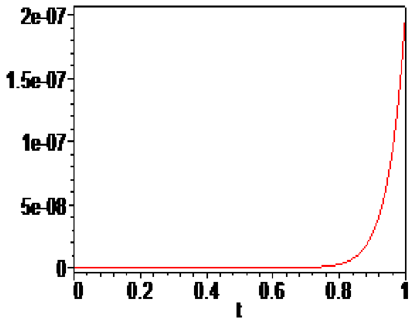
The obtained numerical results are summarized in Tables 1 and 2. In addition, the graphs of the error functions $|u-u_{10}|$ and $|p-p_{10}|$ are plotted in Figure 1.

Table1. The comparison between exact, FDM and VIM solutions for $p(t)$

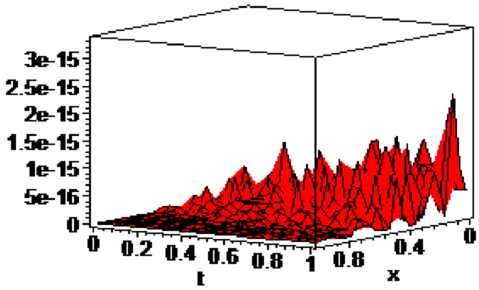
t	Exact value $p(t)$	Frist method [20]	Second method [20]	VIM ($n=5$)	VIM ($n=10$)
0.1	0.8	0.7952548681	0.7987107476	0.7999999992	0.7999999997
0.2	0.6	0.5942632825	0.5987107476	0.6000000014	0.6000000007
0.3	0.4	0.3937116629	0.3985704513	0.4000000269	0.3999999995
0.4	0.2	0.1934449648	0.1984305834	0.2000000269	0.2000000002
0.5	0.0	-0.0065131634	-0.0017537688	0.0000063377	0.0000000000
0.6	-0.2	-0.2061411572	-0.2019831631	-0.199957814	-0.199999997
0.7	-0.4	-0.4054351343	-0.4022583261	-0.399798103	-0.399999992
0.8	-0.6	-0.6044082700	-0.6025801288	-0.599245056	-0.599999983

Table2. The comparison between exact, FDM and VIM solutions for $u(x,0.5)$

x	Exact value	Frist method [20]	Second method [20]	VIM ($n=5$)	VIM ($n=10$)
0.1	1.7328992351	1.7329861820	1.7284586563	1.7328992372	1.7328992351
0.2	1.6635877811	1.6636634535	1.6594361329	1.6635877833	1.6635877811
0.3	1.4637104292	1.4637691547	1.4600169103	1.4637104296	1.4637104292
0.4	1.1689713999	1.1690097460	1.1658643864	1.1689713999	1.1689713999
0.5	0.8243606353	0.8243778730	0.8219140414	0.8243606359	0.8243606353
0.6	0.4797498707	0.4797482318	0.4779764096	0.4797498713	0.4797498707
0.7	0.1850108414	0.1849955463	0.1838627812	0.1850108410	0.1850108414
0.8	-0.0148665104	-0.0148874533	-0.0154891066	-0.014866511	-0.0148665104



a) Graph of $|\, p(t)-p_{10}(t)\,|$



b) Graph of $|\,u(x,t)-u_{10}(x,t)\,|$

Figure 1. Graph of absolute error by using VIM for Example 1

Example 2: Consider the inverse problem (1)-(5) described by:

$$u_t(x,t)=u_{xx}(x,t)+p(t)u(x,t)-(\sin(x)+\cos(x)); \qquad 0<x<1,\ 0<t<1,$$

$$u(x,0)=(\sin(x)+\cos(x)); \qquad 0\le x\le 1,$$

$$u_x(0,t)=e^{-t}; \qquad 0\le t\le 1,$$

$$u_x(1,t)=e^{-t}(\cos(1)-\sin(1)); \qquad 0\le t\le 1,$$

$$\int_0^{\sqrt{t}} u(x,t)dx=-e^{-t}(-1+\cos(\sqrt{t})-\sin(\sqrt{t})); \qquad 0\le t\le 1.$$

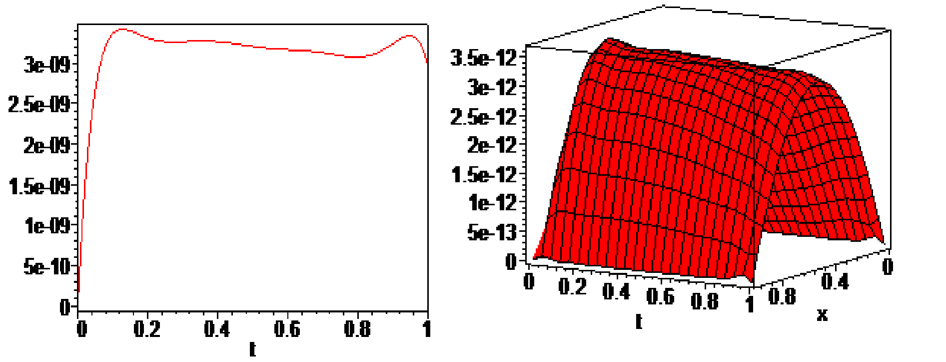
The true solution is $u(x,t)=e^{-t}(\sin(x)+\cos(x))$ while $p(t)=e^{t}$. The obtained numerical results are summarized in Tables 3 and 4. In addition, the graphs of the error functions $|u-u_{12}|$ and $|\,p-p_{12}\,|$ are plotted in Figure 2.

Table3. Absolute errors of p_n for Example 2

t	$\lvert p-p_6\rvert$	$\lvert p-p_8\rvert$	$\lvert p-p_{10}\rvert$	$\lvert p-p_{10}\rvert$	$\lvert p-p_{12}\rvert$
0.1	2.811E-4	1.456E-5	7.498E-7	1.382E-8	4.019E-10
0.2	2.739E-4	1.980E-5	8.349E-7	2.987E-8	7.479E-10
0.3	7.311E-4	1.452E-4	7.010E-6	1.009E-7	1.982E-9
0.4	4.023E-3	2.164E-4	2.992E-6	5.620E-8	9.312E-10
0.5	5.429E-3	3.982E-4	5.982E-6	2.845E-7	6.870E-9
0.6	3.498E-3	1.111E-4	5.111E-6	2.001E-7	4.165E-9
0.7	1.333E-3	1.109E-4	2.009E-7	6.194E-8	1.409E-9
0.8	1.098E-3	1.001E-4	1.101E-6	2.500E-8	1.010E-9

Table4. Absolute errors of u_n at $t = 0.5$ for Example 2

x	$\lvert u-u_6\rvert$	$\lvert u-u_8\rvert$	$\lvert u-u_{10}\rvert$	$\lvert u-u_{10}\rvert$	$\lvert u-u_{12}\rvert$
0.1	5.874E-5	7.410E-6	8.621E-8	6.993E-10	3.412E-12
0.2	4.098E-5	4.496E-6	6.982E-8	6.730E-10	1.960E-12
0.3	3.121E-5	2.196E-6	9.671E-9	1.100E-10	5.783E-13
0.4	3.309E-5	2.433E-6	1.010E-8	2.412E-10	6.628E-13
0.5	5.025E-5	3.982E-6	6.422E-8	5.681E-10	1.201E-12
0.6	3.649E-5	2.670E-6	1.145E-8	2.983E-10	8.881E-13
0.7	1.122E-5	8.333E-7	6.610E-9	7.091E-11	2.512E-13
0.8	8.751E-6	1.001E-7	3.370E-9	5.479E-11	1.200E-13



a) Graph of $|p(t)-p_{12}(t)|$ b) Graph of $|u(x,t)-u_{12}(x,t)|$

Figure2. Graph of absolute error by using VIM for Example 2

References

1. Baiyu W., Liao A., Liu W., "Simultaneous determination of unknown two parameters in parabolic equation", Inter. J. Appl. Math. Comput. 4 (3) (2012) 332-336.

2. Cannon J. R. , Lin Y., "Determination of a parameter $p(t)$ in some quasi-linear parabolic differential equations", Inverse Problems 4 (1998) 35-45.

3. DuChateau P., Thelwell R., Butters G., "Analysis of an adjoint problem approach to the identification of an unknown diffusion coefficient", Inverse Problems 20 (2004) 601 625.

4. Hasanov A., Liu Z. H., "An inverse coefficient problem for a non-linear parabolic variational inequality", Appl. Math. Lett. 21 (6) (2008) 563-570.

5. Liu Z. H. , Wang B. Y., "Coefficient identification in parabolic equations", Appl. Math. Comput. 209 (2009) 379-390.

6. Lesnic D., Yousefi S. A. , Ivanchov M., "Determination of a time-dependent diffusivity from nonlocal conditions", J. Appl. Math. Comput. 41 (2013) 301-320.

7. He J. H., "Approximate analytical solution for seepage flow with fractional derivatives in porous media", Comput. Methods Appl. Mech. Eng. 167 (1998) 57-68.

8. He J. H., "Approximate solution of non-linear differential equations with convolution product non-linearities", Comput. Methods Appl. Mech. Eng. 167 (1998) 69-73.

9. Abdou M. A. , Soliman A. A., "Variational iteration method for solving Burgers and coupled Burgers equations", J. Comput. Appl. Math. 181 (2) (2005) 245-251.

10. Akmaz H. K., "Variational iteration method for elastodynamic Green's function", Non-linear Analysis 71 (2009) 218-223.

11. Xu L., "The variational iteration method for fourth order boundary value problems", Chaos Solitons and Fractals 39 (2009) 1386-1394.

12. Soliman A. A., "A numerical simulation and explicit solutions of KdV-Burgers and Lax's seventh-order KdV equations", Chaos Solitons Fractals 29 (2) (2006) 294-302.

13. Cannon J. R., Lin Y., "Determination of parameter $p(t)$ in some quasi-linear parabolic differential equations", Inverse Problems 4 (1988) 35-45.

14. Cannon J. R.,"The One-Dimensional Heat Equation", Addison Wesley, Reading, MA, 1984.

15. Luushinkov A. A., Ahonen T., Vesala T., Nikinmaa E., Hari P., "Modeling of light-driven RUBP regeneration carboxylation and $C0_2$ diffusion for left photosynthesis", J. Theor. Biol. 188 (1997) 143-151.

16. Wanga W., Hana B., Yamamotob M., "Inverse heat problem of determining time-dependent source parameter in reproducing kernel space", nonlinear Analysis: real world applications 14 (2013) 875-887.

17. Inokuti M., Sekine H., Mura T., "General use of the Lagrange multiplier in non-linear mathematical physics, in: Variational Methods in the Mechanics of Solids", Pergamon Press, New York, 1978, pp.156-162.

18. He J. H., "Generalized Variational Principle in Fluids (in Chinese)", Shanghai University Press, Shanghai, 1998.

19. Tatari M., Dehghan M., "On the convergence of He's variational iteration method", J. Comput. Appl. Math. 207 (2007) 121-128.

20. Bodaghi S., Shahrezaee A. M., "A comparation of two methods for solving a parabolic inverse problem", Quarterly Journal of Science Tarbiat Moallem University (Mathematics) 10 (1) (2012).