Modelling of an Optimal Membrane with Patches

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Extended Abstract

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In this paper, we consider a membrane with shape $\Omega \subset R^2$ which is fixed at its boundary $\partial\Omega$. Assume the membrane is made using two different materials and its mass is prescribed. We need that certain regions $S_1 \dots S_k$ (called patches) in Ω be made out of both materials in a way that the basic frequency of the resulting membrane is either minimal or maximal.

This problem can be mathematically formulated as follows. Let $\alpha, \beta, \alpha > \beta > 0$, be the densities of the materials and *D* be the region occupied by the material with density α. We define $\rho_D = \alpha\,\chi_D + \beta\,\chi_{D^c}$ as the density function. Frequencies of the membrane are eigenvalues of the following elliptic partial differential equation

$$\begin{cases} -\Delta u = \lambda \rho_D\, u & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \qquad (1)$$

It is well-known that (1) has infinitely many eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$. In this study we mainly focus on the first eigenvalue which is denoted by $\lambda(D)$. Hereafter, $|D|$ denotes the Lebesgue measure of set *D*. In order to find the optimal basic frequency, one should consider the following optimization problems

$$inf\{\lambda(D) : D \subset \Omega, |D| = A, |D \cap S_i| = \gamma_i\, |S_i|,\ \ i = 1, \dots, k\}, \qquad (2)$$

$$sup\{\lambda(D) : D \subset \Omega, |D| = A, |D \cap S_i| = \gamma_i\, |S_i|,\ \ i = 1, \dots, k\}, \qquad (3)$$

where $A > 0$, $S_1 \dots S_k$ are disjoint subsets of Ω and $\gamma_i \in (0,1), i = 1 \dots k$. Since we require that the domain *D* cannot fill the set $\Omega \backslash (S_1 \cup S_2 \cup S_3 \cup \dots \cup S_k)$, the following constraint is imposed

$$\sum_{i=1}^{n} \gamma_i |S_i| < A < |\Omega| - \sum_{i=1}^{n} |S_i|, \qquad (4)$$

Problems (2)-(3) have been investigated by Cuccu et al. (cf. Cuccu et al. in Appl. Math. Optim. 62 (2010) 169–184). It has been proved that both problems admit a solution and the solution is unique for the maximization problem.

In optimization problems like (2) and (3), one of challenging mathematical problems after the problem of existence is to determine the optimizer. From the physical point of view, it is important to know the shape of the optimal domain *D*. This class of problems is difficult to solve analytically because of the lack of the topology information of the optimal domain. Therefore, there must be numerical approaches to determine the optimal domain *D*. We Develop two numerical algorithms in order to determine solutions of (3)-(4). Our algorithms are modifications of algorithms which have been developed recently by Kao et al. (cf. Kao et al. J. Sci. Comput. 54 (2013) 492–512) to determine optimal frequencies of a membrane without the patches. The algorithms strongly based upon the variational formula of the principal eigenvalue in (1) and rearrangement techniques.

Assume that at iteration step *n*, there is a guess for the configuration of the optimal domain, D_n , and its corresponding density function is denoted by $\rho_{n=\alpha\, \chi_{D_n}+\beta\, \chi_{D_n^c}}$. We use the finite element method with piecewise linear basis functions to discretize equation (1) with ρ_n as its density function. Let u_n be a solution of (1) associated with ρ_n. In the minimization algorithm, we extract a new density $\rho_{n+1=\alpha\, \chi_{D_{n+1}}+\beta\, \chi_{D_{n+1}^c}}$ using level sets of u_n in view of the rearrangement theory such that $\lambda(D_n) \geq \lambda(D_{n+1})$. The new density $\rho_{n+1=\alpha\, \chi_{D_{n+1}}+\beta\, \chi_{D_{n+1}^c}}$ is extracted for the maximization algorithm in a way that $\lambda(D_n) \leq \lambda(D_{n+1})$. Recall that D_{n+1} should be an admissible subset of Ω such that satisfies (4).

Consequently, we derive a decreasing sequence of eigenvalues to obtain a numerical solution of (2) and an increasing sequence of eigenvalues to compute a numerical solution of (3).

It is established that both algorithms are convergent and various numerical examples reveal the efficiency and applicability of our algorithms.

Keywords: Membrane with patches, Rearrangement of a function, Optimization, Elliptic eigenvalue problem

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