Constrained Interpolation via Cubic Hermite Splines

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Received: 26 Aug 2015 Revised: 17 Sep 2017

Extended Abstract

Paper pages (3544)

Introduction

In industrial designing and manufacturing, it is often required to generate a smooth function approximating a given set of data which preserves certain shape properties of the data such as positivity, monotonicity, or convexity, that is, a smooth shape preserving approximation.

It is assumed here that the data is sufficiently accurate to warrant interpolation, rather than least squares or other approximation methods. The shape preserving interpolation problem seeks a smooth curve/surface passing through a given set of data, in which we priorly know that there is a shape feature in it and one wishes the interpolant to inherit these features. One of the hidden features in a data set may be its boundedness. Therefore, we have a data set, which is bounded, and we already know that. This happens, for example, when the data comes from a sampling of a bounded function or they reflect the probability or efficiency of a process.

Scientists have proposed various shape-preserving interpolation methods and every approach has its own advantages and drawbacks. However, anyone confesses that splines play a crucial role in any shape-preserving technique and every approach to shape-preserving interpolation, more or less, uses splines as a cornerstone.

This study concerns an interpolation problem, which must preserve boundedness and needs a smooth representation of the data so the cubic Hermite splines are employed.

Problem Formulation and the Key Idea

The main problem is stated as follows:

Suppose we have the data $\{(x_i, y_i)\}_{i=1}^{n}$ and the known functions $f_1(x)$ and $f_2(x)$ where we know that $f_1(x_i) \le y_i \le f_2(x_i)$, for $\mathrm{i}=1,\dots,\mathrm{n}$, We wish to find a suitable interpolant $g(x)$ for the data which lies between $f_1(x)$ and $f_2(x)$. We call this as the constrained interpolation problem. This article handles this problem using cubic Splines and we confine $f_1(x)$ and $f_2(x)$ to be quadratic polynomials. For this end, we use the literature and get use of positivity preserving results, to restate the problem and define two functions as follows:

$$\bar{S}_1(x) := S(x) - f_1(x), \quad \bar{S}_2(x) := f_2(x) - S(x),$$

and impose positivity constraints, simultaneously, on them.

To define the cubic Hemite spline we use the notations $h_i = x_{i+1} - x_i$,

$\delta_i = \dfrac{y_{i+1} - y_i}{h_i}$, $\quad t = \dfrac{x - x_i}{h_i}$, for $i = 1,\dots,n-1$. On subinterval $[x_i, x_{i+1}]$ the spline S is defined as

$$S_i(x) = y_i + m_i h_i t + (3\delta_i - 2m_i - m_{i+1}) h_i t^2 + (m_i + m_{i+1} - 2\delta_i) h_i t^3,$$

where $m_i = S'(x_i)$ are unknown slopes which are auxiliary parameters to control the shape constraints. The following lemma is a key result from literature that we employ it to find unknowns according to desired constraints.

Lemma: The function $S_i(x)$ is positive on $[x_i, x_{i+1}]$ if $(m_i, m_{i+1}) \in R_1$ where

$$R_1 = \left\{ (x, y) \middle| x \geq \frac{-3y_i}{h_i}, y \leq \frac{3y_{i+1}}{h_i} \right\}.$$

We employ this lemma on $\bar{S}_1(x) = S(x) - f_1(x)$ و $\bar{S}_2(x) = f_2(x) - S(x)$ and come to some conditions on m_i under which the cubic Hermite spline lies between $f_1(x)$ and $f_2(x)$. (We consider the quadratic case where, $f_1(x) = a_1 x^2 + b_1 x + c_1$ and $f_2(x) = a_2 x^2 + b_2 x + c_2$)

Results and Discussion

Theorem 1: A sufficient condition for the cubic Hermite spline $S(x)$ to lie between $f_1(x)$ and $f_2(x)$ is that the m_i for $i = 1, \ldots, n-1.$ satisfy the following conditions:

$$m_1 \geq \frac{3\left(a_1 x_1^2 + b_1 x_1 + c_1 - y_1\right)}{h_1} + 2a_1 x_1 + b_1,$$

$$m_1 \leq \frac{3\left(a_2 x_1^2 + b_2 x_1 + c_2 - y_2\right)}{h_1} + 2a_2 x_1 + b_2,$$

$$m_i \leq \min\left\{\frac{3\left(a_2 x_i^2 + b_2 x_i + c_2 - y_i\right)}{h_i} + 2a_2 x_i + b_2, \frac{-3\left(a_1 x_{i-1}^2 + b_1 x_{i-1} + c_1 - y_i\right)}{h_{i-1}} - 4a_1 x_{i-1} - 2b_1 - 2a_1 h_{i-1}\right\},$$

$$m_i \geq \max\left\{\frac{3\left(a_1 x_i^2 + b_1 x_i + c_1 - y_i\right)}{h_i} + 2a_1 x_i + b_1, \frac{-3\left(a_2 x_{i-1}^2 + b_2 x_{i-1} + c_2 - y_i\right)}{h_{i-1}} - 4a_2 x_{i-1} - 2b_2 - a_2 h_{i-1}\right\},$$

$$m_n \geq \frac{-3\left(a_2 x_{n-1}^2 + b_2 x_{n-1} + c_2 - y_n\right)}{h_{n-1}} - 4a_2 x_{n-1} - 2b_2 - a_2 h_{n-1},$$

$$m_n \leq \frac{-3\left(a_1 x_{n-1}^2 + b_1 x_{n-1} + c_2 - y_n\right)}{h_{n-1}} - 4a_1 x_{n-1} - 2b_1 - a_1 h_{n-1}.$$

These conditions result in a C^1 spline interpolant. One can add more restrictions to get smoother interpolants, for a C^2 interpolant the following equations must be satisfied:

$$\frac{1}{h_i} m_i + \left(\frac{2}{h_i} + \frac{2}{h_{i+1}}\right) m_{i+1} + \frac{1}{h_{i+1}} m_{i+2} = \frac{3}{h_i}\delta_i + \frac{3}{h_{i+1}}\delta_{i+1},$$

These conditions restrict the feasible region but still can be handled by linear programming techniques.

The problem in both C^1 and C^2 cases may have several solutions, we can put more restrictions to achieve visually pleasing curves. The energy minimization technique proposed by Wolberg and Alfy (G. Wolberg and I. Alfy, An energy-minimization framework for monotonic cubic spline interpolation. Journal of Computational and Applied Mathematics, 2002, 143(2), pp. 145--188.) could be used to find smooth enough splines with minimum curvature.

Keywords: Shape-preserving, Constrained Interpolation, Cubic Hermite Splines

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