A Hoehnke Radical on the Topos *Act*-s

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Received: March 2019 Accepted: August 2019

Extended Abstract

Paper pages (139-148)

Introduction

Recently, applications of Lawvere-Tierney topologies on the category of presheaves in many branches of mathematics such as measure theory and quantum physics are observed. A class of (weak) Lawvere-Tierney topologies on the category of actions over a fixed monoid, is the ``ideal topology" denoted by j^I, which has been introduced and studied by the authors. As an application of the ideal topology, it is observed that some famous categories of separated acts over a monoid are special cases of j^I-separated categories. From this perspective, here we study the ideal topology and separated objects with respect to this topology on the topos **Act**-S. For an idempotent left ideal I of S, we show that the category **Act**-S is a radical extension of the category $Sep_{j^I}(Act-S)$, consisting of all I-separated S-acts.

Main Results

We fix a monoid S with identity element 1. Let I be a left ideal of S. Recall that the corresponding topology to the *ideal closure operator* C^I on **Act**-S is the equivariant map $j^I : RIdl(S) \to RIdl(S)$ given by $j^I(K) = \{s \in S | \forall t \in I, st \in K\}$, for any $K \in RIdl(S)$, called the *weak ideal topology*. Moreover, an S-act A is said to be I-*separated* if, for all $a, b \in A$ we have $(\forall s \in I, as = bs) \Rightarrow a = b$. It is proved that an S-act A is I-separated if and only if it is j^I-separated. Hence, we use the notion of ``I-separated” instead of j^I-separated.

The following theorem gives some characterizations of j^I-sheaves.

Theorem 2.1. Let I be a two sided ideal of the monoid S. For an S-act A, the following are equivalent:

(i) A is j^I-sheaf.

(ii) Every equivariant map $f : I \to A$ can be uniquely extended to an equivariant map $\bar{f} : S \to A$.

(iii) A is uniquely I-complete, in the sense that, for any equivariant map $f : I \to A$ there exists a unique element $a \in A$ such that $f = \lambda_a|_I$, i.e. $f(s) = \lambda_a(s) = as$ for all $s \in I$.

(iv) The map $\lambda: A \longrightarrow Hom_S(I, A)$, given by $\lambda(a) = \lambda_a$ for any $a \in A$, is an isomorphism.

Theorem 2.2. Let $S = G \dot{\cup} I$ be a monoid where G is a group and I a two sided ideal of S and A be an S-act. Then, A as an S-act is (principally) weakly sheaf whenever it is (principally) weakly sheaf as an I^1-act.

Theorem 2.3. Let $S = G \dot{\cup} I$ be a monoid and $h : S \to I^1$ be a nontrivial semigroup homomorphism with h(1) = 1. Then, A is j^I-sheaf in the topos **Act**-I^1 if and only if A is j^I-sheaf in the topos **Act**-S.

We know that for a (left) ideal I of S with $I^2 = I$ ($(IS)^2 = IS$), the inclusion functor $i : Sep_{j^I}(\text{Act-S}) \to \text{Act-S}$ has a left adjoint $L : \text{Act-S} \to Sep_{j^I}(\text{Act-S})$ given by $L(A) = A/\sigma_A$ (on objects) in which σ_A stands for the following S-act congruence on A

(1) $$\sigma_A = \{(a, b) \in A \times A \mid \forall t \in I, at = bt\}.$$

The functor L gives rise to a radical in the category **Act**-S as follows**:**

Theorem 2.4. Let I be a left ideal of S. The assignment $r_I : A \mapsto \sigma_A$ introduced in (1) is a preradical in **Act**-S. Moreover, r_I is a radical in **Act**-S whenever I is idempotent, i.e. $I^2 = I$.

The radical introduced in Theorem 2.4 has more property for S-acts.

Proposition 2.5. Let I be an idempotent left ideal of S. Then the class of all I-separated S-acts is exactly the torsion-free class of the radical r_I. Moreover, the assignment $r_I : A \mapsto \sigma_A$ introduced in (1), is a torsion in **Act**-S.

Now let B be a subact of an arbitrary S-act A and $\pi : A \to A/B$ the canonical projection. To any preradical r in **Act**-S, there is a closure operator C^r given by $C^r_A(B) = \pi^{-1}([B]_{r(A/B)})$. Now, corresponding to preradical r_I (for any left ideal I of S) in **Act**-S we have a closure operator C^{r_I} given by

$$C^{r_I}_A(B) = \{a \in A \mid \exists b \in B; \forall t \in I, at = bt\}.$$

Theorem 2.6. Let I be a left ideal of S. If a subact $B \subseteq A$ is C^{r_I}-dense then it is j^I-dense too. Moreover, any j^I-sheaf is also a C^{r_I}-sheaf.

Conclusion

The following conclusions were drawn from this research.

- Some characterizations of sheaves with respect to the ideal topology on the topos **Act**- S are presented.
- We investigate j^I-sheaves on monoids of the form $S = G \dot{\cup} I$, in which G is a group and I a two sided ideal of S.
- Using the ideal topology on **Act**-S, we construct a Hoehnke radical on this topos and investigate the relationship between the corresponding sheaf to the closure operator obtained from this radical and the corresponding sheaf with respect to the ideal topology.

Keywords: S-act; Ideal closure operator; Hoehnke (pre)radical; Torsion; Ideal topology.

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