

GENERALIZED JOINT HIGHER-RANK NUMERICAL RANGE

H.R. AFSHIN, S. BAGHERI AND M.A. MEHRJOOFARD*

ABSTRACT. The rank- k numerical range has a close connection to the construction of quantum error correction code for a noisy quantum channel. For a noisy quantum channel, a quantum error correcting code of dimension k exists, if and only if the associated joint rank- k numerical range is non-empty. In this paper, the notion of joint rank- k numerical range is generalized, and some statements of [2011, Generalized numerical ranges and quantum error correction, J. Operator Theory, 66: 2, 335-351.] are extended.

1. INTRODUCTION

Let M_n be the set of $n \times n$ complex matrices, and $A \in M_n$. Furthermore, assume that $k \in \{1, \dots, n\}$, $\alpha \subset \{1, \dots, n\}$. Throughout this paper, the following notations are fixed:

$$\omega_k = \exp\left(\frac{2\pi i}{k}\right)$$
$$\Omega_k = \{\omega_k^0, \omega_k^1, \dots, \omega_k^{k-1}\}.$$

Besides, the symbol $\sigma(A)$ stands for the spectrum of the matrix A , and A_α refers to the principal submatrix of A that lies in the rows and columns of A indexed by α .

Recently, the joint higher rank numerical range [5] has played a key role in finding quantum error correcting codes[4], and some researchers

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have taken this into consideration. The present article mainly concentrates on extending the notion of joint rank- k numerical range of $A = (A_1, \dots, A_m) \in M_n^m$, i.e. the set of all $(a_1, \dots, a_m) \in \mathbb{C}^m$, such that there exists the orthogonal projector P of rank k that satisfies

$$PA_jP = a_jP \quad \forall j.$$

Definition 1.1. Let $A = (A_1, \dots, A_m) \in M_n^m$, $B \in M_k$, and $k \leq n$. Then the set

$${}_B\Lambda_k(A) = \{(a_1, \dots, a_m) \in \mathbb{C}^m : \exists U \in M_{n,k}, s.t., U^*U = I_k, U^*A_jU = a_jB \quad \forall j\}$$

is called the joint matrix higher rank numerical range.

When $B = \text{diag}(b_1, \dots, b_k)$, we abbreviate ${}_B\Lambda_k(A)$ as ${}_{b_1, \dots, b_k}\Lambda_k(A)$, and in the case $b_1 = k = 1$, the joint numerical range of A is defined as $W(A) = {}_{b_1, \dots, b_k}\Lambda_k(A)$.

Definition 1.2. Let $A = (A_1, \dots, A_m) \in M_n^m$, $k \leq n$. The k th joint matrix numerical range of A is the set

$$W_k(A) = \{(U^*A_1U, U^*A_2U, \dots, U^*A_mU) : U \in M_{n,k}, U^*U = I_k\}$$

In [3], The authors have introduced " k -generalized projector". They have said that $A \in M_n$ is the k -generalized projector, if $A^k = A^*$ and $k > 1$.

Theorem 1.3. [3] Let $A \in M_n$, and $k \in \mathbb{N}$, $k > 1$. Then the following statements are equivalent:

- (a) A is a k -generalized projector.
- (b) A is a normal matrix, and $\sigma(A) \subset \{0\} \cup \Omega_{k+1}$.

Now, it is natural to extend "joint higher rank numerical range" as follows:

Definition 1.4. Let $A = (A_1, \dots, A_m) \in M_n^m$, and k and k' are positive integers. Then the set

$$\left\{ (a_1, \dots, a_m) \in \mathbb{C}^m : \begin{array}{l} \exists k' \text{ - generalized projector of rank } k \text{ } (P), \\ s.t., PA_jP = a_jP \quad \forall j \end{array} \right\}$$

is called the k' -generalized joint rank- k numerical range, and is abbreviated as $G\Lambda_{k',k}(A)$.

Notice that the recent definition is an obvious extension of "generalized higher rank numerical range," which has been defined in [1].

2. MAIN RESULTS

The proof of the following results is elementary and hence, we leave it to the interested reader.

Proposition 2.1. *Let $A = (A_1, \dots, A_m) \in M_n^m, k \leq n$, and $1 < k'$. The following statements are equivalent:*

- (i) $a = (a_1, a_2, \dots, a_m) \in G\Lambda_{k',k}(A_1, A_2, \dots, A_m)$.
- (ii) *There exist $b_1, \dots, b_k \in \Omega_{k'+1}$ and unitary matrix $U \in M_n$, such that for any $j \in \{1, \dots, m\}$,*

$$(U^* A_j U) \{1, 2, \dots, k\} = a_j \text{diag} \left(\{b_i\}_{i=1}^k \right).$$

- (iii) *There exist $b_1, \dots, b_k \in \Omega_{k'+1}$, and $X = \begin{bmatrix} x_1 & \dots & x_k \end{bmatrix} \in M_{n,k}$, such that $X^* X = I_k$, and*

$$\forall j \in \{1, \dots, m\}, X^* A_j X = a_j \text{diag} \left(\{b_i\}_{i=1}^k \right).$$

- (iv) *There exists $b_1, \dots, b_k \in \Omega_{k'+1}$, and orthonormal vectors $u_1, \dots, u_k \in \mathbb{C}^n$, such that*

$$\forall r \in \{1, \dots, m\} \forall i, j \in \{1, \dots, k\}, \langle A_r u_i, u_j \rangle = a_r b_i \delta_{ij}.$$

Proof. One can deduce, from Theorem 1.3, the equivalency of (i) and (ii). Notice that P is a k' -generalized projector of rank k , if and only if there exists a unitary matrix U , and numbers $b_1, \dots, b_k \in \Omega_{k'+1}$ such that

$$P = U^* \text{diag} \left(b_1, \dots, b_k, \underbrace{0, \dots, 0}_{n-k, 0's} \right) U.$$

Equivalence of parts (ii), (iii), and (iv) is obvious. \square

Corollary 2.2. *Let $A = (A_1, \dots, A_m) \in M_n^m$. Then:*

- (i) $\Lambda_k(A) \subset G\Lambda_{k',k}(A)$;
- (ii) $G\Lambda_{k',k}(A) \subset G\Lambda_{k',k}((A_1, \dots, A_{m-1})) \times \mathbb{C}$;
- (iii) $G\Lambda_{k',k+1}(A) \subset G\Lambda_{k',k}(A)$;

Proof. (i) is trivial, since every orthogonal projector of rank k is a k' -generalized projector of rank k .

(ii) is obvious.

(iii) Assume that $(\lambda_1, \dots, \lambda_m) \in G\Lambda_{k',k+1}(A)$. Then there exist orthonormal vectors $u_1, \dots, u_{k+1} \in \mathbb{C}^n$, and $b_1, \dots, b_{k+1} \in \Omega_{k'+1}$, such that $\langle A_r u_i, u_j \rangle = \lambda_r b_i \delta_{ij}$, for $r \in \{1, \dots, m\}$, $1 \leq i, j \leq k+1$. Therefore, by considering the orthonormal vectors $u_1, \dots, u_k \in \mathbb{C}^n$ and $b_1, \dots, b_k \in \Omega_{k'+1}$, we see that $\langle A_r u_i, u_j \rangle = \lambda_r b_i \delta_{ij}$ for $r \in \{1, \dots, m\}$, $1 \leq i, j \leq k$, and therefore, $(\lambda_1, \dots, \lambda_m) \in G\Lambda_{k',k}(A)$. \square

Corollary 2.3. Let $A = (A_1, \dots, A_m) \in M_n^m$, and $k', k > 1$. If $n \geq (k-1)(m+1)^2$, then $G\Lambda_{k',k}(A) \neq \emptyset$.

Proof. It suffices to consider [5, Proposition 2.4], and Corollary 2.2(i). \square

Proposition 2.4. Let $k' > 1$, and $A \in M_n^m$. Then:

- (i) $G\Lambda_{k',k}(A) = \bigcup_{b_1, \dots, b_k \in \Omega_{k'+1}} b_{1, \dots, b_k} \Lambda_k(A)$;
- (ii) $G\Lambda_{k',1}(A) = \bigcup_{b \in \Omega_{k'+1}} bW(A)$;

Proof. By definition, (i), and (ii) can readily be verified. \square

Corollary 2.5. Consider the Pauli matrices:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and let $A_1, \dots, A_m \in \{I, X, Y, Z\}$, and $k' > 1$. Then:

$$G\Lambda_{k',1}(I_{2^n}, A_1 \otimes \dots \otimes A_m) = \begin{cases} \bigcup_{b \in \Omega_{k'+1}} \{(b, b)\} & : \{A_1, \dots, A_m\} = \{I\} \\ \bigcup_{b \in \Omega_{k'+1}} b \{(1, a) : a \in [-1, 1]\} & : \text{elsewhere} \end{cases}$$

Proof. It suffices to note that Pauli matrices are normal, and:

$$\sigma(X) = \sigma(Y) = \sigma(Z) = \{-1, 1\}.$$

\square

The proofs of the next two results are straightforward, and thus are omitted.

Corollary 2.6.

$$G\Lambda_{k',1}(\{\text{diag}(a_{i1}, \dots, a_{in})\}_{i=1}^m) = \bigcup_{b \in \Omega_{k'+1}} b \text{conv}(\{(a_{1j}, \dots, a_{mj})\}_{j=1}^n)$$

Lemma 2.7. Let $A_j = \text{diag}(a_{1j}, \dots, a_{nj})$, $j = 1, \dots, m$. Then:

$$G\Lambda_{k',n}(A_1, \dots, A_m) \subset \left(\bigcup_{b_1, \dots, b_n \in \Omega_{k'+1}} \left\{ c_1 : c_1 = \frac{a_{11}}{b_1} = \dots = \frac{a_{n1}}{b_n} \right\} \right) \\ \times \left(\bigcup_{b_1, \dots, b_n \in \Omega_{k'+1}} \left\{ c_2 : c_2 = \frac{a_{12}}{b_1} \dots = \frac{a_{n2}}{b_n} \right\} \right) \\ \times \dots \\ \times \left(\bigcup_{b_1, \dots, b_n \in \Omega_{k'+1}} \left\{ c_m : c_m = \frac{a_{1m}}{b_1} \dots = \frac{a_{nm}}{b_n} \right\} \right)$$

Definition 2.8. [2] Let S be a convex set, and $R := S^{\frac{1}{k}} = \{z \in \mathbb{C} : z^k \in S\}$. Then R is called the **convex k th root set**.

Corollary 2.9. Let $A_j = \text{diag}(a_{1j}, \dots, a_{nj})$, $j \in \{1, \dots, m\}$, $k' > 1$, and there exists $i_1, i_2 \in 1, \dots, n$, such that $\frac{a_{i_1, j}}{a_{i_2, j}} \in \mathbb{C} \setminus \left(\mathbb{R}^{\frac{1}{k'+1}}\right)$,. Then:

$$G\Lambda_{k', n}(A_1, \dots, A_m) = \emptyset.$$

Now, we extend [5, Proposition 2.5]:

Proposition 2.10. Let $A = (A_1, \dots, A_m) \in M_n^m$, $B \in M_k$, and $1 \leq r < k \leq n$. Then:

$${}_B\Lambda_k(A) \subset \bigcap_{\substack{X \in M_{n, n-r}, \\ X^*X = I_{n-r}}} \left\{ (a_1, \dots, a_m) \in \mathbb{C}^m : \left(\begin{array}{c} W_{k-r}(a_1B, \dots, a_mB) \cap \\ W_{k-r}(X^*A_1X, \dots, X^*A_mX) \end{array} \right) \neq \emptyset \right\}$$

Proof. Let $a = (a_1, \dots, a_m) \in {}_B\Lambda_k(A)$, and $X \in M_{n, n-r}$ be such that $X^*X = I_{n-r}$. Then there exists $U \in M_{n, k}$, such that:

$$U^*U = I_k, U^*A_jU = a_jB \quad \forall j.$$

We can choose the orthonormal vectors $x_1, \dots, x_{k-r} \in (X\mathbb{C}^{n-r}) \cap (UC^k)$, and therefore, there exist $Y = [y_1, \dots, y_{k-r}] \in M_{k, k-r}$, $Z = [z_1, \dots, z_{k-r}] \in M_{n-r, k-r}$, such that $XZ = [x_1, \dots, x_{k-r}] = UY$, and $Y^*Y = I_{k-r} = Z^*Z$. Therefore,

$$Z^*X^*A_jXZ = a_jY^*BY \quad \forall j,$$

and the proof is completed. \square

The following lemma can directly follow from the definition.

Lemma 2.11. Let $A = (A_1, \dots, A_m) \in M_{n_1}^m$, $C = (C_1, \dots, C_m) \in M_{n_2}^m$, $B \in M_k$, $k \leq \min\{n_2, n_1\}$, and there exists the matrix $V \in M_{n_1, n_2}$, such that $V^*V = I_{n_2}$, and for any $1 \leq j \leq m$, $C_j = V^*A_jV$. Then:

$${}_B\Lambda_k(C) \subset {}_B\Lambda_k(A)$$

Lemma 2.12. Let $A = (A_1, \dots, A_m) \in M_{n_1}^m$, $C = (C_1, \dots, C_m) \in M_{n_2}^m$, $B \in M_k$, $k \leq n_1 \leq n_2$, and for any j , $A_j = C_j \{1, \dots, n_1\}$. Then:

$${}_B\Lambda_k(A) \subset {}_B\Lambda_k(C).$$

Proof. Let $(a_1, \dots, a_m) \in {}_B\Lambda_k(A)$. Therefore, there exist $X \in M_{n_1, k}$, such that $X^*X = I_k$, and $X^*A_jX = a_jB$, for all $j \in \{1, \dots, m\}$.

Now, define $Y = \begin{bmatrix} X \\ 0 \end{bmatrix}_{n_2 \times k}$. Then, one can see that $Y^*Y = I_k$, and $Y^*C_jY = a_jB$, for all $j \in \{1, \dots, m\}$. \square

Corollary 2.13. *Let $A = (A_1, \dots, A_m) \in M_{n_1}^m, C = (C_1, \dots, C_m) \in M_{n_2}^m, k \leq n_1 \leq n_2$, and for any j , $A_j = C_j \{1, \dots, n_1\}, 1 < k'$; and there exists the matrix $V \in M_{n_1, n_2}$, such that $V^*V = I_{n_2}$, and for any $1 \leq j \leq m$, $D_j = V^*A_jV$. Then:*

$$G\Lambda_{k',k}(D) \subset G\Lambda_{k',k}(A) \subset G\Lambda_{k',k}(C).$$

The following theorem is an extension of [5, Theorem 3.1].

Theorem 2.14. *Let $A = (A_1, \dots, A_m) \in M_n^m, \hat{k} \geq (m+2)k, B \in M_k, (0, \dots, 0) \in {}_B\Lambda_{\hat{k}}(A)$, and $(a_1, \dots, a_m) \in {}_B\Lambda_k(A)$. Then for any $t \in [0, 1]$,*

$$t(a_1, \dots, a_k) \in {}_B\Lambda_k(A).$$

Proof. Assume that there exist $X \in M_{n,k}$, and $V \in M_{n,(m+2)k}$, such that:

$$\begin{aligned} X^*X &= I_k, \forall j, X^*A_jX = a_jB, \\ V^*V &= I_{(m+2)k}, \forall j, V^*A_jV = 0_{(m+2)k}. \end{aligned}$$

By the Lemma 2.11 and Lemma 2.12, it suffices to show that there is the non-singular matrix $Z \in M_{n,(m+2)k}$, such that:

$$\left\{ \begin{array}{l} Z^*Z = I_{(m+2)k}, \\ \forall j, Z^*A_jZ = \begin{bmatrix} \begin{bmatrix} a_jB & 0_k \\ 0_k & 0_k \end{bmatrix} & * \\ * & * \end{bmatrix}_{(m+2)k \times (m+2)k} \end{array} \right. . \quad (2.1)$$

Because, in this case, we have:

$$\begin{aligned} & {}_B\Lambda_k \left(\begin{bmatrix} a_1B & 0_k \\ 0_k & 0_k \end{bmatrix}, \dots, \begin{bmatrix} a_mB & 0_k \\ 0_k & 0_k \end{bmatrix} \right) \\ & \subset {}_B\Lambda_k (Z^*A_1Z, \dots, Z^*A_mZ) \\ & \subset {}_B\Lambda_k (A_1, \dots, A_m) \end{aligned}$$

and

$$\forall t \in [0, 1], t(a_1, \dots, a_k) \in {}_B\Lambda_k \left(\begin{bmatrix} a_1B & 0_k \\ 0_k & 0_k \end{bmatrix}, \dots, \begin{bmatrix} a_mB & 0_k \\ 0_k & 0_k \end{bmatrix} \right)$$

(Note that for any $t \in [0, 1]$ there exists $U = \begin{bmatrix} \sqrt{t}I_k \\ \sqrt{1-t}I_k \end{bmatrix}$ such that

$$\text{for all } j, U^* \begin{bmatrix} a_jB & 0_k \\ 0_k & 0_k \end{bmatrix} U = ta_jB.)$$

Now, we want to select $Y \in M_{n,k}$, and $W \in M_{n,mk}$, such that their columns selected from columns of V and $Z = \begin{bmatrix} X & Y & W \end{bmatrix}$ satisfy in 2.1. But, 2.1 is equivalent to:

$$\left\{ \begin{array}{l} Y^*Y = I_k, W^*W = I_{mk}, X^*Y = 0_k, X^*W = 0_{k,mk}, Y^*W = 0_{mk}, \\ \forall j, X^*A_jY = Y^*A_jX = Y^*A_jY = 0_k. \end{array} \right.$$

Thus, in order to find the Y , it is sufficient to find the k columns of columns of V , such that they lie in the space H^\perp , such that:

$$H = \text{span} \left(\begin{array}{c} \{\text{columns of } X\} \cup \{\text{columns of } A_1 X\} \\ \cup \dots \cup \{\text{columns of } A_m X\} \end{array} \right).$$

But $\dim(H) \leq (m+1)k$, while V has $(m+2)k$ columns. Therefore, we can construct Y . Now, $\text{span}(\{\text{columns of } X\} \cup \{\text{columns of } Y\})$ is a space with dimension $2k$ and so we can find the mk columns of V , such that they lie not in this space, and assume W , such that their columns are constructed by those columns. \square

Also, we can extend [5, proposition 2.1], as follows:

Proposition 2.15. *Suppose $A = (A_1, \dots, A_m) \in M_n^m, k \leq n, 1 < k'$ and $S = (s_{ij})$ is an $m \times n$ matrix. If $B_j = \sum_{i=1}^m s_{ij} A_i$, for $j = 1, \dots, n$, then:*

$$\{aT : a \in G\Lambda_{k',k}(A)\} \subset G\Lambda_{k',k}(B).$$

Equality holds, if $\{A_1, \dots, A_m\}$ is linearly independent and:

$$\text{span}\{A_1, \dots, A_m\} = \text{span}\{B_1, \dots, B_n\}.$$

REFERENCES

1. H.R. Afshin and M.A. Mehrjoofard, Generalized higher-rank numerical range, *Journal of Mahani Mathematical Research Center* **1** (2012), 163–168.
2. H.R. Afshin, M.A. Mehrjoofard and A. Salemi, Polynomial numerical hulls of order 3, *Electronic Journal of Linear Algebra* **18** (2009), 253–263.
3. J. Benítez and N. Thome, Characterizations and linear combinations of k -generalized projectors, *Linear Algebra and its Applications* **410** (2005), 150–159.
4. D.W. Kribs, A. Pasieka, M. Laforest, C. Ryan and M.P. Silva, Research problems on numerical ranges in quantum computing, *Linear and Multilinear Algebra* **57** (2009), 491–502.
5. C.K. Li and Y. T. Poon, Generalized numerical ranges and quantum error correction, *J. Operator Theory* **66** (2011), 335–351.

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GENERALIZED JOINT HIGHER-RANK NUMERICAL RANGE

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برد عددی رتبه-بالاتر توأم تعمیم یافته

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برد عددی k رتبه رابطه ی نزدیکی با ساختن کد تصحیح خطای کوانتومی برای کانال کوانتومی پر پارازیت دارد. در کانال کوانتومی پر پارازیت، کد تصحیح خطای کوانتومی از بعد k وجود دارد اگر و تنها اگر برد عددی k رتبه ی توأم مرتبط با آن، \mathbb{A} ناتهی باشد. در این مقاله مفهوم برد عددی k رتبه ی توأم تعمیم داده شده و برخی گزاره های مقاله

[2011, Generalized numerical ranges and quantum error correction, J. Operator Theory, 66: 2, 335-351.]

توسعه داده شده است.

کلمات کلیدی: تصویرگر تعمیم یافته، برد عددی رتبه بالاتر توأم، برد عددی ماتریسی توأم، برد عددی رتبه بالاتر ماتریسی توأم، برد عددی رتبه بالاتر توأم تعمیم یافته.