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ANNIHILATING SUBMODULE GRAPHS FOR MODULES OVER COMMUTATIVE RINGS

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ABSTRACT. In this article, we give several generalizations of the concept of annihilating an ideal graph over a commutative ring with identity to modules. We observe that, over a commutative ring, R, $\mathbb{AG}_*({}_RM)$ is connected, and $\mathrm{diam}\mathbb{AG}_*({}_RM) \leq 3$. Moreover, if $\mathbb{AG}_*({}_RM)$ contains a cycle, then $\mathrm{gr}\mathbb{AG}_*({}_RM) \leq 4$. Also for an R-module M with $\mathbb{A}_*(M) \neq S(M) \setminus \{0\}$, $\mathbb{A}_*(M) = \emptyset$, if and only if M is a uniform module, and $\mathrm{ann}(M)$ is a prime ideal of R.

1. INTRODUCTION

In the literature, there are many papers on assigning a graph to a ring, group, semigroup or module (see for example [1]-[16], [19]and [21]-[25]). The concept of zero-divisor graph of a commutative ring R was first introduced by Beck [11], where he was mainly interested in colorings. In his work, all elements of the ring were vertices of the graph. The investigation of colorings of a commutative ring was then continued by Anderson and Naseer [9]. Let $Z(R)$ be the set of zero-divisors of R. In [8], Anderson and Livingston associated a graph, $\Gamma(R)$, to R with vertices $Z(R) \setminus \{0\}$, the set of non-zero zero-divisors of R, and for distinct x, $y \in Z(R) \setminus \{0\}$, the vertices x, and y are adjacent if and only if $xy = 0$. In [23], Sharma and Bhatwadekar define another graph on R, $G(R)$, with vertices as elements of R, where, two distinct vertices a, and b are adjacent, if and only if $Ra + Rb = R$. (See also [21] and [5], in which, the notion "comaximal graph of commutative

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rings" is investigated.) Recently, Anderson and Badawi, in [6], have introduced and studied the total graph of R, denoted by $T(\Gamma(R))$. It is the (undirected) graph with all elements of R as vertices, and for distinct $x,\ y \in R$, the vertices x and y are adjacent, if and only if $x + y \in Z(R)$. We denote the set of all proper ideals of R by $\mathbb{I}(R)$. In [13], Behboodi and Rakeei named an ideal, I of R, an *annihilating-ideal* if there exists a non-zero ideal J of R, such that $IJ = (0)$, and used the notation $\mathbb{A}(R)$ for the set of all annihilating-ideals of R. They defined *the annihilating-ideal graph* of R, denoted by $\mathbb{AG}(R)$, as a graph with vertices $\mathbb{A}(R)^* = \mathbb{A}(R) \setminus \{(0)\}$, where, distinct vertices I and J are adjacent, if and only if $IJ = (0)$. They extensively investigated the interplay between the graph-theoretic properties of $\mathbb{AG}(R)$ and the ring-theoretic properties of R. There are a few papers on annihilating the ideal graph (see [1] ,[13], and [14]). In the next sections, we introduce and study various module generalizations of the annihilating ideal graphs of commutative rings.

Recall that a graph Γ is connected, if there is a path between any two distinct vertices. For the distinct vertices x and y of Γ, let $d(x, y)$ be the length of the shortest path from x to y ($d(x, y) = \infty$, if there is no such path). The diameter of Γ, $\text{diam}(\Gamma)$, is defined as sup $\{d(x, y) \mid x$ and y are distinct vertices of Γ $\}$. The girth of Γ, denoted by $g(\Gamma)$, is defined as the length of the shortest cycle in Γ ($g(\Gamma) = \infty$; if Γ contains no cycles).

2. Annihilating graphs for modules

We begin with the following definition (we note that for any R-module M, $(N : M) := Ann(M/N)$, for $N \leq M$).

Definition 2.1. Let M be an R-module. A submodule N of M is called:

- *weakly annihilating submodule*, if either $N = 0$ or $(N : M)(K : M)M = 0$, for some non-zero proper submodule K of M.
- *annihilating sub-module*, if either $N = 0$ or $0 \neq (N : M)$ and $(N : M)(K : M)M = 0$ for some non-zero proper submodule K of M with $0 \neq (K : M)$.
- *strongly annihilating submodule*, if either $N = 0$ or $Ann(M) \subset (N : M)$, and $(N : M)(K : M)M = 0$ for some non-zero proper sub-module K of M with $Ann(M) \subset (K : M)$.

For any module M, we denote $\mathbb{A}_*(M)$, $\mathbb{A}(M)$ and $\mathbb{A}^*(M)$, respectively, for the set of *weakly annihilating submodule, annihilating submodule,* and *strongly annihilating submodule* of M. It is clear that

$$\mathbb{A}^*(M) \subseteq \mathbb{A}(M) \subseteq \mathbb{A}_*(M).$$

The following proposition shows that for any module, we only need to consider strongly annihilating and weakly annihilating submodules.

Proposition 2.2. *Let R be a ring and M be an R-module. Then*
1) If M is a faithful R-module, then $\mathbb{A}^*(M) = \mathbb{A}(M)$;
2) If M is a non-faithful R-module, then $\mathbb{A}(M) = \mathbb{A}_*(M)$.

Proof. By Definition 2.1, the results hold. □

The following proposition shows that, for $M = R$, the three parts of Definitions 2.1 are equivalent and they are the generalizations of annihilating ideal.

Proposition 2.3. *Let R be any ring, and I be an ideal of R. Then the following are equivalent:*
1) I is an annihilating ideal of R;
2) I is a weakly annihilating submodule of ${}_RR$;
3) I is an annihilating submodule of ${}_RR$;
4) I is a strongly annihilating submodule of ${}_RR$.

Proof. The proof is easy. □

Now, for an R-module M, we let $\tilde{\mathbb{A}}_*(M) := \mathbb{A}_*(M) \setminus \{0\}$, $\tilde{\mathbb{A}}(M) := \mathbb{A}(M) \setminus \{0\}$, and $\tilde{\mathbb{A}}^*(M) := \mathbb{A}^*(M) \setminus \{0\}$. Then we associate the three undirected (simple) graphs $\mathbb{AG}_*({}_RM)$, $\mathbb{AG}({}_RM)$, and $\mathbb{AG}^*({}_RM)$ to M with vertices $\tilde{\mathbb{A}}_*(M)$, $\tilde{\mathbb{A}}(M)$, and $\tilde{\mathbb{A}}^*(M)$, respectively, and for which, the vertices N, and K are adjacent, if and only if $(N : M)(K : M)M = 0$. It is clear that we have $\mathbb{AG}^*({}_RM) \subseteq \mathbb{AG}({}_RM) \subseteq \mathbb{AG}_*({}_RM)$, as induced subgraphs. In fact, Proposition 2.2 shows that for any R-module M, either $\mathbb{AG}({}_RM) = \mathbb{AG}^*({}_RM)$ or $\mathbb{AG}({}_RM) = \mathbb{AG}_*({}_RM)$.
Let $\mathbb{AG}(R)$ be the annihilating ideal graph of a ring R. By Proposition 2.3, we have $\mathbb{AG}^*({}_RR) = \mathbb{AG}({}_RR) = \mathbb{AG}_*({}_RR) = \mathbb{AG}(R)$. In the following theorem, we determine when $\mathbb{AG}_*({}_RM) = \mathbb{AG}({}_RM) = \mathbb{AG}^*({}_RM)$.

Theorem 2.4. *Let M be an R-module. Then $\mathbb{AG}_*({}_RM) = \mathbb{AG}({}_RM) = \mathbb{AG}^*({}_RM)$, if and only if $Ann(M) \subset (N : M)$, for every non-zero submodule N of M.*

Proof. ($\Rightarrow$) If for some non-zero proper submodule N of M, $(N : M) =$Ann(M), then for every non-zero submodule K of M, we have $(K : M)(N : M)M = 0$, so that N —— K is a path in $\mathbb{AG}_*(M)$, and

hence, is a path in $\mathbb{AG}^*(M)$, which implies $\mathrm{Ann}(M) \subset (N : M)$, which is a contradiction.
($\Leftarrow$) By definition 2.1. □

Recall that an R-module M is called *multiplication,* in case for every non-zero submodule N of M, there exists an ideal I of R, such that $N = IM$. One can show that if M is a multiplication module, then for every submodule N of M, we have $N = (N : M)M$.

Corollary 2.5. *Let M be a multiplication R-module. Then $\mathbb{AG}_*({}_RM) = \mathbb{AG}({}_RM) = \mathbb{AG}^*({}_RM)$.*

Proof. The result holds, since multiplication modules have the property that for every non-zero submodule N of M, $\mathrm{Ann}(M) \subset (N : M)$. □

Proposition 2.6. *Let M be an R-module with $0 \neq I = Ann(M)$. Then the following statements hold.*
(1) $\mathbb{AG}({}_RM) = \mathbb{AG}_*({}_RM) = \mathbb{AG}_*({}_{R/I}M)$;
(2) $\mathbb{AG}^*({}_RM) = \mathbb{AG}^*({}_{R/I}M) = \mathbb{AG}({}_{R/I}M)$.

Proof. Let $N \in \tilde{\mathbb{AG}}_*(M)$. Then there exists $0 \neq K \leq M$ such that $(N : M)(K : M)M = 0$. It is clear that $I = Ann(M) \subsetneq (N : M) \cap (K : M)$, $Ann_{R/I}(M/N) = (N : M)/I$, $Ann_{R/I}(M/K) = (K : M)/I$, and $((N : M)/I)(K : M)/I)M = 0$. This follows that $N \in \mathbb{A}_*({}_RM)$, if and only if $N \in \mathbb{A}_*({}_{R/I}M)$, and the vertices N and K are adjacent in $\mathbb{AG}_*({}_RM)$, if and only if N and K are adjacent in $\mathbb{AG}_*({}_{R/I}M)$. Therefore, $\mathbb{AG}_*({}_RM) = \mathbb{AG}_*({}_{R/I}M)$. Similarly, we can show that $\mathbb{AG}^*({}_RM) = \mathbb{AG}^*({}_{R/I}M)$. □

Proposition 2.7. *Let M be a homogeneous sem-isimple R-module. Then $\mathbb{AG}^*({}_RM)$ is the empty graph.*

Proof. Since $Ann(M)$ is a maximal ideal, the result holds. □

Proposition 2.8. *Let M be an R-module. Then $\mathbb{AG}^*({}_RM)$ is the empty graph, if and only if $Ann(M)$ is a prime ideal of R.*

Proof. Since for every non-zero submodules N, K of M, $(N : M)(K : M)M = 0$ if and only if $(N : M)M = 0$ or $(K : M)M = 0$, if and only if $\mathrm{Ann}(M)$ is a prime ideal of R, we are done. □

Corollary 2.9. *Let M be an R-module. Then $\mathbb{AG}_*(M) = \mathbb{AG}(M) = \mathbb{AG}^*(M) = \emptyset$, if and only if $Ann(M)$ is a prime ideal of R, and $Ann(M) \subset (N : M)$, for every non-zero submodule N of M.*

Proof. It follows from Theorem 2.4 and Proposition 2.8. □

3. Weakly annihilating submodule graph

Now, one may ask a question; when two submodules of an R-module M maybe connected to each other in $\mathbb{AG}_*(M)$?

Lemma 3.1. *Let M be an R-module, and N, K be the submodules of M.*
1) If $N \cap K = 0$, then N —— K is a path in $\mathbb{AG}_*(M)$.
2) If N —— K is a path in $\mathbb{AG}_*(M)$, then for each $0 \neq N_1 \leq N$ and $0 \neq K_1 \leq K$, N_1 —— K_1 is also a path in $\mathbb{AG}_*(M)$.

Proof. 1) The result holds, since $(N : M)(K : M)M \subseteq N \cap K$.
2) Let $N, K \in \tilde{\mathbb{AG}}_*(M)$, and $0 \neq N_1 \leq N$. Assume that N —— K is a path in $\mathbb{AG}_*(M)$, and $0 \neq K_1 \leq K$. Then $(N : M)(K : M)M = 0$. It is clear that $(N_1 : M) \subseteq (N : M)$, and $(K_1 : M) \subseteq (K : M)$. Therefore, $(N_1 : M)(K_1 : M)M \subseteq (N : M)(K : M)M = 0$. Thus N_1 —— K_1 is also a path in $\mathbb{AG}_*(M)$. □

Corollary 3.2. *Let M be an R-module. Then $N \in \mathbb{AG}_*(M)$, for every non-zero non-essential submodule N of M.*

In [7, Theorem 2.3], it is shown that, for any commutative ring R, $\Gamma(R)$ is connected, and diam$\Gamma(R) \leq 3$. Furthermore, if $\Gamma(R)$ contains a cycle, then $g(\Gamma(R)) \leq 7$. Moreover, in [22], it is shown that, for any commutative ring R, the girth of the zero-divisor graph of R is less than (or equal to) 4. In the next theorem, we give a generalization of these result for modules.

Theorem 3.3. *Let M be any R-module.*
1) The graph $\mathbb{AG}_*(_RM)$ is a connected graph, and diam$\mathbb{AG}_*(_RM) \leq 3$.
2) If $\mathbb{AG}_*(_RM)$ contains a cycle, then $g(\mathbb{AG}_*(_RM)) \leq 4$.

Proof. (1) Let $N, K \in \tilde{\mathbb{AG}}_*(M)$ be distinct. If $(N : M)(K : M)M = 0$, then $d(N, K) = 1$. So suppose that $(N : M)(K : M)M \neq 0$. Hence, there are $A, B \in \tilde{\mathbb{AG}}_*(M) \setminus \{N, K\}$ with $(A : M)(N : M)M = (B : M)(K : M)M = 0$. If $(A : M)(B : M)M = 0$, then $N — A — B — K$ is a path of length 3. Thus we may assume that $(A : M)(B : M)M \neq 0$; then $T = A \cap B \neq 0$. Hence by Lemma 2.1, $N — T — K$ is a path of length 2, and hence, $d(N, K) \leq 3$. Thus $diam(\mathbb{AG}(_RM)) \leq 3$.
(2) Let $N_1 — N_2 — ... — N_{k-1} — N_k$ be a cycle with length $k \geq 3$. Put $N_{k+1} := N_1$, and $N_0 := N_k$. If N_i has a proper non-zero submodule T_i (for some $1 \leq i \leq k$), then, by Lemma 2.1, $N_{i-1} — T_i — N_{i+1}$ is a path, and $N_{i-1} — T_i — N_{i+1} — N_i — N_{i-1}$ is a cycle of length at most 4. If every N_i has no proper non-zero submodule, then every N_i is a simple module. If $N_1 \cap N_4 = 0$ then $N_1 — N_2 — N_3 — N_4 — N_1$

is a cycle of length 4. If $N_1 \cap N_4 \neq 0$, then $N_1 = N_4$, and $N_1 — N_2 - - N_3 — N_4$ is a cycle of length 3.Thus $g(\mathbb{AG}_*(_RM)) \leq 4$.

□

Corollary 3.4. *Let M be any non-faithful R-module. Then $\mathbb{AG}(_RM)$ is connected, and diam$\mathbb{AG}(_RM) \leq 3$. Moreover, if $\mathbb{AG}(_RM)$ contains a cycle, then $g(\mathbb{AG}(_RM)) \leq 4$.*

Proof. If M is a non-faithful R-module, then, by Proposition 2.2, $\mathbb{AG}(_RM) = \mathbb{AG}_*(_RM)$. Now, apply Theorem 3.3. □

The following result assures us when $\mathbb{AG}_*(_RM)$ contains a cycle. As we can see it happens when $\mathbb{AG}_*(_RM)$ contains a path of length 4. In fact, when $\mathbb{AG}_*(_RM)$ has a path of length 4, then gr$(\mathbb{AG}_*(_RM)) \leq 4$.

Proposition 3.5. *Let M be an R-module. If $\mathbb{AG}_*(_RM)$ contains a path of length 4, then $\mathbb{AG}_*(_RM)$ contains a cycle.*

Proof. Let $N_1 - N_2 - N_3 - N_4 - N_5$ be a path of length 4. If $N_2 \cap N_4 = 0$, then N_2 and $N_4 = 0$ are adjacent, and hence, $N_2 - N_3 - N_4 - N_2$ is a cycle. Now, assume that $0 \neq K \leqslant N_2 \cap N_4$. One of the following cases holds:
(Case 1). If $K = N_1$, then, by Lemma 3.1, $N1 - N_2 - N3 - N_1$ is a cycle.
(Case 2). If $K = N_2$, then, by Lemma 3.1, $N_2 - N_3 - N_4 - N_5 - N_2$ is a cycle.
(Case 3). If $K = N_3$, then, by Lemma 3.1, $N_1 - N_2 - N_3 - N_1$ is a cycle.
(Case 4). If $K = N_4$, then by Lemma 3.1, $N_3 - N_4 - N_1 - N_2 - N_3$ is a cycle.
(Case 5). If $K = N_5$, then by Lemma 3.1, $N_3 - N_4 - N_5 - N_3$ is a cycle.
(Case 6). If $K \notin \{N_1, N_2, N_3, N_4, N_5\}$, then by Lemma 3.1, $N_1 - K - N_3 - N_2 - N_1$ is a cycle. □

Corollary 3.6. *Let R be a ring. If $\mathbb{AG}(R)$ contains a path of length 4, then $\mathbb{AG}(R)$ contains a cycle.*

Proof. By Proposition 3.5, the verification is immediate. □

Let Γ be a graph with vertices V, and let $\emptyset \neq A,\ B \subseteq V$. Then $A \leftrightsquigarrow B$ means that, for each $a \in A$, $b \in B$, a —— b is a path in Γ. Also, for each non-zero R-module M, we denote the set of all non-zero proper submodules of M by $\tilde{S}(M)$ (i.e., $\tilde{S}(M) = S(M)\setminus\{0\}$). Let $M = M_1 \oplus M_2$, where $M_i \neq 0$, $i = 1, 2$. Then $\tilde{S}(M_1), \tilde{S}(M_2) \subseteq \tilde{\mathbb{AG}}_*(M)$, and $\tilde{S}(M_1) \leftrightsquigarrow \tilde{S}(M_2)$ in $\mathbb{AG}_*(M)$.

Theorem 3.7. *Let M be an R-module with $\mathbb{A}_*(M) \neq S(M) \setminus \{0\}$. Then $\mathbb{A}_*(M) = \emptyset$, if and only if M is a uniform module, and $Ann(M)$ is a prime ideal of R.*

Proof. Let $\mathbb{A}_*(M) = \emptyset$. Then, by Lemma 3.1 for non-zero elements $K, N \in S(M)$, $N \cap K$ must be non-zero. This implies that M is a uniform R-module. Now, suppose that I and J are ideals of R, such that $IJ \subseteq$Ann(M), but neither $I \subseteq$Ann(M) nor $J \subseteq$Ann(M). Therefore,

$$(JM : M)(IM : M)M \subseteq (JM : M)IM \subseteq IJM = 0.$$

Hence, IM and JM belong to $\mathbb{A}_*(M)$. This is a contradiction. Conversely, assume that M is a uniform module with, prime annihilator such that $0 \neq N \in \mathbb{A}_*(M)$. There exists $0 \neq K \in \mathbb{A}_*(M)$, such that $(N : M)(K : M)M = 0$. Therefore $(N : M)(K : N) \subseteq$Ann(M), and hence, either $(N : M) \subseteq$Ann(M) or $(K : M) \subseteq$Ann(M) because Ann(M) is a prime ideal. Hence, for each non-zero submodule T of M, either $(T : M)(N : M)M = 0$ or $(T : M)(K : M)M = 0$. Thus $\mathbb{A}_*(M) = S(M) \setminus \{0\}$. This is a contradiction. □

Corollary 3.8. *Let R be a ring. R is a domain, if and only if there exists a faithful R-module M with $\Gamma_*(M) = \emptyset$.*

Proof. By Theorem 3.7, the verification is immediate. □

Proposition 3.9. *Let M be a non-simple semisimple R-module. Then $\mathbb{AG}_*({}_RM)$ is a connected graph with vertex set $\tilde{S}(M)$.*

Proof. Since every proper submodule of a semisimple module M is a direct summand of M, by Lemma 3.1 is evident. □

Lemma 3.10. *Let $M = M_1 \oplus M_2$, and $0 \neq N \in \tilde{\mathbb{A}}_*(M_1)$. Then $N \oplus 0 \in \tilde{\mathbb{A}}_*(M)$. Moreover, if the vertices N and K are adjacent in $\mathbb{AG}_*(M_1)$, then $N \oplus 0$, $K \oplus 0$ are adjacent in $\mathbb{AG}_*({}_RM)$.*

Proof. It is clear that for every $N \leq M_1$;

$$\frac{M_1 \oplus M_2}{N \oplus 0} \cong \frac{M_1}{N} \oplus M_2.$$

Therefore, if $N \in \tilde{\mathbb{A}}_*(M_1)$, then there exists $0 \neq K \leq M_1$, such that $(N : M_1)(K : M_1)M_1 = 0$. Now, $(N \oplus 0 : M_1 \oplus M_2) = Ann(\frac{M_1}{N} \oplus M_2)$, and $(K \oplus 0 : M_1 \oplus M_2) = Ann(\frac{M_1}{K} \oplus M_2)$. Thus $(N \oplus 0 : M_1 \oplus M_2)(K \oplus 0 : M_1 \oplus M_2)M = 0$, and it follows that $N \oplus 0 \in \tilde{\mathbb{A}}_*(M)$. Now, the "moreover" statement is clear. □

Theorem 3.11. *Let $M = M_1 \oplus M_2$, such that $\mathbb{AG}_*(M_1) \neq \emptyset$. Then $\mathbb{AG}_*(M_1) \cong G$, where G is an induced subgraph of $\mathbb{AG}_*({}_RM)$ with vertex set $\{N \oplus 0 \in \tilde{\mathbb{A}}_*(M) |\ N \in \tilde{\mathbb{A}}_*(M_1)\}$.*

Proof. The result is a consequence of Lemma 3.10. □

Lemma 3.12. *Let M be an R-module, and $f \in \mathrm{End}_R(M)$ be a non-monic and non-zero endomorphism. Then $\ker(f)$ is adjacent to $\mathrm{Im}(f)$ in $\mathbb{AG}_*(M)$.*

Proof. Let $K = \ker(f)$, and $I = \mathrm{Im}(f)$. Then:

$$(K : M)(I : M)M \subseteq (K : M)f(M) \subseteq f((K : M)M) \subseteq f(K) = 0.$$

Thus $\ker(f)$ is adjacent to $\mathrm{Im}(f)$. □

Corollary 3.13. *Let M be an R-module, and f be a non-monic epimorphism of M. Then $\mathbb{A}_*(M) = S(M) \setminus \{0\}$.*

Proof. Since f is non-monic, $\ker(f) \neq 0$. By Lemma 3.12, $\mathrm{Im}(f) = M$ is adjacent to $\ker(f)$. Now, by Lemma 3.1, any sub-module of M is adjacent to $\ker(f)$. Therefore, $\mathbb{A}_*(M) = S(M) \setminus \{0\}$. □

Corollary 3.14. *Let M be an R-module. If $\mathbb{A}_*(M) \neq S(M) \setminus \{0\}$, then M is a Hopfian module.*

Proof. Let $f : M \to M$ be a non-zero epimorphism. Then f must be monic. Otherwise, by Corollary 3.13, $\mathbb{A}_*(M) = S(M) \setminus \{0\}$, which is a contradiction. □

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گراف های زیر مدول پوچ ساز برای مدول ها روی حلقه های جابجایی

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در این مقاله چند تعمیم از مفهوم گراف ایده ال های پوچ ساز روی حلقه های جابجایی و یکدار به مدول ها ارائه خواهیم داد. ما دریافتیم که روی یک حلقه R گراف $AG_*({}_RM)$ همبند و $\mathrm{diam}AG_*({}_RM) \leq ۳$ است. به عبارت بیشتر، اگر $AG_*({}_RM)$ شامل یک دور باشد آن گاه $\mathrm{gr}AG_*({}_RM) \leq ۴$ است. همچنین برای هر R -مدول مانند M با این خاصیت که $A_*(M) \neq S(M) \setminus \{۰\}$ داریم $A_*(M) = \emptyset$ اگر و تنها اگر M یک مدول یکنواخت و $\mathrm{Ann}(M)$ یک ایده ال اول از حلقه R باشد.

کلمات کلیدی: گراف مقسوم علیه صفر، گراف زیر مدول های پوچ ساز، زیر مدول بطور ضعیف پوج ساز.