

H_v MV-ALGEBRAS II

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ABSTRACT. In this paper, we continue our study on H_v MV-algebras. The quotient structure of an H_v MV-algebra by a suitable type of congruence is studied, and some properties and related results are given. Some homomorphism theorems are given, as well. Also the fundamental H_v MV-algebra, and the direct product of a family of H_v MV-algebras are investigated, and some related results are obtained.

1. INTRODUCTION

In 1958, Chang [2] introduced the concept of an MV-algebra as an algebraic proof of completeness theorem for \aleph_0 -valued Łukasiewicz propositional calculus; also see [3]. Many mathematicians have worked on MV-algebras, and obtained significant results. The hyperstructure theory (also called multialgebras) was introduced in 1934 by Marty [9]. Around the 40's, several authors worked on the hypergroups, especially in France and in the United States, but also in Italy, Russia, and Japan.

Recently, Ghorbani et al. [7] have applied the hyperstructures to MV-algebras, introduced the concept of hyper MV-algebra and investigated some related results; also see [8, 10]. Hyperstructures have many applications to several sectors of both the pure and applied sciences. A short review of the theory of hyperstructures has appeared in [4]. In [5], a wealth of applications can also be found. There are applications

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to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy set and rough sets, automata, cryptography, combinatorics, codes, artificial intelligence, and probabilities.

H_v -structures were introduced by Vougiouklis in the 4th AHA congress [11]; also see [12] and [13]. The concept of H_v -structure constitute a generalization of the well-known algebraic hyperstructures (hypergroup, hyperring, hypermodule, and so on). Actually, some axioms concerning the above hyperstructures such as the associative law, and distributive law have been replaced by their corresponding weak axioms. The reader finds in [12] some basic definitions and theorems about H_v -structures. Since then, the study of the H_v -structure theory has been pursued in many directions by Vougiouklis, Davvaz, Spartalis and others. A survey of the most results and applications of H_v -structure theory is based up on many papers, some of which contain more detailed presentations (see [6]).

In this paper, the quotient structure of H_v MV-algebras, direct product of H_v MV-algebras, and their direct product are introduced, and their properties are investigated, as mentioned in the abstract.

2. PRELIMINARIES

This section is devoted to give some preliminaries from the literature. For more details, we refer to the references.

Definition 2.1. An MV-algebra is an algebra $(M; +, *, 0)$ of type $(2,1,0)$, satisfying the following properties:

- (MV1) $+$ is associative,
- (MV2) $+$ is commutative,
- (MV3) $x + 0 = x$,
- (MV4) $(x^*)^* = x$,
- (MV5) $x + 0^* = 0^*$,
- (MV6) $(x^* + y)^* + y = (y^* + x)^* + x$.

On any MV-algebra M , a binary relation ' \leq ' can be defined as $x \leq y$, if and only if $x^* + y = 0^*$. Then \leq is a partial ordering in M .

In this section, the concept of an H_v MV-algebra is introduced, and some basic results are given.

Definition 2.2. An H_v MV-algebra is a non-empty set, H endowed with a binary hyperoperation ' \oplus ', a unary operation ' $*$ ', and a constant, ' 0 ' satisfying the following conditions:

- $(H_v$ MV1) $x \oplus (y \oplus z) \cap (x \oplus y) \oplus z \neq \emptyset$, (weak associativity)
- $(H_v$ MV2) $x \oplus y \cap y \oplus x \neq \emptyset$, (weak commutativity)
- $(H_v$ MV3) $(x^*)^* = x$,

- (H_vMV4) $(x^* \oplus y)^* \oplus y \cap (y^* \oplus x)^* \oplus x \neq \emptyset$,
(H_vMV5) $0^* \in x \oplus 0^* \cap 0^* \oplus x$,
(H_vMV6) $0^* \in x \oplus x^* \cap x^* \oplus x$,
(H_vMV7) $x \in x \oplus 0 \cap 0 \oplus x$,
(H_vMV8) $0^* \in x^* \oplus y \cap y \oplus x^*$ and $0^* \in y^* \oplus x \cap x \oplus y^*$ imply $x = y$.

Remark 2.3. On any H_vMV-algebra H , we can define a binary relation ‘ \preceq ’ by

$$x \preceq y \Leftrightarrow 0^* \in x^* \oplus y \cap y \oplus x^*.$$

Hence, the condition (H_vMV8) can be redefined as follows:

$$x \preceq y \text{ and } y \preceq x \text{ imply } x = y.$$

Let A and B be non-empty subsets of H . By $A \preceq B$, we mean that there exist $a \in A$, and $b \in B$, such that $a \preceq b$. For $A \subseteq H$, denote the set $\{a^* : a \in A\}$ by A^* and 0^* by 1.

On H , we define a hyperoperation ‘ \odot ’ as $x \odot y = (x^* \oplus y^*)^*$. The next theorem gives some properties.

Proposition 2.4. *In any H_vMV-algebra H , the following hold: $\forall x, y \in H$ and $\forall A, B \subseteq H$,*

- (1) $x \odot (y \odot z) \cap (x \odot y) \odot z \neq \emptyset$,
- (2) $x \odot y \cap y \odot x \neq \emptyset$,
- (3) $0 \in x \odot 0 \cap 0 \odot x$,
- (4) $0 \in x \odot x^* \cap x^* \odot x$,
- (5) $x \in x \odot 1 \cap 1 \odot x$,

Definition 2.5. Let $(H; \oplus, *, 0_H)$, and $(K; \otimes, *, 0_K)$ be H_vMV-algebras, and let $f : H \rightarrow K$ be a function satisfying the following conditions:

- (1) $f(0_H) = 0_K$,
- (2) $f(x^*) = f(x)^*$,
- (3) $f(x^*) \preceq f(x)^*$,
- (4) $f(x \oplus y) = f(x) \otimes f(y)$,
- (5) $f(x \oplus y) \subseteq f(x) \otimes f(y)$.

f is called a *homomorphism*, if it satisfies the conditions (1), (2), and (4); it is called a *weak homomorphism* if it satisfies the conditions (1), (3), and (5). Clearly, if f is a homomorphism, $f(1) = 1$. For convenience, we use the same operations for H and K .

By $\ker f$, we mean the set $\{x \in H : f(x) = 0\}$. As usual, a homomorphism that is one-to-one (resp. onto) is called a *monomorphism* (resp. *epimorphism*). A homomorphism which is both an epimorphism and a monomorphism is called an *isomorphism*. If $f : H \rightarrow K$ is an isomorphism, we say that H and K are isomorphic, and we write $H \simeq K$.

Theorem 2.6. *Let $f : H \longrightarrow K$ be a homomorphism.*

- (1) *f is one-to-one, if and only if $\ker f = \{0\}$.*
- (2) *f is an isomorphism, if and only if there exists a homomorphism $f^{-1} : K \longrightarrow H$, such that $ff^{-1} = 1_K$ and $f^{-1}f = 1_H$.*

Definition 2.7. A non-empty subset S of H is called an H_v MV-subalgebra of H , if $(S; \oplus, *, 0)$ is itself an H_v MV-algebra.

The next proposition gives an equivalent condition for an H_v MV-subalgebra.

Proposition 2.8. *A nonempty subset S of H is an H_v MV-subalgebra of H if and only if*

- (1) *$x \oplus y \subseteq S$, for all $x, y \in S$,*
- (2) *$x^* \in S$, for all $x \in S$.*

Corollary 2.9. *A non-empty subset S of H is an H_v MV-subalgebra, if and only if*

- (1) *$0 \in S$,*
- (2) *$x^* \oplus y \subseteq S$, for all $x, y \in S$.*

Definition 2.10. Let I be a non-empty subset of H , satisfying (I_0) . $x \preceq y$, and $y \in I$ imply $x \in I$.

Then I is called

- (1) *an H_v MV-ideal, if $x \oplus y \subseteq I$, for all $x, y \in I$,*
- (2) *a weak H_v MV-ideal, if $x \oplus y \preceq I$, for all $x, y \in I$.*

It is easy to see that, in any H_v MV-algebra H , $\{0\}$ is a weak H_v MV-ideal, and obviously, H is an H_v MV-ideal of H . Also, every H_v MV-ideal is a weak H_v MV-ideal.

Theorem 2.11. *Let $f : H \longrightarrow K$ be a homomorphism.*

- (1) *$\ker f$ is a weak H_v MV-ideal of H .*
- (2) *If I is an H_v MV-ideal of K , $f^{-1}(I)$ is an H_v MV-ideal of H .*

From now on, in this paper, H is denoted by an H_v MV-algebra, unless otherwise stated.

3. QUOTIENT STRUCTURES

In this section, it is shown that how we can construct the quotient H_v MV-algebra from the old one, and some homomorphism theorems are stated and proved. We start with a definition.

Definition 3.1. Let θ be a binary relation in H , and $A, B \subseteq H$. We say that

- (1) $A\theta_s B$, if for all $a \in A$, and for all $b \in B$, $a\theta b$,
- (2) $A\theta B$, if for all $a \in A$, there exists $b \in B$, and for all $b \in B$, there exists $a \in A$, such that $a\theta b$,
- (3) $A\theta_{sw} B$, if for all $a \in A$ there exists $b \in B$, such that $a\theta b$,
- (4) $A\theta_w B$, if there exist $a \in A$, and $b \in B$, such that $a\theta b$.

Obviously, $\theta_s \subseteq \theta \subseteq \theta_{sw} \subseteq \theta_w$.

It must be noticed that when A and B are singleton, $\theta = \theta_s = \theta_{sw} = \theta_w$.

Proposition 3.2. *Let θ be a transitive relation in H , and $A, B, C \subseteq H$.*

- (1) *If $A\theta_w B$ and $B\theta_{sw} C$, then $A\theta_w C$.*
- (2) *If $A\theta_{sw} B$ and $B\theta_{sw} C$, then $A\theta_{sw} C$.*
- (3) *If $A\theta_s B$ and $B\theta_s C$, then $A\theta_s C$.*
- (4) *If $A\theta_s B$ and $B\theta_{sw} C$, then $A\theta_{sw} C$.*
- (5) *If $A\theta_s B$ and $B\theta_w C$, then $A\theta_{sw} C$.*
- (6) *If $A\theta_s B$ and $B\theta C$, then $A\theta C$.*
- (7) *If $A\theta B$ and $B\theta C$, then $A\theta C$.*

Proof. Routine. □

Definition 3.3. Let θ be a binary relation in H with the property

$$x\theta y \text{ implies that } x^*\theta y^*. \quad (3.1)$$

θ is said to be

- *strongly compatible*, if $x\theta y$ and $u\theta v$ imply that $x \oplus u \theta_s y \oplus v$.
- *compatible*, if $x\theta y$ and $u\theta v$ imply that $x \oplus u \theta y \oplus v$.
- *s-weak compatible*, if $x\theta y$ and $u\theta v$ imply that $x \oplus u \theta_{sw} y \oplus v$.
- *weakly compatible*, if $x\theta y$ and $u\theta v$ imply that $x \oplus u \theta_w y \oplus v$.

It is clear that every strongly compatible relation is compatible, every compatible relation is s-weak compatible, and every s-weak compatible relation is weakly compatible.

Theorem 3.4. *Let θ be a reflexive and transitive binary relation in H . Then θ is compatible, if and only if (3.1) holds, and,*

$$x\theta y \text{ implies that } x \oplus a \theta y \oplus a \text{ and } a \oplus x \theta a \oplus y, \quad (3.2)$$

for all $x, y, a \in H$.

Proof. Assume that θ is compatible, $x\theta y$, and $a \in H$. Since θ is reflexive, so $a\theta a$, whence $x \oplus a \theta y \oplus a$ and $a \oplus x \theta a \oplus y$.

Conversely, assume that θ satisfies (3.2), $x\theta y$, and $u\theta v$. Hence, $x \oplus u \theta y \oplus u$ and $y \oplus u \theta y \oplus v$, whence, by Proposition 3.2(7), $x \oplus u \theta y \oplus v$. □

Remark 3.5. In virtue of Proposition 3.2, it is easy to see that an analogous result holds for strongly compatible relations, and s-weak compatible relations.

Proposition 3.6. *Every reflexive weakly compatible relation θ in H satisfies:*

$$x\theta y \text{ implies that } x \oplus a \theta_w y \oplus a \text{ and } a \oplus x \theta_w a \oplus x. \quad (3.3)$$

Proof. The proof is similar to the proof of Theorem 3.4. \square

Proposition 3.7. *Let θ be a symmetric binary relation in H . Then θ is compatible, if and only if it is s-weak compatible.*

Proof. In virtue of the observation just after Definition 3.3, it is enough to prove that every symmetric s-weak compatible relation is compatible. Assume that θ is a symmetric s-weak compatible, and $x\theta y$ and $u\theta v$, for $x, y, u, v \in H$. Then $x \oplus u \theta_{sw} y \oplus v$, which means that for all $a \in x \oplus u$, there exists $b \in y \oplus v$, such that $a\theta b$. Since θ is symmetric, so $y\theta x$ and $v\theta u$, whence $y \oplus v \theta_{sw} x \oplus u$, i.e., for all $b \in y \oplus v$, there exists $a \in x \oplus u$, such that $a\theta b$. This implies that $x \oplus u \theta y \oplus v$, i.e., θ is compatible. \square

In virtue of Definition 3.3 and Proposition 3.7, we define three types of congruences in H .

Definition 3.8. Let θ be an equivalence relation in H that satisfies (3.1). θ is called a:

- *strong congruence*, if it is strongly compatible.
- *congruence*, if it is compatible.
- *weak congruence*, if it is weakly compatible.

Corollary 3.9. *Let θ be an equivalence relation in H . θ is a congruence, if and only if it satisfies (3.1) and (3.2).*

Example 3.10. (i) Obviously, in any H_v MV-algebra H , ∇_H is a strong congruence in H .

(ii) Let $H = \{0, a, b, 1\}$, and let the operations \oplus and $*$ be defined as shown in Table 1. Then $(H; \oplus, *, 0)$ is a proper H_v MV-algebra (see [1]).

Let $\theta = \{(0, 0), (a, a), (b, b), (1, 1), (a, b), (b, a)\}$. Obviously, θ is an equivalence relation in H , which satisfies (3.1). Also it is easily verified that θ is weakly compatible. Hence, θ is a weak congruence in H .

(iii) Let $H = \{0, a, b, c, 1\}$, and consider Table 2. Then $(H; \oplus, *, 0)$ is an H_v MV-algebra (see [1]). Let

$$\theta = \{(0, 0), (a, a), (b, b), (c, c), (1, 1), (a, b), (b, a), (a, c), (c, a), (b, c), (c, b), (0, 1), (1, 0)\}.$$

\oplus	0	a	b	1
0	{0,a}	{0,a,b}	{0,a,b}	{0,a,b,1}
a	{0,a,b,1}	{0,b}	{0,1}	{a,b,1}
b	{a,b}	{0,a,b,1}	{0}	{0,a,b,1}
1	{0,a,1}	{0,a,b,1}	{1}	{0,a,b,1}
*	1	b	a	0

TABLE 1. The Cayley table of \oplus and $*$

\oplus	0	a	b	c	1
0	{0}	{0,a}	{0,b}	{0,c}	{0,a,b,c,1}
a	{0,a}	{0,a}	{0,a,b,c,1}	{0,a,b,c,1}	{0,a,b,c,1}
b	{0,b}	{0,a,b,c,1}	{0,a,b,c,1}	{0,a,b,c}	{0,a,b,c,1}
c	{0,c}	{0,a,b,c,1}	{0,a,b,c}	{0,a,b,c,1}	{0,a,b,c,1}
1	{0,a,b,c,1}	{0,a,b,c,1}	{0,a,b,c,1}	{0,a,b,c,1}	{0,a,b,c,1}
*	1	b	a	c	0

TABLE 2. The Cayley table of ' \oplus ' and '*'

It is not difficult to verify that θ is a congruence in H .

In virtue of Remark 3.5, we can check that an analogous result holds for strong congruences and weak congruences.

Definition 3.11. A binary relation θ in H is called *regular*, if $x^* \oplus y\theta_w\{0^*\}$ and $y^* \oplus x\theta_w\{0^*\}$ imply $x\theta y$.

For a congruence θ in H , let x/θ be the congruence class of x , and $H/\theta = \{x/\theta : x \in H\}$. We define the operations ' \oplus ' and '*'' on H/θ by

$$x/\theta \oplus y/\theta = \{a/\theta : a \in x \oplus y\} \text{ and } (x/\theta)^* = x^*/\theta.$$

Then we have the following theorem:

Theorem 3.12. *Let H be an H_v MV-algebra, and θ be a regular congruence in H . Then $(H/\theta, \oplus, *, 0/\theta)$ forms an H_v MV-algebra.*

Proof. We first prove that ' \oplus ' and '*'' are well-defined. Let $x, y \in H$ be such that $x/\theta = y/\theta$. This implies that $x\theta y$, and so $x^*\theta y^*$, whence $x^*/\theta = y^*/\theta$. This means that $(x/\theta)^* = (y/\theta)^*$. Let $x_1, x_2, y_1, y_2 \in H$ be such that $x_1/\theta = y_1/\theta$, and $x_2/\theta = y_2/\theta$. Then $x_1 \oplus x_2 \theta y_1 \oplus x_2$, and $y_1 \oplus x_2 \theta y_1 \oplus y_2$ whence $x_1 \oplus x_2 \theta y_1 \oplus y_2$, by Proposition 3.2(7). If $a/\theta \in x_1/\theta \oplus x_2/\theta$, then $a/\theta = b/\theta$, for some $b \in x_1 \oplus x_2$ and so $a\theta b$ and $b\theta c$, where $c \in y_1 \oplus y_2$. Thus $a/\theta = c/\theta \in y_1/\theta \oplus y_2/\theta$, proving

$x_1/\theta \oplus x_2/\theta \subseteq y_1/\theta \oplus y_2/\theta$. In a similar way, we can prove that the converse inclusion holds. Thus \oplus is well-defined.

The proof of the properties (H_vMV1) - (H_vMV7) follows directly. The proof of (H_vMV8) follows from the regularity. \square

Theorem 3.13. *If θ is a regular congruence in H , $0/\theta$ is a weak H_vMV -ideal of H .*

Proof. Let $x, y \in H$ be such that $x \preceq y$, and $y \in 0/\theta$. Then $0^* \in x^* \oplus y \cap y \oplus x^*$ and $y/\theta = 0/\theta$, whence:

$$0^*/\theta \in x^*/\theta \oplus y/\theta \cap y/\theta \oplus x^*/\theta = x^*/\theta \oplus 0/\theta \cap 0/\theta \oplus x^*/\theta.$$

Hence, $x/\theta \preceq 0/\theta$, and so $x/\theta = 0/\theta$ means that $x \in 0/\theta$.

Now, let $x, y \in 0/\theta$. Then $x\theta 0$ and $0\theta y$, and so $x \oplus y \theta 0 \oplus y$, and $0 \oplus y \theta 0 \oplus 0$, whence $x \oplus y\theta 0 \oplus 0$. Since $0 \in 0 \oplus 0$, so there exists $a \in x \oplus y$, such that $a\theta 0$, i.e., $a \in 0/\theta$, and so $x \oplus y \cap 0/\theta \neq \emptyset$, whence $x \oplus y \preceq 0/\theta$, proving that $0/\theta$ is a weak H_vMV -ideal of H . \square

Open Problem 3.14. Let I be a (weak) H_vMV -ideal of H . Is there a congruence θ in H , such that $0/\theta = I$?

The next theorem is easily proved, and so the proof is omitted.

Theorem 3.15. *If θ is a regular congruence in H , the mapping $\natural : H \rightarrow H/\theta$ with $\natural(x) = x/\theta$ is an epimorphism with $\ker \natural = 0/\theta$.*

The mapping \natural is called the *canonical epimorphism*.

Theorem 3.16. *If θ is a regular congruence in H , and $f : H \rightarrow K$ is a homomorphism of H_vMV -algebras, such that $0/\theta \subseteq \ker f$, there exists a unique homomorphism $\bar{f} : H/\theta \rightarrow K$, such that $\bar{f}(a/\theta) = f(a)$, for all $a \in H$, $Im \bar{f} = Im f$, and $\ker \bar{f} = \ker f/\theta$. \bar{f} is an isomorphism, if and only if f is onto and $\ker f = 0/\theta$.*

Proof. We first prove that \bar{f} is well-defined. Let $a, b \in H$ be such that $a/\theta = b/\theta$. Then $0^*/\theta \in a^*/\theta \oplus b/\theta \cap b/\theta \oplus a^*/\theta$. This implies that $x/\theta = 0^*/\theta = y/\theta$, for some $x \in a^* \oplus b$ and $y \in b \oplus a^*$ whence $x^*, y^* \in 0/\theta \subseteq \ker f$, i.e., $f(x^*) = f(0) = f(y^*)$. Hence,

$$0^* = f(0^*) = f(x) \in f(a)^* \oplus f(b)$$

and similarly, $0^* \in f(b) \oplus f(a)^*$, and hence, $f(a) \preceq f(b)$. In a similar way, we can show that $f(b) \preceq f(a)$. Thus $\bar{f}(a/\theta) = f(a) = f(b) = \bar{f}(b/\theta)$, i.e., \bar{f} is well-defined. Obviously, \bar{f} is a homomorphism, and $Im \bar{f} = Im f$. Now,

$$a/\theta \in \ker \bar{f} \Rightarrow f(a) = \bar{f}(a/\theta) = 0 \Rightarrow a \in \ker f \Rightarrow a/\theta \in \ker f/\theta,$$

whence $\ker \bar{f} \subseteq \ker f/\theta$. Conversely, if $a/\theta \in \ker f/\theta$, so $a/\theta = b/\theta$, for some $b \in \ker f$, and hence, $f(a) = \bar{f}(a/\theta) = \bar{f}(b/\theta) = f(b) = 0$, i.e., $a \in \ker f$. This implies that $\ker f/\theta \subseteq \ker \bar{f}$, proving $\ker \bar{f} = \ker f/\theta$. \bar{f} is unique, because it is determined completely by f .

Finally, \bar{f} is an isomorphism, if and only if it is an epimorphism and a monomorphism. Obviously, \bar{f} is an epimorphism, if and only if f is an epimorphism, and by Theorem 2.6, \bar{f} is a monomorphism, if and only if $\ker f/\theta = \ker \bar{f} = \{0/\theta\}$, i.e., if and only if $\ker f = 0/\theta$. \square

Corollary 3.17. (*Fundamental Homomorphism Theorem*) *Let θ be a regular congruence in H . Then every homomorphism $f : H \rightarrow K$ of H_v MV-algebras induces an isomorphism $H/\theta \simeq \text{Im} f$, where $0/\theta = \ker f$.*

Proof. Since $f : H \rightarrow \text{Im} f$ is an epimorphism, and $\ker f = 0/\theta$, so, by Theorem 3.16, the mapping $\bar{f} : H/\theta \rightarrow \text{Im} f$ with $a/\theta \mapsto f(a)$ is an isomorphism. \square

Corollary 3.18. *Let θ and ϑ be the regular congruences in H_v MV-algebras H and K , respectively, and let $f : H \rightarrow K$ be a homomorphism with $f(0/\theta) \subseteq 0/\vartheta$. Then f induces a homomorphism $\bar{f} : H/\theta \rightarrow K/\vartheta$ with $\bar{f}(a/\theta) = f(a)/\vartheta$. \bar{f} is an isomorphism, if and only if $\text{Im} f/\vartheta = K$, and $f^{-1}(0/\vartheta) \subseteq 0/\theta$.*

Proof. Obviously, the composition $H \xrightarrow{f} K \xrightarrow{\natural} K/\vartheta$ is a homomorphism, and $0/\theta \subseteq f^{-1}(0/\vartheta) = \ker \natural f$ because:

$$\begin{aligned} x \in \ker \natural f &\Leftrightarrow \natural f(x) = 0/\vartheta \Leftrightarrow f(x)/\vartheta = 0/\vartheta \Leftrightarrow f(x) \in 0/\vartheta \\ &\Leftrightarrow x \in f^{-1}(0/\vartheta). \end{aligned}$$

Now, by Theorem 3.16, for $\natural f$ instead of f , and K/ϑ instead of K , the mapping $H/\theta \rightarrow K/\vartheta$ with $a/\theta \mapsto (\natural f)(a) = f(a)/\vartheta$ is a homomorphism, which is an isomorphism, if and only if $\natural f$ is an epimorphism and $\ker \natural f = 0/\theta$. But $\ker \natural f = 0/\theta$, if and only if $f^{-1}(0/\vartheta) \subseteq 0/\theta$. Now, assume that $\natural f$ is an epimorphism, and $x \in K$. Then $x/\vartheta \in K/\vartheta$, and so $x/\vartheta = \natural f(h) = f(h)/\vartheta$, where $h \in H$. This implies that $x \in f(h)/\vartheta$, whence $K \subseteq \text{Im} f/\vartheta$. Obviously, $\text{Im} f/\vartheta \subseteq K$. Hence, $K = \text{Im} f/\vartheta$. Conversely, assume that $K = \text{Im} f/\vartheta$, and $x/\vartheta \in K/\vartheta$. Then $x \in K$, and so $x/\vartheta = f(a)/\vartheta = \natural f(a)$, for some $a \in H$, proving $\natural f$ is onto. Particulary, if f is onto, $\text{Im} f = K$, and so $\text{Im} f/\vartheta = K$. \square

Let θ and ϑ be regular congruences in H such that $\vartheta \subseteq \theta$. Define a binary relation θ/ϑ in H/ϑ by

$$a/\vartheta(\theta/\vartheta)b/\vartheta \Leftrightarrow a\theta b.$$

It is obvious that θ/ϑ is a regular congruence in H/ϑ .

Corollary 3.19. *Let θ and ϑ be regular congruences in H and $\vartheta \subseteq \theta$. Then $(H/\vartheta)/(\theta/\vartheta) \simeq H/\theta$.*

Proof. Obviously, the mapping $f : H/\vartheta \longrightarrow H/\theta$ with $f(a/\vartheta) = a/\theta$ is an epimorphism, and

$$\begin{aligned} \ker f &= \{a/\vartheta : a/\theta = f(a/\vartheta) = 0/\theta\} = \{a/\vartheta : a\theta 0\} \\ &= \{a/\vartheta : a/\vartheta(\theta/\vartheta)0/\vartheta\} \\ &= 0/(\theta/\vartheta) \end{aligned}$$

whence, by Corollary 3.17, $(H/\vartheta)/(\theta/\vartheta) \simeq \text{Im}f = H/\theta$. \square

4. FUNDAMENTAL MV-ALGEBRAS

In this section, we introduce the concept of fundamental relation on H_v MV-algebras. We first give an application of strong congruences.

Theorem 4.1. *If θ is a regular strong congruence in H , $(H/\theta, \oplus, *, 0/\theta)$ is an MV-algebra.*

Proof. In virtue of Theorem 3.12, it is enough to prove that, for all $x, y \in H$, the set $x/\theta \oplus y/\theta$ is singleton. Let $a/\theta, b/\theta \in x/\theta \oplus y/\theta$. Then there exist $c, d \in x \oplus y$, such that $a/\theta = c/\theta$ and $b/\theta = d/\theta$. Since θ is reflexive, so $x\theta x$ and $y\theta y$, whence $x \oplus y\theta_s x \oplus y$. This implies that $c\theta d$, i.e., $a/\theta = c/\theta = d/\theta = b/\theta$, proving $|x/\theta \oplus y/\theta| = 1$. \square

Let \mathcal{U}_{ps} and \mathcal{U}_{sp} be the set of all finite sums of finite products and the set of all finite products of finite sums of the elements of H , respectively, and let $\mathcal{U} = \mathcal{U}_{ps} \cup \mathcal{U}_{sp}$. Define a binary relation γ in H by

$$a\gamma b, \text{ if and only if } \{a, b\} \subseteq u, \text{ for some } u \in \mathcal{U}.$$

It is obvious that γ is reflexive and symmetric.

Define a binary relation γ^* in H by $a\gamma^*b$, if and only if there exist $n \in \mathbb{N}$, and $z_1, z_2, \dots, z_{n+1} \in H$, such that $a = z_1$, $b = z_{n+1}$, and for all $i \in \{1, 2, \dots, n\}$, $\{z_i, z_{i+1}\} \subseteq u_i$, for some $u_i \in \mathcal{U}$.

Theorem 4.2. *The relation γ^* is an equivalence relation in H .*

Proof. Let $a \in H$. From $a \in a \odot 1$, for $n = 2$, $z_1 = z_2 = a$, and $u = (0 \odot 1) \oplus (a \odot 1)$, we get

$$\{a\} \subseteq a \odot 1 \subseteq 0 \oplus (a \odot 1) \subseteq (0 \odot 1) \oplus (a \odot 1) = u,$$

whence $a\gamma^*a$, i.e., γ^* is reflexive. Now, let $a, b \in H$ be such that $a\gamma^*b$. Then there exist $n \in \mathbb{N}$, $z_1, z_2, \dots, z_{n+1} \in H$, such that $a = z_1$, $b = z_{n+1}$, and for all $i \in \{1, 2, \dots, n\}$, $\{z_i, z_{i+1}\} \subseteq u_i$, where $u_i \in \mathcal{U}$.

Let $y_i = z_{n-i+2}$, for all $i \in \{1, 2, \dots, n\}$. Then $y_1 = b$, $y_{n+1} = a$, and $\{y_i, y_{i+1}\} \subseteq v_i$, where $v_i = u_{n-i+1} \in \mathcal{U}$, proving $b\gamma^*a$, i.e. γ^* is symmetric. For transitivity, let $a\gamma^*b$ and $b\gamma^*c$, for $a, b, c \in H$. Then there exist $n, m \in \mathbb{N}$, $z_1, \dots, z_{n+1}, w_1, \dots, w_{m+1} \in H$, such that $a = z_1$, $z_{n+1} = b = w_1$, $c = w_{m+1}$, and $\{z_i, z_{i+1}\} \subseteq u_i$ and $\{w_j, w_{j+1}\} \subseteq v_j$, where $u_i, v_j \in \mathcal{U}$, for all $i \in \{1, 2, \dots, n\}$ and $j \in \{1, 2, \dots, m\}$. Let $x_i = z_i$, for $i \in \{1, 2, \dots, n\}$, and $x_i = w_j$, where $i = n + j$, for $j \in \{1, 2, \dots, m\}$. Then $a = x_1$, $c = x_{m+1}$ and $\{x_i, x_{i+1}\} \subseteq r_i$, where $r_i \in \mathcal{U}$. Thus γ^* is an equivalence relation in H . \square

Theorem 4.3. γ^* is the smallest regular strong congruence in H with the property that H/γ^* is an MV-algebra.

Proof. In virtue of Theorem 3.4 and Remark 3.5, to prove that γ^* is a strong congruence, it is enough to prove that (3.1) holds, and for all $x, y, a \in H$,

$$x\gamma^*y \text{ implies that } x \oplus a\gamma_s^*y \oplus a, \text{ and } a \oplus x\gamma_s^*a \oplus y.$$

Assume that $x\gamma^*y$, for $x, y \in H$. Then there exist $n \in \mathbb{N}$, $a_1, \dots, a_{n+1} \in H$, such that $x = a_1$, $y = a_{n+1}$, and for all $i \in \{1, 2, \dots, n\}$, $\{a_i, a_{i+1}\} \subseteq u_i$, for some $u_i \in \mathcal{U}$. This implies that $x^* = a_1^*$, $y^* = a_{n+1}^*$, and $\{a_i^*, a_{i+1}^*\} \subseteq u_i^* \in \mathcal{U}$, whence $x^*\gamma^*y^*$. Thus (3.1) holds. Also, if $a \in H$, for $s_i \in a_i \oplus a$, we have:

$$\{s_i, s_{i+1}\} \subseteq (a_i \oplus a) \cup (a_{i+1} \oplus a) \subseteq u_i \oplus a \subseteq u_i \oplus (a \odot 1) = v_i \in \mathcal{U}$$

or $\{s_i, s_{i+1}\} \subseteq u_i \oplus a \subseteq u_i \odot (a \oplus 0) \in \mathcal{U}$. Thus for $s_1 \in x \oplus a = a_1 \oplus a$ and $s_{n+1} \in y \oplus a = a_{n+1} \oplus a$, we have $s_1\gamma^*s_{n+1}$. This implies that $x \oplus a\gamma_s^*y \oplus a$. Similarly, we can show that $x\gamma^*y$ implies that $a \oplus x\gamma_s^*a \oplus y$. Thus γ^* is a strong congruence in H .

For regularity, assume that $x^* \oplus y\gamma_w^*\{0^*\}$, and $y^* \oplus x\gamma_w^*\{0^*\}$, for $x, y \in H$. Then $(x^* \oplus y)^*\gamma_w^*\{0\}$, and $(y^* \oplus x)^*\gamma_w^*\{0\}$, and so $(x^* \oplus y)^* \oplus y\gamma_s^*0 \oplus y$, and $(y^* \oplus x)^* \oplus x\gamma_s^*0 \oplus x$, whence $0 \oplus x\gamma_s^*0 \oplus y$. Since $y \in 0 \oplus y$ and $x \in 0 \oplus x$, so $x\gamma^*y$ proving γ^* is regular. Therefore, γ^* is a regular strong congruence in H , and H/γ^* is an MV-algebra.

Now, let δ be a regular strong congruence in H with the property that H/δ is an MV-algebra, and $x\gamma y$, for $x, y \in H$. Then $\{x, y\} \subseteq u \in \mathcal{U}$. Assume that $u = \bigoplus_{i=1}^n (\bigodot_{j=1}^m x_{ij})$, where $x_{ij} \in H$. Since δ is a strong congruence, so $u/\delta = \bigoplus_{i=1}^n (\bigodot_{j=1}^m x_{ij}/\delta)$ is singleton, and since $x/\delta, y/\delta \in u/\delta$ implies that $x/\delta = y/\delta$, i.e., $x\delta y$. Hence, $\gamma \subseteq \delta$. Now, if $x\gamma^*y$, there exist $n \in \mathbb{N}$ and $a_1, a_2, \dots, a_{n+1} \in H$, such that $x = a_1$, $y = a_{n+1}$ and $a_i\gamma a_{i+1}$, whence $a_i\delta a_{i+1}$. Since δ is transitive, so $x\delta y$, proving $\gamma^* \subseteq \delta$. Therefore, γ^* is the smallest regular strong congruence in H , such that H/γ^* is an MV-algebra. \square

Remark 4.4. The relation γ^* is called the *fundamental relation* in H , and H/γ^* is called the *fundamental MV-algebra*.

5. DIRECT PRODUCTS

In this section, we define the direct product of a family of H_v MV-algebras, characterize the H_v MV-subalgebras and (weak) H_v MV-ideals of it, and give some homomorphism theorems.

Let $\{(H_i; \oplus_i, *^i, 0_i) : i \in I\}$ be a non-empty family of H_v MV-algebras. The cartesian product $\prod_{i \in I} H_i$ of H_i 's is defined as the set of all functions $f : I \rightarrow \cup H_i$ with $f(i) \in H_i$, for all $i \in I$. For $f, g \in \prod_{i \in I} H_i$, define $f = g$, if and only if $f(i) = g(i)$, for all $i \in I$, and

$$f^*(i) = f(i)^{*^i} \quad \text{and} \quad (f \oplus g)(i) = f(i) \oplus_i g(i), \forall i \in I.$$

Also define $0(i) = 0_i$, for all $i \in I$. It is easy to check that $\prod_{i \in I} H_i$ together with ' \oplus ', ' $*$ ' satisfies $(H_v$ MV1)-(H_vMV8). Thus we get

Theorem 5.1. *If $\{(H_i; \oplus_i, *^i, 0_i) : i \in I\}$ is a non-empty family of H_v MV-algebras;*

- (1) $(\prod_{i \in I} H_i; \oplus, *, 0)$ is an H_v MV-algebra,
- (2) for each $k \in I$, the mapping $\pi_k : \prod_{i \in I} H_i \rightarrow H_k$ with $f \mapsto f(k)$ is an epimorphism.

$\prod_{i \in I} H_i$ is called the *direct product* of H_i 's. If H_i is an H_v MV-algebra with the order \preceq_i , the order on $\prod_{i \in I} H_i$ is given by $f \preceq g$, if and only if $f(i) \preceq_i g(i)$.

The image of f can be written as $\{a_i\}$, where $a_i \in H_i$. In this case, the hyperoperation ' \oplus ' is written as $\{a_i\} \oplus \{b_i\} = \{a_i \oplus_i b_i\}$. If $I = \{1, 2, \dots, n\}$ is finite, $\prod_{i \in I} H_i$ is written as $H_1 \times H_2 \times \dots \times H_n$.

In the sequel, in this section, $\{H_i : i \in I\}$ is a non-empty family of H_v MV-algebras, and $\prod_{i \in I} H_i$ is the direct product of H_i 's.

Theorem 5.2. *Let H_i be an H_v MV-algebra, and S_i be a non-empty subset of H_i with $i \in I$.*

- (1) *If S_i is an H_v MV-subalgebra, $\prod_{i \in I} S_i$ is an H_v MV-subalgebra of $\prod_{i \in I} H_i$.*
- (2) *If S_i is an H_v MV-ideal, $\prod_{i \in I} S_i$ is an H_v MV-ideal of $\prod_{i \in I} H_i$.*
- (3) *If S_i is a weak H_v MV-ideal, $\prod_{i \in I} S_i$ is a weak H_v MV-ideal of $\prod_{i \in I} H_i$.*

Proof. Routine. □

Theorem 5.3. *Let H_i ($i \in I$) be an H_v MV-algebra and S be a non-empty subset of $\prod_{i \in I} H_i$.*

- (1) If S is an H_v MV-subalgebra, there exists unique H_v MV-subalgebra S_i of H_i , for all $i \in I$, such that $S = \prod_{i \in I} S_i$.
- (2) If S is an H_v MV-ideal, there exists unique H_v MV-ideal S_i of H_i , for all $i \in I$, such that $S = \prod_{i \in I} S_i$.
- (3) If S is a weak H_v MV-ideal, there exists unique weak H_v MV-ideal I_i of H_i , for all $i \in I$, such that $S = \prod_{i \in I} S_i$.

Proof. We first observe that if S is a non-empty subset of $\prod_{i \in I} H_i$, for

$$S_i = \{a_i \in H_i : \exists f \in S, \text{ such that } f(i) = a_i\},$$

we get $\prod_{i \in I} S_i = S$.

(1) Assume that S is an H_v MV-subalgebra of $\prod_{i \in I} H_i$. We show that S_i is an H_v MV-subalgebra of H_i . Obviously, $S_i \neq \emptyset$ because $0_i \in S_i$, for all $i \in I$. Let $a_i, b_i \in S_i$, for $i \in I$. Then there exist $f, g \in S$, such that $f(i) = a_i$ and $g(i) = b_i$, whence $a_i^{*i} \oplus_i b_i = f^{*i}(i) \oplus_i g(i) = (f^* \oplus g)(i) \subseteq S_i$, proving S_i is an H_v MV-subalgebra of H_i .

Now, let T_i be an H_v MV-subalgebra of H_i , for all $i \in I$, such that $S = \prod_{i \in I} T_i$. We show that $T_i = S_i$, for all $i \in I$. Let $a_i \in H_i$. Then $a_i \in T_i$, if and only if there exists $f \in \prod_{i \in I} T_i = S = \prod_{i \in I} S_i$ such that $f(i) = a_i$, if and only if $a_i \in S_i$ means that $T_i = S_i$.

(2) By (1), S_i is closed with respect to \oplus_i , for all $i \in I$. Now, let $a_i \preceq_i b_i$ and $b_i \in S_i$. Let $f, g \in \prod_{i \in I} H_i$ be such that $f(i) = a_i$, and $g(i) = b_i$. Then:

$$\begin{aligned} 0_i^{*i} \in a_i^{*i} \oplus_i b_i \cap b_i \oplus_i a_i^{*i} &= f^{*i}(i) \oplus_i g(i) \cap g(i) \oplus_i f^{*i}(i) \\ &= (f^* \oplus g)(i) \cap (g \oplus f^*)(i) \end{aligned}$$

whence $0^* \in f^* \oplus g \cap g \oplus f^*$, i.e. $f \preceq g \in \prod_{i \in I} S_i = S$. Since S is an H_v MV-ideal, so $f \in S$, and hence, $a_i = f(i) \in S_i$. The uniqueness is proved similar to the proof of (1).

(3) The proof is similar to the proof of (2). □

Definition 5.4. Let $\{H_i : i \in I\}$ be a non-empty family of H_v MV-algebras. The *weak direct product* of H_i 's, denoted by $\prod_{i \in I}^w H_i$, is defined as the set of all $f \in \prod_{i \in I} H_i$, such that for all but a finite number of $i \in I$, we have $f(i) = 0_i$.

Remark 5.5. Note that when I is finite, the weak direct product and direct product are equal.

Theorem 5.6. Let $\{H_i : i \in I\}$ be a non-empty family of H_v MV-algebras.

- (1) If $0_i \oplus 0_i = \{0_i\}$, for all $i \in I$, $\prod_{i \in I}^w H_i$ is an H_v MV-ideal of $\prod_{i \in I} H_i$.

- (2) the mapping $\iota_k : H_k \longrightarrow \prod_{i \in I}^w H_i$ given by $\iota_k(a) = \{a_i\}_{i \in I}$, in which $a_k = a$ and $a_i = 0_i$, for all $i \neq k$, is a weak monomorphism.
- (3) If $0_i \oplus 0_i = \{0_i\}$, for all $i \in I$, $\iota_i(H_i)$ is an H_v MV-ideal of $\prod_{i \in I}^w H_i$.

Proof. (1) Let $f, g \in \prod_{i \in I} H_i$ be such that $f \preceq g$, and $g \in \prod_{i \in I}^w H_i$. Then $f(i) \preceq_i g(i)$ and $g(i) = 0_i$, for all but a finite number of $i \in I$ whence $0_i^{*i} \in f^{*i}(i) \oplus_i g(i) \cap g(i) \oplus_i f^{*i}(i)$. Now, for $i \in I$ with $g(i) = 0_i$, we have $0_i^{*i} \in f^{*i}(i) \oplus_i 0_i \cap 0_i \oplus_i f^{*i}(i)$, which implies that $f(i) \preceq_i 0_i$, i.e., $f(i) = 0_i$, proving $f \in \prod_{i \in I}^w H_i$. Let $f, g \in \prod_{i \in I}^w H_i$. Then $f(i) = 0_i$ and $g(j) = 0_j$, for all but a finite number of $i, j \in I$. Let $k \in I$ be the smallest element, for which $f(k) = g(k) = 0_k$. Then $(f \oplus g)(k) = f(k) \oplus_k g(k) = 0_k \oplus 0_k = \{0_k\}$, for all but a finite number of $k \in I$. Thus $f \oplus g \in \prod_{i \in I}^w H_i$. Therefore, $\prod_{i \in I}^w H_i$ is an H_v MV-ideal of $\prod_{i \in I} H_i$.

(2) Let $a \in H_k$. Then $\iota_k(a^{*k}) = \{a_i\}_{i \in I}$ in which $a_k = a^{*k}$ and $a_i = 0_i$, for all $i \neq k$. On the other hand, $\iota_k(a)^* = \{a_i\}_{i \in I}^* = \{a_i^{*i}\}_{i \in I}$, in which $a_k^{*k} = a^{*k}$ and $a_i^{*i} = 1$, for all $i \neq k$. This implies that $\iota_k(a^{*k}) \preceq \iota_k(a)^*$. Now, let $a, b \in H_i$. Then:

$$\begin{aligned}
\iota_k(a \oplus_k b) &= \{\iota_k(c) : c \in a \oplus_k b\} \\
&= \{\{a_i\}_{i \in I} : a_k = c \in a \oplus_k b, a_i = 0_i, \forall i \neq k\} \\
&\subseteq \{a_i \oplus b_i\} \text{ with } a_k = a, b_k = b, a_i = b_i = 0_i, \forall i \neq k \\
&= \{a_i\}_{i \in I} \oplus \{b_i\}_{i \in I} \text{ with } a_k = a, b_k = b, a_i = b_i = 0_i, \forall i \neq k \\
&= \iota_k(a) \oplus \iota_k(b)
\end{aligned}$$

proving that ι_k is a weak homomorphism. Obviously, ι_k is one-to-one.

(3) Let $\{a_i\} \preceq \{b_i\}$, and $\{b_i\} \in \iota_k(H_k)$. Then $0^* \in \{a_i\}^* \oplus \{b_i\} \cap \{b_i\} \oplus \{a_i\}^*$, and $b_k = a \in H_k$ and $b_i = 0_i$, for all $i \neq k$, whence $0_i^{*i} \in a_i^{*i} \oplus_i b_i$, $b_k = a$, and $b_i = 0_i$, for all $i \neq k$. This implies that $0^{*i} \in a_i^{*i} \oplus_i 0_i$, for all $i \neq k$, and hence, $a_i \preceq 0_i$, i.e., $a_i = 0_i$, for all $i \neq k$. Thus $\{a_i\} \in \iota_k(H_k)$. Now, let $\{a_j\}, \{b_j\} \in \iota_i(H_i)$. Then $a_j = b_j = 0_j$, for all $j \neq i$, and so, $\{a_j\} \oplus \{b_j\} = \{a_j \oplus_j b_j\}$ in which $a_j \oplus_j b_j = 0_j \oplus_j 0_j = \{0_j\}$, proving $\{a_j\} \oplus \{b_j\} \subseteq \iota_i(H_i)$. \square

Theorem 5.7. Let $\{f_i : H_i \longrightarrow K_i : i \in I\}$ be a non-empty family of homomorphisms of H_v MV-algebras, and the mapping $f = \prod f_i : \prod_{i \in I} H_i \longrightarrow \prod_{i \in I} K_i$ be given by $\{a_i\} \mapsto \{f_i(a_i)\}$. Then f is a homomorphism such that $f(\prod_{i \in I}^w H_i) \subseteq \prod_{i \in I}^w K_i$, $\ker f = \prod_{i \in I} \ker f_i$ and $\text{Im} f = \prod_{i \in I} \text{Im} f_i$. Thus f is a monomorphism (resp. an epimorphism) if and only if so is each f_i .

Proof. Routine. \square

Theorem 5.8. *Let $\{H_i : i \in I\}$ be a non-empty family of H_v MV-algebras, and β_i^{*i} be the fundamental equivalence relation in H_i , for all $i \in I$, and β^* be the fundamental equivalence relation in $\prod_{i \in I} H_i$. Then $(\prod_{i \in I} H_i)/\beta^* \simeq \prod_{i \in I} H_i/\beta_i^{*i}$.*

Proof. Consider the relation $\tilde{\beta}$ in $\prod_{i \in I} H_i$, as follows:

$$\{a_i\}\tilde{\beta}\{b_i\} \Leftrightarrow a_i\beta_i^{*i}b_i, \forall i \in I.$$

Obviously, $\tilde{\beta}$ is a congruence relation in $\prod_{i \in I} H_i$. To prove regularity, let $\{a_i\}, \{b_i\} \in \prod_{i \in I} H_i$ be such that $\{a_i\}^* \oplus \{b_i\}\tilde{\beta}\{0_i\}$ and $\{b_i\}^* \oplus \{a_i\}\tilde{\beta}\{0_i\}$. Then $\{a_i^{*i} \oplus_i b_i\}\tilde{\beta}\{0_i\}$ and $\{b_i^{*i} \oplus_i a_i\}\tilde{\beta}\{0_i\}$, whence $\{c_i\}\tilde{\beta}\{0_i\}$ and $\{d_i\}\tilde{\beta}\{0_i\}$, for all $c_i \in a_i^{*i} \oplus_i b_i$ and $d_i \in b_i^{*i} \oplus_i a_i$. Hence, $c_i\beta_i^{*i}0_i$ and $d_i\beta_i^{*i}0_i$, i.e., $a_i^{*i} \oplus_i b_i\beta_i^{*i}\{0_i\}$ and $b_i^{*i} \oplus_i a_i\beta_i^{*i}\{0_i\}$, whence $a_i\beta_i^{*i}b_i$, proving $\{a_i\}\tilde{\beta}\{b_i\}$.

Now, define ‘*’ and ‘ \oplus ’ on $(\prod_{i \in I} H_i)/\tilde{\beta}$ by

$$(\{a_i\}/\tilde{\beta})^* = \{a_i^{*i}\}/\tilde{\beta}, \{a_i\}/\tilde{\beta} \oplus \{b_i\}/\tilde{\beta} = \{\{c_i\}/\tilde{\beta} : c_i \in a_i/\beta_i^{*i} \oplus_i b_i/\beta_i^{*i}\}.$$

It is easy to check that $\tilde{\beta}$ is the smallest regular congruence relation in $\prod_{i \in I} H_i$, such that $(\prod_{i \in I} H_i)/\tilde{\beta}$ is an H_v MV-algebra, so $\tilde{\beta} = \beta^*$.

Now, by Theorem 3.15, the mapping $\natural_i : H_i \rightarrow H_i/\beta_i^{*i}$ is an epimorphism with $\ker \natural_i = 0_i/\beta_i^{*i}$ and so, by Theorem 5.7, $\prod \natural_i : \prod_{i \in I} H_i \rightarrow \prod_{i \in I} H_i/\beta_i^{*i}$ is an epimorphism with

$$\ker(\prod \natural_i) = \prod_{i \in I} \ker \natural_i = \prod_{i \in I} 0_i/\beta_i^{*i} = 0/\beta^*.$$

Thus, by Corollary 3.17, $(\prod_{i \in I} H_i)/\beta^* \simeq \prod_{i \in I} H_i/\beta_i^{*i}$. □

6. CONCLUSIONS AND THE FUTURE WORKS

Based on the first work on H_v MV-algebras, in this paper, we introduced some types of congruences, studied the quotient H_v MV-algebra, and obtained some homomorphism theorems. Moreover, we obtained the fundamental equivalence relation on an H_v MV-algebra, the smallest equivalence relation on an H_v MV-algebra to make it to an MV-algebra. Finally, we introduced the direct product of a non-empty family of H_v MV-algebras. We proved that the direct product of a family of H_v MV-subalgebras (H_v MV-ideals, weak H_v MV-ideals) is again an H_v MV-subalgebra (H_v MV-ideal, weak H_v MV-ideal). Also we characterized the H_v MV-subalgebras (H_v MV-ideals, weak H_v MV-ideals) of the direct product of a family of H_v MV-algebras via the H_v MV-subalgebras (H_v MV-ideals, weak H_v MV-ideals) of any member of the family. Then

using the fundamental equivalence relation, we obtained some homomorphism theorems.

The category of H_v MV-algebras and fuzzy H_v MV-ideals could be the topics for further research works.

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H_vMV -ALGEBRAS II

M. BAKHSI

 H_vMV -جبرها ۲محمود بخشی
گروه ریاضی دانشگاه بجنورد

در این مقاله، تحقیق در مورد H_vMV -جبرها را ادامه می دهیم. به این ترتیب که ساختار خارج قسمتی آنها را مطالعه و برخی خواص و نتایج مرتبط را بدست می آوریم. همچنین چند قضیه همریختی را بیان و اثبات می کنیم. بعلاوه H_vMV -جبرهای اساسی را معرفی و نیز حاصلضرب مستقیم H_vMV -جبرها را بررسی و نتایج مرتبط با آنها را ارائه می دهیم.

کلمات کلیدی: MV -جبر، H_vMV -جبر، MV -جبر اساسی.