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ON GRADED LOCAL COHOMOLOGY MODULES DEFINED BY A PAIR OF IDEALS

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Abstract. Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ be a standard graded ring, M be a finitely-generated graded R-module, and J be a homogeneous ideal of R. In this work, we study the graded structure of the i-th local cohomology module of M, defined by a pair of ideals (R_+, J), i.e. $H^i_{R_+,J}(M)$. More precisely, we discuss the finiteness property and vanishing of the graded components $H^i_{R_+,J}(M)_n$. Also, we study the Artinian property and tameness of certain submodules and quotient modules of $H^i_{R_+,J}(M)$.

1. Introduction

Let R denotes a commutative Noetherian ring, M be an R-module, and I and J stand for two ideals of R. Takahashi et al. in [10], introduced the i-th local cohomology functor with respect to (I, J), denoted by $H^i_{I,J}(-)$, as the i-th right derived functor of the (I, J)-torsion functor $\Gamma_{I,J}(-)$, where $\Gamma_{I,J}(M) := \{x \in M : I^n x \subseteq Jx$ for $n \gg 1\}$. This notion is the ordinary local cohomology functor when $J = 0$ (see [3]). The main motivation for this generalization comes from

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the study of a dual of ordinary local cohomology modules $H^i_I(M)$ (see [9]). Basic facts and more information about local cohomology defined by a pair of ideals can be obtained from [4], [5], and [10].

Now let $R = \oplus_{n\in\mathbb{N}_0} R_n$ be a standard graded Noetherian ring, i.e. the base ring R_0 is a commutative Noetherian ring, and R is generated, as an R_0-algebra, by finitely many elements of R_1. Also, let J be a homogeneous ideal of R, M be a graded R-module, and $R_+ := \oplus_{n\in\mathbb{N}} R_n$ be the irrelevant ideal of R. It is well-known ([3, Section 12]) that for all $i \geq 0$, the i-th local cohomology module $H^i_J(M)$ of M with respect to J has a natural grading, and that in the case where M is finitely generated, $H^i_{R_+}(M)_n$ is a finitely generated R_0-module for all $n \in \mathbb{Z}$, and it vanishes for all $n \gg 0$ ([3, Theorem 15.1.5]).

In this paper, first, we show that $H^i_{I,J}(M)$ has a natural grading, when I and J are homogeneous ideals of R, and M is a graded R-module. Then, we show that, although in spite of the ordinary case, $H^i_{R_+,J}(M)_n$ might be non-finitely generated over R_0 for some $n \in \mathbb{Z}$ and non-zero for all $n \gg 0$, in some special cases, they are finitely-generated for all $n \in \mathbb{Z}$ and vanish for all $n \gg 0$. More precisely, we show that if $(R_0, \mathfrak{m}_0)$ is local, $R_+ \subseteq \mathfrak{b}$ is an ideal of R, J_0 is an ideal of R_0, and $\bigcap_{k=0}^{\infty} \mathfrak{m}_0^k H^i_{\mathfrak{b},J_0R}(M)_n = 0$ for all $n \gg 0$, then $H^i_{\mathfrak{b},J_0R}(M)_n = 0$ for all $n \gg 0$ (Theorem 3.2). Also, we present an equivalent condition for the finiteness of components $H^i_{R_+,J}(M)_n$ (Theorem 3.3).

In the last section, first, we study the asymptotic stability of the set $\{\mathrm{Ass}_{R_0}(H^i_{R_+,J}(M)_n)\}_{n\in\mathbb{Z}}$ for $n \to -\infty$ in a special case (Theorem 4.1). Then we present some results about Artinianness of some quotients of $H^i_{R_+,J}(M)$. In particular, we show that if R_0 is a local ring with maximal ideal $\mathfrak{m}_0$ and $c \in \mathbb{Z}$ such that $H^i_{R_+,\mathfrak{m}_0R}(M)$ is Artinian for all $i > c$, then the R-module $H^c_{R_+,\mathfrak{m}_0R}(M)/\mathfrak{m}_0 H^c_{R_+,\mathfrak{m}_0R}(M)$ is Artinian (Theorem 4.2). Finally, we show that $H^i_{R_+,J}(M)$ is "tame" in a special case (Corollary 4.4).

2. GRADED LOCAL COHOMOLOGY MODULES DEFINED BY A PAIR OF IDEALS

Throughout this section, let $R = \oplus_{n\in\mathbb{Z}} R_n$ be a graded ring, I and J be two homogeneous ideals of R and $M = \oplus_{n\in\mathbb{Z}} M_n$ be a graded R-module. Then it is natural to ask whether the local cohomology modules $H^i_{I,J}(M)$ for all $i \in \mathbb{N}_0$, also carry structures as graded R-modules. In this section, we show that it has an affirmative answer.

First we show that the (I, J)-torsion functor $\Gamma_{I,J}(-)$ can be viewed as a (left exact, additive) functor from $^*\mathcal{C}$ to itself. Since the category $^*\mathcal{C}$ of all graded R-modules and homogeneous R-homomorphisms, is an Abelian category that has enough projective objects and enough injective objects, we can, therefore, carry out standard techniques of homological algebra in this category. Hence, we can form the right derived functors $^*H^i_{I,J}(-)$ of $\Gamma_{I,J}(-)$ on the category $^*\mathcal{C}$.

Lemma 2.1. *Let $x = x_{i_1} + \cdots + x_{i_k} \in M$ be such that $x_{i_j} \in M_{i_j}$ for all $j = 1, \cdots, k$. Then:*

$$r(\mathrm{Ann}\,(x)) = \bigcap_{j=1}^{k} r(\mathrm{Ann}\,(x_{i_j})).$$

Proof. $\supseteq$: Is clear.

$\subseteq$: First we show that if $y = y_{l_1} + \cdots + y_{l_m} \in \mathrm{Ann}\,(x)$ such that $y_{l_k} \in R_{l_k}$ for all $k = 1, \cdots, m$ and $l_1 < l_2 < \cdots < l_m$, then $y_{l_1} \in \cap_{j=1}^{k} r(\mathrm{Ann}\,(x_{i_j}))$. We have:

$$0 = yx = \sum_{j=1}^{n}\sum_{k=1}^{m} y_{l_k} x_{i_j}, \qquad (*)$$

comparing degrees, we get $y_{l_1} x_{i_1} = 0$. Let $j_0 > 1$, and suppose, inductively, that for all $j' < j_0$, $y^{j'}_{l_1} x_{i_{j'}} = 0$. Then using $(*)$, we get:

$$\sum_{j=1}^{n}\sum_{k=1}^{m} y_{l_k} y^{j_0-1}_{l_1} x_{i_j} = 0.$$

Again, comparing degrees, we have: $y^{j_0}_{l_1} x_{i_{j_0}} = 0$. Thus $y_{l_1} \in \cap_{j=1}^{k} r(\mathrm{Ann}\,(x_{i_j}))$.

Now let $y = y_{l_1} + \cdots + y_{l_m} \in r(\mathrm{Ann}\,(x))$ such that $l_1 < l_2 < \ldots < l_m$ and $y_{l_j} \in R_{l_j}$ for all $j = 1, \cdots, m$. Then, there exists $s \in \mathbb{N}_0$ such that $y^s x = 0$.

In order to show that $y \in \cap_{j=1}^{k} r(\mathrm{Ann}\,(x_{i_j}))$, we proceed by induction on m. The result is clear in the case of $m = 1$. Now suppose, inductively, that $m > 1$, and the result has been proved for all values less than m. Using the above agreement, we know that $y_{l_1} \in \bigcap_{j=1}^{k} r(\mathrm{Ann}\,(x_{i_j})) \subseteq r(\mathrm{Ann}\,(x))$. Then $y_{l_2} + \cdots + y_{l_m} = y - y_{l_1} \in r(\mathrm{Ann}\,(x))$. By inductive hypothesis, $y_{l_2} + \cdots + y_{l_m} \in \bigcap_{j=1}^{k} r(\mathrm{Ann}\,(x_{i_j}))$, and so $y \in \bigcap_{j=1}^{k} r(\mathrm{Ann}\,(x_{i_j}))$. □

Lemma 2.2. *$\Gamma_{I,J}(M)$ is a graded R-module.*

Proof. Let $x \in \Gamma_{I,J}(M)$. Assume that $x = x_{i_1} + \cdots + x_{i_k}$, where, for all $j = 1, 2, \cdots, k$, $x_{i_j} \in M_{i_j}$ and $i_1 < i_2 < \cdots < i_k$. We show that $x_{i_1}, \cdots, x_{i_k} \in \Gamma_{I,J}(M)$. Since R is Noetherian, there is $t' \in \mathbb{N}$ such that $(r(\mathrm{Ann}\,(x_{i_j})))^{t'} \subseteq \mathrm{Ann}\,(x_{i_j})$ for all $j = 1, 2, \cdots, k$. Let $n \in \mathbb{N}_0$ be such that $I^n \subseteq \mathrm{Ann}\,(x) + J$. Thus by Lemma 2.1, for all $j = 1, 2, \cdots, k$, we have:

$$I^{2nt'} \subseteq (\mathrm{Ann}\,(x) + J)^{2t'} \subseteq (\mathrm{Ann}\,(x))^{t'} + J^{t'} \subseteq (r(\mathrm{Ann}\,(x)))^{t'} + J^{t'}$$
$$= (\cap_{j=1}^{k} r(\mathrm{Ann}\,(x_{i_j})))^{t'} + J^{t'} \subseteq \mathrm{Ann}\,(x_{i_j}) + J.$$

Thus $x_{i_j} \in \Gamma_{I,J}(M)$, as required. □

To calculate the graded local cohomology module ${}^*H^i_{I,J}(M)$ $(i \in \mathbb{N}_0)$, one proceeds as follows:

Taking an *injective resolution,

$$E^\bullet : 0 \to E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \to \cdots \to E^i \xrightarrow{d^i} E^{i+1} \to \cdots,$$

of M in ${}^*\mathcal{C}$, applying the functor $\Gamma_{I,J}(-)$ to it, and taking the i-th cohomology module of this complex, we get:

$$\frac{ker\Gamma_{I,J}(d^i)}{im\Gamma_{I,J}(d^{i-1})},$$

which is denoted by ${}^*H^i_{I,J}(M)$, and is a graded R-module.

Remark 2.3. *Let* $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ *be an exact sequence of* R*-modules and* R*-homomorphisms. Then, for each* $i \in \mathbb{N}_0$*, there is a homogeneous connecting homomorphism,* ${}^*H^i_{I,J}(N) \to {}^*H^{i+1}_{I,J}(L)$*, and these connecting homomorphisms make the resulting homogeneous long exact sequence:*

$$\begin{array}{llll} 0 \to {}^*H^0_{I,J}(L) & \xrightarrow{{}^*H^0_{I,J}(f)} {}^*H^0_{I,J}(M) & \xrightarrow{{}^*H^0_{I,J}(g)} {}^*H^0_{I,J}(N) \\ \to {}^*H^1_{I,J}(L) & \xrightarrow{{}^*H^1_{I,J}(f)} {}^*H^1_{I,J}(M) & \xrightarrow{{}^*H^1_{I,J}(g)} {}^*H^1_{I,J}(N) \\ \to \cdots & & \\ \to {}^*H^i_{I,J}(L) & \xrightarrow{{}^*H^i_{I,J}(f)} {}^*H^i_{I,J}(M) & \xrightarrow{{}^*H^i_{I,J}(g)} {}^*H^i_{I,J}(N) \\ \to {}^*H^{i+1}_{I,J}(L) \to \cdots . & & \end{array}$$

The reader should also be aware of the 'natural' or 'functorial' properties of these long exact sequences.

Definition 2.4. *We define a partial order on the set:*

$$\begin{aligned} {}^*\widetilde{W}(I,J) := \{ & \mathfrak{a} : \mathfrak{a} \text{ is a homogeneous ideal of } R; I^n \subseteq \mathfrak{a} + J \\ & \text{for some integer } n \geq 1\}, \end{aligned}$$

by letting $\mathfrak{a} \leq \mathfrak{b}$ *if* $\mathfrak{a} \supseteq \mathfrak{b}$ *for* $\mathfrak{a}, \mathfrak{b} \in {}^*\widetilde{W}(I,J)$*. If* $\mathfrak{a} \leq \mathfrak{b}$*, then we have* $\Gamma_{\mathfrak{a}}(M) \subseteq \Gamma_{\mathfrak{b}}(M)$*. Therefore, the relation* $\leq$ *on* ${}^*\widetilde{W}(I,J)$ *with together the inclusion maps make* $\{\Gamma_{\mathfrak{a}}(M)\}_{\mathfrak{a} \in {}^*\widetilde{W}(I,J)}$ *into a direct system of graded* R*-modules.*

As Takahashi et al. showed, in [10], the relation between the local cohomology functor $H^i_I(-)$ and $H^i_{I,J}(-)$, we show the same relation between their graded version as follows.

Proposition 2.5. *There is a natural graded isomorphism,*

$$\left({}^*H^i_{I,J}(-)\right)_{i\in\mathbb{N}_0} \cong \left(\varinjlim_{\mathfrak{a} \in {}^*\widetilde{W}(I,J)} {}^*H^i_{\mathfrak{a}}(-)\right)_{i\in\mathbb{N}_0},$$

of strongly-connected sequences of covariant functors.

Proof. First of all, we show that $\Gamma_{I,J}(M) = \bigcup_{\mathfrak{a} \in {}^*\widetilde{W}(I,J)} \Gamma_{\mathfrak{a}}(M)$.

$\supseteq$: Suppose that $x \in \bigcup_{\mathfrak{a} \in {}^*\widetilde{W}(I,J)} \Gamma_{\mathfrak{a}}(M)$. Then there are $\mathfrak{a} \in {}^*\widetilde{W}(I,J)$ and integer n such that $I^n \subseteq \mathfrak{a} + J$ and $x \in \Gamma_{\mathfrak{a}}(M)$. Let $t \in \mathbb{N}_0$ be such that $\mathfrak{a}^t \subseteq \operatorname{Ann}(x)$. Therefore, $I^{2nt} \subseteq (\mathfrak{a}+J)^{2t} \subseteq \mathfrak{a}^t + J^t \subseteq \operatorname{Ann}(x) + J$, and so $x \in \Gamma_{I,J}(M)$.

$\subseteq$: Conversely, let $x \in \Gamma_{I,J}(M)$. Then $I^n \subseteq \operatorname{Ann}(x) + J$ for some $n \in \mathbb{N}$. We show that $x \in \Gamma_{\mathfrak{a}}(M)$ such that $\mathfrak{a} = r(\operatorname{Ann}(x))$. As $r(\operatorname{Ann}(x))$ is homogeneous by Lemma 2.1, and $I^n \subseteq \operatorname{Ann}(x) + J$, we have $r(\operatorname{Ann}(x)) \in {}^*\widetilde{W}(I,J)$ and $x \in \Gamma_{r(\operatorname{Ann}(x))}(M)$.

Now, [3, Exercise 12.1.7] implies the desired isomorpism.

□

Remark 2.6. *If one forgets the grading on* ${}^*H^i_{I,J}(M)$*, the resulting* R*-module is isomorphic to* $H^i_{I,J}(M)$*. More precisely, using* [3, Proposition 12.1.3] *and the fact that the direct systems* $\widetilde{W}(I,J)$ *and* ${}^*\widetilde{W}(I,J)$ *are cofinal, we have:*

$$H^i_{I,J}(E) \cong \varinjlim_{\mathfrak{a} \in \widetilde{W}(I,J)} H^i_{\mathfrak{a}}(E) \cong \varinjlim_{\mathfrak{a} \in {}^*\widetilde{W}(I,J)} H^i_{\mathfrak{a}}(E) = 0,$$

for all $i > 0$ *and all* **injective* R*-modules* E*. Now, using similar argument as used in* [3, Corollary 12.3.3]*, one can see that there exists an equivalent of functors:*

$$H^i_{I,J}(- \rceil_{{}^*\mathcal{C}}) \cong {}^*H^i_{I,J}(-),$$

from ${}^*\mathcal{C}$ *to itself.*

As a consequence of the above remark and [3, Remark 13.1.9(ii)], we have the following.

Corollary 2.7. *Let* $t \in \mathbb{Z}$*, then:*

$$H^i_{I,J}(M(t)) \cong (H^i_{I,J}(M))(t),$$

for all $i \in \mathbb{N}$*, where* $(.)(t) : {}^*\mathcal{C} \to {}^*\mathcal{C}$ *is the* t*-th shift functor.*

3. VANISHING AND FINITENESS OF COMPONENTS

A crucial role in the study of the graded local cohomology is vanishing and finiteness of their components. As one can see in [3, Theorem 15.1.5], $H^i_{R_+}(M)_n$ is a finitely-generated R_0-module for all $n \in \mathbb{Z}$, and it vanishes for all $n \gg 0$. In this section, we show that, although it is not the same for $H^i_{R_+,J}(M)$, it holds in some special cases.

In the rest of this paper, we assume that $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ is a standard graded ring, and M is a finitely generated-graded R-module.

Local cohomology with respect to a pair of ideals does not satisfy in [3, Theorem 15.1.5], in general, as the first part of the following counterexample shows.

Remark 3.1. *(i) Let $R = \mathbb{Z}[X]$, and $R_+ = (X)$. We can see that $\Gamma_{R_+,R_+}(\mathbb{Z}[X])_n = \mathbb{Z}[X]_n \neq 0$ for all $n \in \mathbb{N}_0$.*

(ii) Assume that J is an ideal of R generated by elements of degree zero such that $JR_+ = 0$. It is easy to see that in this condition $\Gamma_{R_+,J}(M) = \Gamma_{R_+}(M)$, and therefore, [3, Theorem 15.1.5] *holds for $H^i_{R_+,J}(M)$.*

(iii) Let $(R_0, \mathfrak{m}_0)$ be a local ring, and $\dim R_0 = 0$. Then $\Gamma_{R_+,\mathfrak{m}_0R}(M) = \Gamma_{R_+}(M)$, and, again, [3, Theorem 15.1.5] *holds for $H^i_{R_+,\mathfrak{m}_0R}(M)$.*

The following proposition, indicates a vanishing property on the graded components of $H^i_{\mathfrak{b},\mathfrak{m}_0R}(M)$ for ideal $\mathfrak{b} = \mathfrak{b}_0 + R_+$, where $\mathfrak{b}_0$ is an ideal of R_0, and $\mathfrak{m}_0$ is the unique maximal ideal of R_0. Vanishing of the components $H^i_{\mathfrak{b}}(M)_n$ for $n \gg 0$ has already been studied in [7].

Theorem 3.2. *Assume that $(R_0, \mathfrak{m}_0)$ is local, and $i \in \mathbb{N}_0$. Let $\mathfrak{b}_0$ and J_0 be two proper ideals of R_0 and $\mathfrak{b} := \mathfrak{b}_0 + R_+$, such that for all finitely generated graded R-modules M, $\bigcap_{k=0}^{\infty} \mathfrak{m}_0^k H^i_{\mathfrak{b},J_0R}(M)_n = 0$ for all $n \gg 0$. Then $H^i_{\mathfrak{b},J_0R}(M)_n = 0$ for all $n \gg 0$ and all finitely generated graded R-modules M.*

Proof. We proceed by induction on $\dim M$.

Let $J := J_0R$. If $\dim M = 0$, then, using [8, Theorem 1], $\Gamma_{\mathfrak{b},J}(M)_n = M_n = 0$ for all $n \gg 0$.

Now let $\dim M > 0$. Considering the long exact sequence

$$H^i_{\mathfrak{b},J}(\Gamma_J(M))_n \to H^i_{\mathfrak{b},J}(M)_n \to H^i_{\mathfrak{b},J}(\overline{M})_n \to H^{i+1}_{\mathfrak{b},J}(\Gamma_J(M))_n,$$

where $\overline{M} = M/\Gamma_J(M)$, by [7, Proposition 1.1], we get $H^i_{\mathfrak{b},J}(M)_n \cong H^i_{\mathfrak{b},J}(\overline{M})_n$ for all $n \gg 0$. Therefore, we may assume that M is J-torsion free, and so there exists $x_0 \in J_0 \backslash Z_{R_0}(M)$. Now, the exact sequence

$$0 \to M \xrightarrow{x_0} M \to M/x_0M \to 0,$$

implies the exact sequence

$$H^i_{\mathfrak{b},J}(M)_n \xrightarrow{x_0} H^i_{\mathfrak{b},J}(M)_n \to H^i_{\mathfrak{b},J}(M/x_0M)_n.$$

Then, by the assumptions and the inductive hypothesis,

$$H^i_{\mathfrak{b},J}(M/x_0M)_n = 0,$$

for all $n \gg 0$. Thus:

$$H^i_{\mathfrak{b},J}(M)_n = x_0 H^i_{\mathfrak{b},J}(M)_n,$$

for all $n \gg 0$. Therefore,

$$H^i_{\mathfrak{b},J}(M)_n = \bigcap_{k=0}^{\infty} x_0^k H^i_{\mathfrak{b},J}(M)_n = 0,$$

for all $n \gg 0$. Now the result follows by induction.

□

In the following, we present an equivalent condition for the finiteness of components $H^i_{R_+,J}(M)_n$.

Theorem 3.3. *Let $(R_0, \mathfrak{m}_0)$ be local, and $J_0 \subseteq \mathfrak{m}_0$ be an ideal of R_0. Then the following statements are equivalent.*

a) *For all finitely-generated graded R-modules M and all $i \in \mathbb{N}_0$, $H^i_{R_+,J_0R}(M)_n = 0$ for $n \gg 0$.*
b) *For all finitely-generated graded R-modules M, all $i \in \mathbb{N}_0$ and $n \in \mathbb{Z}$, $H^i_{R_+,J_0R}(M)_n$ is a finitely generated R_0-module.*

Proof. Let $J := J_0R$.

a)$\Rightarrow$ b) Let M be a non-zero finitely-generated graded R-module. We proceed by induction on i. It is clear that $H^0_{R_+,J}(M)$ is a finitely generated R-module, and then $H^0_{R_+,J}(M)_n$ is finitely generated as an R_0-module for each $n \in \mathbb{Z}$.

Now suppose that $i > 0$, and the result is proved for values smaller than i. As $H^i_{R_+,J}(M) \cong H^i_{R_+,J}(M/\Gamma_{R_+,J}(M))$, we may assume that M is an (R_+, J)-torsion free R-module, and so R_+-torsion free R-module. Hence, R_+ contains a non zero-divisor on M. As $M \neq R_+M$, there exists a homogeneous element $x \in R_+$ of degree t, which is a non zero-divisor on M, by [3, Lemma 15.1.4]. We use the exact sequence $0 \to M \xrightarrow{x} M(t) \to (M/xM)(t) \to 0$ of graded R-modules and homogeneous homomorphisms to obtain the exact sequence

$$H^{i-1}_{R_+,J}(M/xM)_{n+t} \to H^i_{R_+,J}(M)_n \xrightarrow{x} H^i_{R_+,J}(M)_{n+t},$$

for all $n \in \mathbb{Z}$. It follows from the inductive hypothesis that $H^{i-1}_{R_+,J}(M/xM)_j$ is a finitely-generated R_0-module for all $j \in \mathbb{Z}$. Let $s \in \mathbb{Z}$ be such that $H^i_{R_+,J}(M)_m = 0$ for all $m \geq s$. Fixing an integer n, then for some $k \in \mathbb{N}_0$, we get $n + kt \geq s$, and then $H^i_{R_+,J}(M)_{n+kt} = 0$. Now, for all $j = 0, \cdots, k-1$, we have the exact sequence

$$H^{i-1}_{R_+,J}(M/xM)_{n+(j+1)t} \to H^i_{R_+,J}(M)_{n+jt} \xrightarrow{x} H^i_{R_+,J}(M)_{n+(j+1)t}.$$

Since $H^i_{R_+,J}(M)_{n+kt} = 0$, and $H^{i-1}_{R_+,J}(M/xM)_{n+kt}$ is a finitely-generated R_0-module, so $H^i_{R_+,J}(M)_{n+(k-1)t}$ is a finitely-generated R_0-module. Therefore, $H^i_{R_+,J}(M)_{n+jt}$ is a finitely-generated R_0-module for $j = 0, \ldots, k-1$. Now the result follows by induction.

b)$\Rightarrow$ a) The result follows from the above theorem.

□

4. Asymptotic behavior of $H^i_{R_+,J}(M)_n$ for $n \ll 0$

In this section, we consider the asymptotic behavior of components $H^i_{R_+,J}(M)_n$ when $n \to -\infty$. More precisely, first we study the asymptotic stability of the set $\{\mathrm{Ass}_{R_0}(H^i_{R_+,J}(M)_n)\}_{n\in\mathbb{Z}}$, in a special case.

Then we investigate the Artinianness and tameness of some quotients and submodules of $H^i_{R_+,J}(M)$.

Let us recall that for a given sequence $\{S_n\}_{n\in\mathbb{Z}}$ of sets $S_n \subseteq \operatorname{Spec}(R_0)$, we say that $\{S_n\}_{n\in\mathbb{Z}}$ is asymptotically stable for $n \to -\infty$, if there is some $n_0 \in \mathbb{Z}$ such that $S_n = S_{n_0}$ for all $n \leq n_0$ (see [1]). Let the base ring R_0 be local, and $i \in \mathbb{N}_0$ be such that the R-module $H^j_{R_+}(M)$ is finitely-generated for all $j < i$. In [2, Lemma 5.4], it has been shown that $\{\operatorname{Ass}_{R_0}(H^i_{R_+}(M)_n)\}_{n\in\mathbb{Z}}$ is asymptotically stable for $n \to -\infty$. The next theorem use similar argument to improve this result to local cohomology modules defined by a pair of ideals.

Theorem 4.1. *Let $(R_0, \mathfrak{m}_0)$ be a local ring with infinite residue field, and $i \in \mathbb{N}_0$ be such that the R-module $H^j_{R_+,J}(M)$ is finitely-generated for all $j < i$. If one of the equivalent conditions of the Theorem 3.3 holds, then $\operatorname{Ass}_{R_0}(H^i_{R_+,J}(M)_n)$ is asymptotically stable for $n \to -\infty$.*

Proof. We use induction on i. For $i = 0$, the result is clear from the fact that $H^0_{R_+,J}(M)_n = 0$ for all $n \ll 0$. Now let $i > 0$. In view of the natural graded isomorphism, $H^i_{R_+,J}(M) \cong H^i_{R_+,J}(M/\Gamma_{R_+,J}(M))$, for all $i \in \mathbb{N}_0$, and using [3, Lemma 15.1.4], we may assume that there exists a homogeneous element $x \in R_1$ that is a non zero-divisor on M. Now, by the long exact sequence

$$H^{j-1}_{R_+,J}(M) \to H^{j-1}_{R_+,J}(M/xM) \to H^j_{R_+,J}(M)(-1) \xrightarrow{x} H^j_{R_+,J}(M),$$

for all $j \in \mathbb{Z}$, we have $H^j_{R_+,J}(M/xM)$ is finitely-generated for all $j < i-1$. Hence, by the inductive hypothesis,

$$\operatorname{Ass}_{R_0}(H^{i-1}_{R_+,J}(M/xM)_n) = \operatorname{Ass}_{R_0}(H^{i-1}_{R_+,J}(M/xM)_{n_1}) =: X,$$

for some $n_1 \in \mathbb{Z}$ and all $n \leq n_1$. Furthermore, there is some $n_2 < n_1$ such that $H^{i-1}_{R_+,J}(M)_{n+1} = 0$ for all $n \leq n_2$. Then for all $n \leq n_2$, we have the exact sequence

$$0 \to H^{i-1}_{R_+,J}(M/xM)_{n+1} \to H^i_{R_+,J}(M)_n \xrightarrow{x} H^i_{R_+,J}(M)_{n+1}.$$

Thus it shows that:

$$X \subseteq \operatorname{Ass}_{R_0}(H^i_{R_+,J}(M)_n) \subseteq X \cup \operatorname{Ass}_{R_0}(H^i_{R_+,J}(M)_{n+1}),$$

for all $n \leq n_2$. Hence,

$$\mathrm{Ass}_{R_0}(H^i_{R_+,J}(M)_n) \subseteq \mathrm{Ass}_{R_0}(H^i_{R_+,J}(M)_{n+1}),$$

for all $n < n_2$, and, using the assumption, the proof is complete.

□

In the rest of the paper, we pay attention to the Artinianness property of the graded modules $H^i_{R_+,J}(M)$. The following proposition gives a graded analogue of [5, Theorem 2.2].

Theorem 4.2. *Assume that R_0 is a local ring with maximal ideal $\mathfrak{m}_0$. If $c \in \mathbb{Z}$ and $H^i_{R_+,\mathfrak{m}_0R}(M)$ is Artinian for all $i > c$, then the R-module $H^c_{R_+,\mathfrak{m}_0R}(M)/\mathfrak{m}_0 H^c_{R_+,\mathfrak{m}_0R}(M)$ is Artinian.*

Proof. Let $\mathfrak{m} := \mathfrak{m}_0 + R_+$ be the unique graded maximal ideal of R, and let $J := \mathfrak{m}_0R$. We have $H^i_{R_+,J}(M) = H^i_{\mathfrak{m},J}(M)$ for all i. Thus we can replace R_+ by $\mathfrak{m}$. We proceed the assertion by induction on $n := \dim M$. The result is clear in the case of $n = 0$. Let $n > 0$, and that the statement is proved for all values less than n. Now, using the long exact sequence:

$$H^i_{\mathfrak{m},J}(\Gamma_J(M)) \to H^i_{\mathfrak{m},J}(M) \to H^i_{\mathfrak{m},J}(M/\Gamma_J(M)) \to H^{i+1}_{\mathfrak{m},J}(\Gamma_J(M)),$$

and the fact that $H^i_{\mathfrak{m},J}(\Gamma_J(M)) = H^i_{\mathfrak{m}}(\Gamma_J(M))$ is Artinian for all i, replacing M with $M/\Gamma_J(M)$, we may assume that $\Gamma_J(M) = 0$. Therefore, there exists $x_0 \in \mathfrak{m}_0 \backslash Z_{R_0}(M)$. Now, the long exact sequence

$$H^i_{\mathfrak{m},J}(M) \xrightarrow{x_0} H^i_{\mathfrak{m},J}(M) \xrightarrow{\alpha_i} H^i_{\mathfrak{m},J}(M/x_0M) \xrightarrow{\beta_i} H^{i+1}_{\mathfrak{m},J}(M),$$

implies that $H^i_{\mathfrak{m},J}(M/x_0M)$ is Artinian for all $i > c$, and so, by inductive hypothesis, $H^c_{\mathfrak{m},J}(M/x_0M)/\mathfrak{m}_0 H^c_{\mathfrak{m},J}(M/x_0M)$ is Artinian. Considering the exact sequences:

$$0 \to Im\alpha_c \to H^c_{\mathfrak{m},J}(M/x_0M) \to Im\beta_c \to 0,$$

and

$$H^c_{\mathfrak{m},J}(M) \xrightarrow{x_0} H^c_{\mathfrak{m},J}(M) \xrightarrow{\alpha_c} Im\alpha_c \to 0,$$

we get the following exact sequences:

$$Tor_1^R(R_0/\mathfrak{m}_0, Im\beta_c) \to Im\alpha_c/\mathfrak{m}_0 Im\alpha$$

$$\to H^c_{\mathfrak{m},J}(M/x_0M)/\mathfrak{m}_0 H^c_{\mathfrak{m},J}(M/x_0M), \qquad (A)$$

and

$$H^c_{\mathfrak{m},J}(M)/\mathfrak{m}_0 H^c_{\mathfrak{m},J}(M) \xrightarrow{x_0} H^c_{\mathfrak{m},J}(M)/\mathfrak{m}_0 H^c_{\mathfrak{m},J}(M)$$

$$\to Im\alpha_c/\mathfrak{m}_0 Im\alpha_c \to 0. \qquad (B)$$

These two exact sequences imply that $H^c_{\mathfrak{m},J}(M)/\mathfrak{m}_0 H^c_{\mathfrak{m},J}(M)$ is Artinian, and the assertion follows. □

Let I, J be ideals of R. Chu and Wang, in [5], defined $\operatorname{cd}(I, J, R) := \sup\{i; H^i_{I,J}(M) \neq 0\}$.

The following corollary is an immediate consequence of Theorem 4.2.

Corollary 4.3. *Assume that R_0 is local with maximal ideal $\mathfrak{m}_0$. If $c := \operatorname{cd}(R_+, \mathfrak{m}_0 R, M)$, then $H^c_{R_+,\mathfrak{m}_0 R}(M)/\mathfrak{m}_0 H^c_{R_+,\mathfrak{m}_0 R}(M)$ is Artinian.*

Let $T = \bigoplus_{n\in\mathbb{N}_0} T_n$ be a graded R-module. Following [1], we say that T is tame or asymptotically gap free, if either $T_n = 0$ for all $n \ll 0$ or else $T_n \neq 0$ for all $n \ll 0$. Now, as an application of the above Corollary, we have the following:

Corollary 4.4. *Let $(R_0, \mathfrak{m}_0)$ be local and $c := \operatorname{cd}(R_+, \mathfrak{m}_0 R, M)$. If one of the equivalent conditions of Theorem 3.3 holds, then $H^c_{R_+,\mathfrak{m}_0 R}(M)$ is tame.*

Proof. Since $H^c_{R_+,\mathfrak{m}_0 R}(M)/\mathfrak{m}_0 H^c_{R_+,\mathfrak{m}_0 R}(M)$ is Artinian, it is tame. Now, the result follows using Nakayama's lemma. □

Proposition 4.5. *Let $(R_0, \mathfrak{m}_0)$ be local, $J \subseteq R_+$ be a homogeneous ideal of R, and $g(M) := \sup\{i : \forall j < i, \ell_{R_0}(H^j_{R_+}(M)_n) < \infty, \forall n \ll 0\}$ be finite. Then, the graded R-module $H^i_{\mathfrak{m}_0 R,J}(H^j_{R_+}(M))$ is Artinian for $i = 0, 1$ and all $j \leq g(M)$.*

Proof. Since $J \subseteq R_+$, so $H^j_{R_+}(M)$ is J-torsion, therefore, $H^i_{\mathfrak{m}_0 R,J}(H^j_{R_+}(M)) \cong H^i_{\mathfrak{m}_0 R}(H^j_{R_+}(M))$. Now the result follows from [6, Theorem 2.4]. □

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ON GRADED LOCAL COHOMOLOGY MODULES DEFINED BY A PAIR OF IDEALS

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بررسی مدول‌های کوهمولوژی موضعی مدرج نسبت به دو ایده‌آل

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فرض کنیم $R = \bigoplus_{n \in \mathbb{N}_\circ} R_n$ یک حلقه مدرج استاندارد، M یک R-مدول متناهی مولد مدرج و J یک ایده‌آل همگن از R باشد. در این مقاله ما ساختار مدرج i-امین مدول کوهمولوژی موضعی M نسبت به جفت ایده‌آل (R_+, J)، یعنی $H^i_{R_+,J}(M)$، را مطالعه می‌کنیم. به‌عبارت دقیقتر، پیرامون خاصیت متناهی مولد بودن و صفر شدن مؤلفه‌های مدرج $H^i_{R_+,J}(M)_n$ بحث می‌کنیم. همچنین به مطالعه خاصیت آرتینی و همنوایی بعضی زیرمدول‌ها و مدول‌های خارج قسمتی از $H^i_{R_+,J}(M)$ خواهیم پرداخت.

کلمات کلیدی: مدول‌های مدرج، مدول‌های کوهمولوژی موضعی نسبت به دوایده‌آل، مدول‌های آرتینی.