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ON THE VANISHING OF DERIVED LOCAL HOMOLOGY MODULES

M. HATAMKHANI

Abstract. Let R be a commutative Noetherian ring, $\mathfrak{a}$ be an ideal of R, and $\mathcal{D}(R)$ denote the derived category of R-modules. For any homologically-bounded complex X, we conjecture that $\sup \mathbf{L}\Lambda^{\mathfrak{a}}(X) \leq \mathrm{mag}_R X$. We prove this in several cases.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring with non-zero identity, and $\mathcal{D}(R)$ denotes the derived category of R-modules. The full subcategory of homologically-bounded complexes is denoted by $\mathcal{D}_{\square}(R)$, and that for complexes homologically-bounded to the right (resp. left) is denoted by $\mathcal{D}_{\sqsupset}(R)$ (resp. $\mathcal{D}_{\sqsubset}(R)$). Also $\mathcal{D}^{f}_{\square}(R)$(resp. $\mathcal{D}^{Art}_{\square}(R)$) consists of homologically-bounded complexes with finitely-generated (resp. Artinian) homologies. The symbol $\simeq$ denotes an isomorphism in the category $\mathcal{D}(R)$. For any complex X in $\mathcal{D}_{\sqsupset}(R)$ (resp. $\mathcal{D}_{\sqsubset}(R)$), there is a bounded to the right (resp. left) complex U of projective (resp. injective) R-modules such that $U \simeq X$. A such

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complex U is called a projective (resp. injective) resolution of X. The left-derived tensor product functor $-\otimes^{\mathbf{L}}_R \sim$ is computed by taking a projective resolution of the first argument or of the second one. Also, the right-derived homomorphism functor $\mathbf{R}\operatorname{Hom}_R(-,\sim)$ is computed by taking a projective resolution of the first argument or by taking an injective resolution of the second one.

Let $\mathfrak{a}$ be an ideal of R, and $\mathcal{C}_0(R)$ denote the full sub-category of R-modules. It is known that the $\mathfrak{a}$-adic completion functor

$$\Lambda^{\mathfrak{a}}(-) = \varprojlim_n (R/\mathfrak{a}^n \otimes_R -) : \mathcal{C}_0(R) \to \mathcal{C}_0(R)$$

is not right exact, in general. The left-derived functor of $\Lambda^{\mathfrak{a}}(-)$ exists in $\mathcal{D}(R)$, and so, for any complex $X \in \mathcal{D}_{\sqsupset}(R)$, the complex $\mathbf{L}\Lambda^{\mathfrak{a}}(X) \in \mathcal{D}_{\sqsupset}(R)$ is defined by $\mathbf{L}\Lambda^{\mathfrak{a}}(X) := \Lambda^{\mathfrak{a}}(P)$, where P is a (every) projective resolution of X. Let $X \in \mathcal{D}_{\sqsupset}(R)$. For any integer i, the i-th local homology module of X with respect to $\mathfrak{a}$ is defined by $H^{\mathfrak{a}}_i(X) := H_i(\mathbf{L}\Lambda^{\mathfrak{a}}(X))$. First E. Matlis [12], in 1974, studied the theory of the local homology. Next Simon in [18] and [19] continued the study of this theory. Later, J.P.C. Greenlees and J.P. May [9] defined local homology groups of a module M using a new approach. Then came the works of Alonso Tarrío, Jeremías López and Lipman [1]. After the works of [17], [4], [5], [8], and [14], started a new era in the study of local homology.

The most essential vanishing result for the local cohomology modules $H^i_{\mathfrak{a}}(M)$ is Grothendieck's Vanishing Theorem, which asserts that $H^i_{\mathfrak{a}}(M) = 0$ for all $i > \dim_R M$. Letting $X \in \mathcal{D}_{\square}(R)$, Foxby generalized this result for derived local cohomology modules $H^i_{\mathfrak{a}}(X)$. He proved that $H^i_{\mathfrak{a}}(X) = 0$ for all $i > \dim_R X$ [7, Theorem 7.8, Corollary 8.29]. We intend to establish the dual of this result for the derived local homology modules. Let $\check{C}(\underline{\mathfrak{a}})$ denote the $\check{C}$ech complex of R on a set $\underline{\mathfrak{a}}$ of generators of $\mathfrak{a}$. By [1, (0.3), aff,p.4] (see also [17, Section 4] for corrections),

$$\mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq \mathbf{R}\operatorname{Hom}_R(\check{C}(\underline{\mathfrak{a}}), X).$$

Using this isomorphism Frankild [8, Theorem 2.11] proved that $\inf \mathbf{L}\Lambda^{\mathfrak{a}}(X) = \operatorname{width}_R(\mathfrak{a}, X)$, where $\operatorname{width}_R(\mathfrak{a}, X) := \inf(R/\mathfrak{a} \otimes^{\mathbf{L}}_R X)$.

The aim of this work was to find an upper bound for $\sup \mathbf{L}\Lambda^{\mathfrak{a}}(X)$. Finding a good upper bound for $\sup \mathbf{L}\Lambda^{\mathfrak{a}}(X)$ was considered in [17] and [8]. In fact, we conjecture that $H^{\mathfrak{a}}_i(X) = 0$ for all $i > \operatorname{mag}_R X$. Our investigation on this conjecture is the core of this paper. We show the correctness of this conjecture in several cases. Namely, we prove that if for all $i \in \mathbb{Z}$, either:

$\operatorname{Coass}_R H_i(X) = \mathcal{A}tt_R H_i(X)$,

$H_i(X)$ is finitely-generated, Artinian or Matlis reflexive,

$H_i(X)$ is linearly-compact,

R is complete local, and $H_i(X)$ has finitely many minimal coassociated prime ideals; or:

R is complete local with the maximal ideal $\mathfrak{m}$, and $\mathfrak{m}^n H_i(X)$ is minimax for some integer $n \geq 0$,

then $H^{\mathfrak{a}}_i(X) = 0$ for all $i > \operatorname{mag}_R X$.

First, Sazeedeh [16] studied connections between the Gorenstein injective modules and the local cohomology modules. The Gorenstein flat dimension of X is defined by

$$\begin{aligned}\operatorname{Gfd}_R X \ &:= \inf\{\sup\{l \in \mathbb{Z} | Q_l \neq 0\} | Q \text{ is a bounded to the right} \\ &\quad \text{complex of Gorenstein flat R-modules and } Q \simeq X\}.\end{aligned}$$

For more details on the theory of Gorenstein homological dimensions for complexes, we refer the reader to [2].

2. Results

In what follows, we denote the faithful exact functor,

$$\operatorname{Hom}_R(-, \bigoplus_{\mathfrak{m} \in \operatorname{Max} R} E(R/\mathfrak{m}))$$

by $(-)^{\vee}$. Let M be an R-module. A prime ideal $\mathfrak{p}$ of R is said to be a coassociated prime ideal of M if there is an Artinian quotient L of M such that $\mathfrak{p} = (0 :_R L)$. The set of all coassociated prime ideals of M

is denoted by $\operatorname{Coass}_R M$. Also, $\mathcal{A}tt_R M$ is defined by

$$\mathcal{A}tt_R M := \{\mathfrak{p} \in \operatorname{Spec} R|\ \mathfrak{p} = (0 :_R L) \ \text{ for some quotient } \ L \text{ of } \ M\}.$$

Clearly, $\operatorname{Coass}_R M \subseteq \mathcal{A}tt_R M$ and the equality holds if either R or M is Artinian. More generally, if M is representable, then it is easy to check that $\operatorname{Coass}_R M = \mathcal{A}tt_R M$. If $0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$ is an exact sequence of R-modules and R-homomorphisms, then it is easy to check that:

$$\operatorname{Coass}_R L \subseteq \operatorname{Coass}_R N \subseteq \operatorname{Coass}_R L \cup \operatorname{Coass}_R M,$$

and:

$$\mathcal{A}tt_R L \subseteq \mathcal{A}tt_R N \subseteq \mathcal{A}tt_R L \cup \mathcal{A}tt_R M.$$

Also if R is local, then one can see that $\operatorname{Coass}_R M = \operatorname{Ass}_R M^\vee$.

For an R-module M, set $\operatorname{cd}_\mathfrak{a} M := \sup\{i | H^i_\mathfrak{a}(M) \neq 0\}$.

By [9, Corollary 3.2], $H_i^\mathfrak{a}(M) = 0$ for all $i > \operatorname{cd}_\mathfrak{a} R$.

Next, we recall the definition of the notion $\operatorname{mag}_R M$.

Definition 2.1. Let M be an R-module.

i) (See [20]) The magnitude of M is defined by

$$\operatorname{mag}_R M := \sup\{\dim R/\mathfrak{p} | \mathfrak{p} \in \operatorname{Coass}_R M\}.$$

If $M = 0$, then we put $\operatorname{mag}_R M = -\infty$.

ii) (See [15]) The Noetherian dimension of M is defined inductively as follows: when $M = 0$, put $\operatorname{Ndim}_R M = -1$. Then, by induction, for an integer $d \geq 0$, we put $\operatorname{Ndim}_R M = d$ if $\operatorname{Ndim}_R M < d$ is false, and for every ascending sequence $M_0 \subseteq M_1 \subseteq \ldots$ of submodules of M, there exists n_0 such that $\operatorname{Ndim}_R M_{n+1}/M_n < d$ for all $n > n_0$.

iii) (See [14]) The co-localization of M at a prime ideal $\mathfrak{p}$ of R is defined by

$$^{\mathfrak{p}}M := \operatorname{Hom}_{R_\mathfrak{p}}((M^\vee)_\mathfrak{p}, E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})).$$

Then $\operatorname{Cosupp}_R M$ is defined by

$$\operatorname{Cosupp}_R M := \{\mathfrak{p} \in \operatorname{Spec} R | {}^{\mathfrak{p}}M \neq 0\}.$$

iv) (See [3]) M is said to be N-critical if $\operatorname{Ndim}_R N < \operatorname{Ndim}_R M$ for all proper submodules N of M.

If $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is an exact sequence of R-modules and R-homomorphisms, then it is easy to verify that:

$$\operatorname{mag}_R Y = \max\{\operatorname{mag}_R X, \operatorname{mag}_R Z\}.$$

Recall that an R-module M is said to be *Matlis reflexive* if the natural homomorphism $M \longrightarrow M^{\vee\vee}$ is an isomorphism.

Now we recall some definitions, which are required in the following statements.

We begin by recalling the definition of linearly-compact modules from [11]. Let M be a topological R-module. Then M is said to be *linearly-topologized* if M has a base $\mathcal{M}$ consisting of sub-modules for the neighborhoods of its zero element. A Hausdorff linearly-topologized R-module M is said to be *linearly-compact* if for any family $\mathcal{F}$ of cosets of closed submodules of M which has the finite intersection property, the intersection of all cosets in $\mathcal{F}$ is non-empty. A Hausdorff linearly topologized R-module M is called *semi-discrete* if every submodule of M is closed. The class of semi-discrete linearly-compact modules is very large it contains many important classes of modules such as the class of Artinian modules or the class of finitely-generated modules over a complete local ring.

An R-module M is called *minimax* if it has a finitely-generated submodule N such that M/N is Artinian. By [21, Lemma 1.1], over a complete local ring R, an R-module M is minimax if and only if M is semi-discrete linearly-compact and if and only if M is Matlis reflexive. These definitions can be extended to complexes in obvious ways.

In the case $(R, \mathfrak{m})$ is a local ring, by [20, Lemma 2.2], we have $\operatorname{mag} M = \dim M^{\vee}$ for any R-module M, where $(-)^{\vee} := \operatorname{Hom}_R(-, E(\frac{R}{\mathfrak{m}}))$. Following this idea, one could expect $\operatorname{mag}_R X = \dim_R X^{\vee}$.

Let $X \in D(R)$. We know that $\dim_R X^{\vee} = \sup\{\dim_R H_i(X^{\vee}) - i \mid i \in \mathbb{Z}\} = \sup\{\dim_R H_{-i}(X)^{\vee} - i \mid i \in \mathbb{Z}\} = \sup\{\dim_R H_j(X)^{\vee} + j \mid \ j \in \mathbb{Z}\}$.

Therefore, we define $\operatorname{mag}_R X$ as follows:

Definition 2.2. Let R be a commutative Noetherian ring, and $X \in \mathcal{D}(R)$. We define $\operatorname{mag}_R X := \sup\{\operatorname{mag}_R H_i(X) + i \mid \ i \in \mathbb{Z}\}$.

Lemma 2.3. *Let $(R, \mathfrak{m})$ be a local ring, and $X \in \mathcal{D}(R)$. Then:*

$$mag_R X = \sup\{\dim \tfrac{R}{\mathfrak{p}} + \sup{}^{\mathfrak{p}} X \mid \ \mathfrak{p} \in Cosupp_R X\}.$$

Proof. By definition, we have $\operatorname{mag}_R X = \dim_R X^\vee$. Now we know that $\dim_R X^\vee = \sup\{\dim \frac{R}{\mathfrak{p}} - \inf(X^\vee)_\mathfrak{p} | \ \mathfrak{p} \in \operatorname{Supp}_R X^\vee\}$. Let $\mathfrak{p} \in \operatorname{Spec} R$. By definition, we have ${}^{\mathfrak{p}}X = \operatorname{Hom}_{R_\mathfrak{p}}((X^\vee)_\mathfrak{p}, E(\frac{R}{\mathfrak{p}}))$, and so:

$$\sup{}^{\mathfrak{p}} X = -\inf(X^\vee)_\mathfrak{p}.$$

Also, by [14, Theorem 2.7] , $Cosupp_R X = \operatorname{Supp}_R X^\vee$. Hence,

$$\operatorname{mag}_R X = \sup\{\dim \tfrac{R}{\mathfrak{p}} + \sup{}^{\mathfrak{p}} X | \ \mathfrak{p} \in Cosupp_R X\}.$$

□

Remark 2.4. Let X and Y be complexes in $\mathcal{D}(R)$. Observe that the isomorphism $\mathbf{L}\Lambda^\mathfrak{a}(X) \simeq \mathbf{R}\operatorname{Hom}_R(\check{C}(\underline{\mathfrak{a}}), X)$ immediately gives:

$$\mathbf{L}\Lambda^\mathfrak{a}(\mathbf{R}\operatorname{Hom}_R(X, Y)) \simeq \mathbf{R}\operatorname{Hom}_R(X, \mathbf{L}\Lambda^\mathfrak{a}(Y)) \simeq \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\Gamma_\mathfrak{a}(X), Y).$$

Definition 2.5. (See [6]) Let $X \in \mathcal{D}(R)$.

i) If m is an integer, $\Sigma^m X$ denotes the complex X shifted (or translated) m degrees (to the left); it is given by

$$(\Sigma^m X)_l = X_{l-m} \quad , \quad d_l^{\Sigma^m X} = (-1)^m d_{l-m}^X,$$

for $l \in \mathbb{Z}$.

ii) If $m, n \in \mathbb{Z}$, the truncated complexes $\tau_{m\sqsubset} X$ and $\tau_{n\sqsupset} X$ are given by

$$\tau_{m\sqsubset} X = 0 \longrightarrow C_m^X \xrightarrow{\hat{d}_m^X} X_{m-1} \xrightarrow{d_{m-1}^X} X_{m-2} \xrightarrow{d_{m-2}^X} \ldots,$$

and

$$\tau_{n\sqsupset} X = \ldots \xrightarrow{d_{n+3}^X} X_{n+2} \xrightarrow{d_{n+2}^X} X_{n+1} \xrightarrow{\tilde{d}_{n+1}^X} Z_n^X \longrightarrow 0,$$

where $\hat{d}_m^X$ and $\tilde{d}_{n+1}^X$ are the induced maps.

Lemma 2.6. *Let R be a commutative Noetherian ring, and $X \in \mathcal{D}(R)$.*

i) *If* $X \in \mathcal{D}_{\square}(R)$, *then* $\sup \mathbf{L}\Lambda^{\mathfrak{a}}(X) \leq \sup\{\sup \mathbf{L}\Lambda^{\mathfrak{a}}(H_i(X)) + i \mid \ i \in \mathbb{Z}\}$.

ii) *If* $(R, \mathfrak{m})$ *is a complete local ring and* $X \in \mathcal{D}_{\square}^{Art}(R)$, *then*

$$\sup \mathbf{L}\Lambda^{\mathfrak{a}}(X) = \sup\{\sup \mathbf{L}\Lambda^{\mathfrak{a}}(H_i(X)) + i \ | i \in \mathbb{Z}\}.$$

Proof. i) Let $s := \sup \mathbf{L}\Lambda^{\mathfrak{a}}(X)$, and assume that $s > \sup(\mathbf{L}\Lambda^{\mathfrak{a}}(H_l(X))) + l$ for all l. By descending induction on l, we show that, $\sup(\mathbf{L}\Lambda^{\mathfrak{a}}(\tau_{l\sqsubset}X)) = s$ for all $l \in \mathbb{Z}$. This gives the desired contradiction, since $\tau_{l\sqsubset}X \simeq 0$ for l small enough. Since $\tau_{l\sqsubset}X \simeq X$ for l large enough, the equality $\sup(\mathbf{L}\Lambda^{\mathfrak{a}}(\tau_{l\sqsubset}X)) = s$ certainly holds for large l.

In the inductive step, note that the exact triple,

$$(\Sigma^l H_l(X), \tau_{l\sqsubset}X, \tau_{l-1\sqsubset}X)$$

(see [6, Corollary 1.43]) induces an exact sequence:

$$\ldots \longrightarrow H^{\mathfrak{a}}_{m-l}(H_l(X)) \longrightarrow H^{\mathfrak{a}}_m(\tau_{l\sqsubset}X) \longrightarrow H^{\mathfrak{a}}_m(\tau_{l-1\sqsubset}X) \longrightarrow H^{\mathfrak{a}}_{m-l-1}(H_l(X)) \longrightarrow \ldots$$

from which the desired assertion, $\sup(\mathbf{L}\Lambda^{\mathfrak{a}}(\tau_{l-1\sqsubset}X)) = s$, follows from the inductive assumption,

$\sup(\mathbf{L}\Lambda^{\mathfrak{a}}(\tau_{l\sqsubset}X)) = s$, and the assumption $\sup(\mathbf{L}\Lambda^{\mathfrak{a}}(H_l(X))) < s - l$ made earlier.

ii) As $X \in \mathcal{D}_{\square}^{Art}(R)$, it follows that $X^{\vee\vee} \simeq X$ in $\mathcal{D}(R)$. Hence, we have:

$$\begin{aligned}
\sup \mathbf{L}\Lambda^{\mathfrak{a}}(X) &= \sup \mathbf{L}\Lambda^{\mathfrak{a}}(X^{\vee\vee}) \\
&= \sup \mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\operatorname{Hom}_R(X^{\vee}, E(\frac{R}{\mathfrak{m}})) \\
&\overset{(a)}{=} \sup(\mathbf{R}\operatorname{Hom}_R(X^{\vee}, \mathbf{L}\Lambda^{\mathfrak{a}}(E(\frac{R}{\mathfrak{m}}))) \\
&\overset{(b)}{=} \sup\{\sup \mathbf{R}\operatorname{Hom}_R(H_i(X^{\vee}), \mathbf{L}\Lambda^{\mathfrak{a}}(E(\frac{R}{\mathfrak{m}}))) - i \mid \ i \in \mathbb{Z}\} \\
&\overset{(c)}{=} \sup\{\sup \mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\operatorname{Hom}_R(H_i(X^{\vee}), E(\frac{R}{\mathfrak{m}}))) - i \mid \ i \in \mathbb{Z}\} \\
&= \sup\{\sup \mathbf{L}\Lambda^{\mathfrak{a}}(H_i(X^{\vee})^{\vee}) - i \mid \ i \in \mathbb{Z}\} \\
&= \sup\{\sup \mathbf{L}\Lambda^{\mathfrak{a}}((H_{-i}(X))^{\vee\vee}) - i \mid \ i \in \mathbb{Z}\} \\
&\overset{(d)}{=} \sup\{\sup \mathbf{L}\Lambda^{\mathfrak{a}}(H_j(X)) + j \mid \ j \in \mathbb{Z}\}
\end{aligned}$$

The equalities (a) and (c) follow by Remark 2.4, and since $X^\vee \in \mathcal{D}^f_\Box(R)$, (b) follows from [6, Lemma 16.26]. Since $X \in \mathcal{D}^{Art}_\Box(R)$, the equality (d) holds. □

Now we recall the following definition of Zoschinger:

Definition 2.7. Let M be an R-module. Then $\mathrm{Coass}(M)$ has a finite final subset, when the set of minimal elements of $\mathrm{Coass}(M)$ is finite, or equivalently, there exists a finite subset, $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ of $\mathrm{Coass}(M)$ such that $\bigcap \mathrm{Coass}(M) = \bigcap\limits_{i=1}^{n} \mathfrak{p}_i$.

Theorem 2.8. *Let $\mathfrak{a}$ be an ideal of R, and $X \in \mathcal{D}_\Box(R)$. Assume that for all $i \in \mathbb{Z}$, either:*

i) $\mathrm{Coass}_R\, H_i(X) = \mathcal{A}tt_R H_i(X)$,
ii) *$H_i(X)$ is N-critical,*
iii) *$H_i(X)$ is finitely-generated or Matlis reflexive,*
iv) *$H_i(X)$ is linearly-compact,*
v) *R is complete local and $H_i(X)$ has finitely many minimal coassociated prime ideals; or:*
vi) *R is complete local with the maximal ideal $\mathfrak{m}$, and $\mathfrak{m}^n H_i(X)$ is minimax for some integer $n \geq 0$.*

Then $\sup(\mathbf{L}\Lambda^{\mathfrak{a}}(X) \leq mag_R X$ *and equality holds if R is complete local with the maximal ideal $\mathfrak{a}$ and $X \in \mathcal{D}^{Art}_\Box(R)$.*

Proof. From Lemma 2.6 i), we have:

$$\sup \mathbf{L}\Lambda^{\mathfrak{a}}(X) \leq \sup\{\sup \mathbf{L}\Lambda^{\mathfrak{a}}(H_i(X)) + i \mid \; i \in \mathbb{Z}\}.$$

On the other hand, by [10, Theorem 2.8], in each of these cases, $\sup \mathbf{L}\Lambda^{\mathfrak{a}}(H_i(X)) \leq \mathrm{mag}_R H_i(X)$ for all $i \in \mathbb{Z}$. Now the result follows by the definition of $\mathrm{mag}_R X$.

Now let $(R, \mathfrak{m})$ be a complete local ring. From [5, Theorem 4.8, 4.10], for each Artinian module M, $\mathrm{Ndim}_R\, M = max\{i \mid H^{\mathfrak{m}}_i(M) \neq 0\}$. Hence, the last part follows from Lemma 2.6 ii) and [20, Theorem 2.10]. □

Corollary 2.9. *Let $(R, \mathfrak{m})$ be a complete local ring, and $X \in \mathcal{D}_{\square}^{Art}(R)$. Then $mag_R X \leq \operatorname{Gfd}_R X$.*

Proof. From [13, Theorem 2.5], $\sup(\mathbf{L}\Lambda^{\mathfrak{a}}(X)) \leq \operatorname{Gfd}_R X$ for any ideal $\mathfrak{a}$ of R. Now, the result follows by Theorem 2.8. ☐

Definition 2.10. (See [6])

i) Let $X \in \mathcal{D}_{\sqsupset}(R)$, and $Y \in \mathcal{D}_{\sqsubset}(R)$. The module:

$$H_{-i}(\mathbf{R}\operatorname{Hom}_R(X, Y))$$

is often denoted by $\operatorname{Ext}_R^i(X, Y)$, and called the i-th hyper *Ext* module of the complexes X and Y.

ii) Let $X, Y \in \mathcal{D}_{\sqsupset}(R)$. The module: $H_i(X \otimes_R^{\mathbf{L}} Y)$ is sometimes denoted by $\operatorname{Tor}_i^R(X, Y)$, and called the i-th hyper *Tor* module of the complexes X and Y.

Assume that M and N are two R-modules, and X and Y are two complexes. The following result is deduced from Theorem 2.8.

Corollary 2.11. *Assume that M is a linearly-compact R-module, N an R-module, and $X, Y \in \mathcal{D}(R)$.*

i) *If $\mathbf{R}\operatorname{Hom}_R(N, M) \in \mathcal{D}_{\square}(R)$, then $H_i^{\mathfrak{a}}(\mathbf{R}\operatorname{Hom}_R(N, M)) = 0$ for all $i >$mag$(\mathbf{R}\operatorname{Hom}_R(N, M))$.*

ii) *If N is finitely-generated, and $N \otimes_R^{\mathbf{L}} M \in \mathcal{D}_{\square}(R)$, then:*

$$H_i^{\mathfrak{a}}(N \otimes_R^{\mathbf{L}} M) = 0$$

for all $i >$mag$(N \otimes_R^{\mathbf{L}} M)$.

Proof. By [4, Lemma 2.5 and 2.6], the R-modules $\operatorname{Ext}_R^i(N, M)$ and $\operatorname{Tor}_i^R(N, M)$ are linearly-compact for all non-negative integers i. Hence, the result follows by Theorem 2.8. ☐

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M. Hatamkhani

Department of Mathematics,

Faculty of Science,

Arak University,

Arak, 38156-8-8349, Iran.

`Email: m-hatamkhani@araku.ac.ir`

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M. HATAMKHANI

صفر شدن مدول‌های همولوژی موضعی مشتق‌شده

مرضیه حاتم خانی
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فرض کنید R یک حلقه جابجایی نوتری و a ایده‌آلی از R و $D(R)$ نشان‌دهنده کاتگوری مشتق‌شده از R–مدول‌ها باشد. ما حدس می‌زنیم برای هر همبافت بطور همولوژیکی کراندار X، بزرگی همبافت X، کران بالایی برای سوپریموم فانکتور مشتق شده چپ نسبت به ایده‌آل a روی X باشد و در چند حالت مختلف آن را ثابت میکنیم.

کلمات کلیدی: کوهمولوژی موضعی، مدول‌های همولوژی موضعی، بزرگی مدول، بعد نوتری.