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AN INDUCTIVE FUZZY DIMENSION

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Abstract. Using a system of axioms among with a modified definition of boundary on the basis of the intuitionistic fuzzy sets, we formulate an inductive structure for the dimension of fuzzy spaces which has been defined by Coker. This new definition of boundary allows to characterize an intuitionistic fuzzy clopen set as a set with zero boundary. Also, some critical properties and applications are established.

1. Introduction

Science and technology are usually featured with complex processes and complete information about scientific phenomena is not always available. For such cases, mathematical models are developed to handle various types of systems, containing elements of uncertainty. A large part of these models are based on a recent extension of the ordinary set theory, so-called intuitionistic fuzzy sets. The concept of intuitionistic fuzzy sets has been introduced by Atanassov [3, 4, 5], as a generalization of fuzzy sets. This concept has wide applications, including GIS fields, medical diagnosis and microelectronic fault analysis [14, 15, 17, 18]. Taking the advantage of intuitionistic fuzzy sets, the notion of intuitionistic fuzzy topological spaces was realized by Coker [7]. Some concepts of fuzzy topology namely, covering dimension, separation axioms, fuzzy compactness, Tychonoff theorem, fuzzy continuity, fuzzy metric spaces and fuzzy connectedness have been generalized for intuitionistic fuzzy topological spaces [2, 6, 9, 10, 12, 20, 21]. Tang [19]

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made a heavy use of the notion of fuzzy boundary for studying land cover changes. Considering the inherent nature of GIS phenomena, it seems more suitable to study the problem of land cover changes using intuitionistic fuzzy topology. Thus, the study of intuitionistic fuzzy boundary is imperative for recasting the GIS problems in terms of intuitionistic fuzzy topology. Recently, intuitionistic fuzzy boundaries have been investigated [16], however, the definition of intuitionistic fuzzy boundary lacks the following properties that one may wish:

(a) If an intuitionistic fuzzy set is both open and closed, then the intuitionistic fuzzy boundary is $0_{\widetilde{X}}$.
(b) If an intuitionistic fuzzy set is closed (or open), then the interior of the intuitionistic fuzzy boundary is $0_{\widetilde{X}}$.

In our recent paper [1], we investigated the zero dimensionality of fuzzy topological spaces in the sense of Lowen, and showed how the concept might be sensitive to the choice of definition of fuzzy topology. In this paper, and with almost the same purpose, a new definition of boundary of an intuitionistic fuzzy set is established that allows us to characterize the intuitionistic fuzzy clopen sets as those whose boundaries are $0_{\widetilde{X}}$. It is then followed by introducing the concept of small inductive dimension for intuitionistic fuzzy topological spaces. Some properties such as subspace theorem are also proven. By virtue of property (a), the zero dimensionality of an intuitionistic fuzzy topological space is defined through the existence of a neighborhood basis consisting of fuzzy clopen sets.

2. Preliminaries

We give here some basic preliminaries.

Definition 2.1. [3, 4] Let X be a non-empty fixed set. An intuitionistic fuzzy set A (IFS, for short) in X is an object having the form $A = \{< x, \mu_A(x), \gamma_A(x) >: x \in X\}$, where $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for every $x \in X$. The functions $\mu_A : X \to \mathbb{I}$ and $\gamma_A : X \to \mathbb{I}$ are called the degree of membership and the degree of non-membership, respectively.

Notice that every fuzzy set $A = \{< x, \mu_A(x) >: x \in X\}$ on X is an intuitionistic fuzzy set of the form $A = \{< x, \mu_A(x), 1 - \mu_A(x) >: x \in X\}$.

Definition 2.2. [5] Let X be a non-empty set, and consider the intuitionistic fuzzy sets $A = \{< x, \mu_A(x), \gamma_A(x) >: x \in X\}$, $B = \{< x, \mu_B(x), \gamma_B(x) >: x \in X\}$ and let $\{A_\lambda\}_\lambda$ be an arbitrary family of intuitionistic fuzzy sets in X. Then,

(a) $A \leq B$ if for every $x \in X, \mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$,

(b) $A = B$ if $A \leq B$ and $B \leq A$,
(c) $A^c = \{< x, \gamma_A(x), \mu_A(x) >: x \in X\}$,
(d) $\bigwedge A_\lambda = \{< x, \bigwedge \mu_{A_\lambda}(x), \bigvee \gamma_{A_\lambda}(x) >: x \in X\}$,
in particular

$$A \wedge B = \{< x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x) >: x \in X\},$$

(e) $\bigvee A_\lambda = \{< x, \bigvee \mu_{A_\lambda}(x), \bigwedge \gamma_{A_\lambda}(x) >: x \in X\}$,
in particular

$$A \vee B = \{< x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x) >: x \in X\},$$

(f) $0_{\widetilde{X}} = \{< x, 0, 1 >: x \in X\}$ and $1_{\widetilde{X}} = \{< x, 1, 0 >: x \in X\}$.

Definition 2.3. (Intuitionistic fuzzy sets induced by mapping) Let X and Y be two non-empty sets and let $f : X \to Y$ be a function. Then,

(a) If $B = \{< y, \mu_B(y), \gamma_B(y) >: y \in Y\}$ is an intuitionistic fuzzy set in Y, then the preimage of B under f denoted by $f^{-1}(B)$ is the IFS in X defined by

$$f^{-1}(B) = \{< x, f^{-1}(\mu_B)(x), f^{-1}(\gamma_B)(x) >: x \in X\},$$

(b) If $A = \{< x, \lambda_A(x), \nu_A(x) >: x \in X\}$ is an intuitionistic fuzzy set in X, then the image of A under f denoted by $f(A)$ is the intuitionistic fuzzy set in Y defined by

$$f(A) = \{< y, f(\lambda_A)(y), 1 - f(1 - \nu_A)(y) >: y \in Y\},$$

where,

$$f(\lambda_A)(y) = \begin{cases} \mathrm{sup}_{\mathrm{x \in f^{-1}(y)}}\{\lambda_\mathrm{A}(\mathrm{x})\} & ;\ f^{-1}(y) \neq \emptyset, \\ 0 & ;\ \text{otherwise} \end{cases}$$

and

$$1 - f(1 - \nu_A)(y) = \begin{cases} \mathrm{inf}_{\mathrm{x \in f^{-1}(y)}}\{\nu_\mathrm{A}(\mathrm{x})\} & ;\ f^{-1}(y) \neq \emptyset, \\ 1 & ;\ \text{otherwise} \end{cases}$$

See [13] for more informations.

Definition 2.4. [8] Let $\alpha, \beta \in [0, 1]$, such that $\alpha + \beta \leq 1$. An intuitionistic fuzzy point (IFP, for short) of non-empty set X is an IFS of X defined by:

$$x_{(\alpha,\beta)}(y) = \begin{cases} (\alpha, \beta) & ;\ y = x, \\ (0, 1) & ;\ y \neq x, \end{cases}$$

In this case, x is called the support of $x_{(\alpha,\beta)}$ and α, β are called the value and non-value of $x_{(\alpha,\beta)}$, respectively.

Clearly, an intuitionistic fuzzy point can be represented by an ordered pair of fuzzy point as follows:

$$x_{(\alpha,\beta)} = (x_\alpha, 1 - x_{1-\beta}).$$

An IFP $x_{(\alpha,\beta)}$ is said to belong to an IFS, $A = \{< x, \mu_A, \gamma_A >: x \in X\}$, denoted by $x_{(\alpha,\beta)} \in A$, if $\alpha \leq \mu_A(x)$ and $\beta \geq \gamma_A(x)$.

Definition 2.5. [7] An intuitionistic fuzzy topology (IFT, for short) on a non-empty set X is a family τ of intuitionistic fuzzy sets in X with the following axioms:

(i) $0_{\widetilde{X}}, 1_{\widetilde{X}} \in \tau$,
(ii) If $A_1, A_2 \in \tau$, then $A_1 \wedge A_2 \in \tau$,
(iii) If $A_\lambda \in \tau$ for each $\lambda \in \Lambda$, then $\bigvee_{\lambda\in\Lambda} A_\lambda \in \tau$.

The pair (X, τ) is called an intuitionistic fuzzy topological space (IFTS, for short). An intuitionistic fuzzy set in τ is known as an intuitionistic fuzzy open set (IFOS, for short) of X, and its complement as a closed intuitionistic fuzzy set (IFCS, for short).

An intuitionstic fuzzy set in X is said to be a clopen if it is closed and open. The intuitionistic fuzzy interior and intuitionistic fuzzy closure of intuitionstic fuzzy set $A =< x, \mu_A(x), \gamma_A(x) >$ are defined by

$$\mathrm{cl(A)} = \bigwedge\{\mathrm{K : K \text{ is an IFCS in X and } A \subset K}\},$$
$$\mathrm{int(A)} = \bigvee\{\mathrm{G : G \text{ is an IFOS in X and } G \subset A}\}.$$

It can be easily shown that cl(A) is an IFCS and int(A) is an IFOS in X, and

(a) A is an IFCS in X iff cl(A) = A,
(b) A is an IFOS in X iff int(A) = A.

See [7] for more informations.

Definition 2.6. [9] If (X, τ) and (Y, ϕ) are intuitionistic fuzzy topological spaces, then a map $f : X \to Y$ is said to be;

(a) Continuous if $f^{-1}(B)$ is an intuitionistic fuzzy open set in X, for each intuitionistic fuzzy open set B in Y, or equivalently, $f^{-1}(B)$ is an intuitionistic fuzzy closed set in X, for each intuitionistic fuzzy closed set B in Y,
(b) Open if $f(A)$ is an intuitionistic fuzzy open set in Y, for each intuitionistic fuzzy open set A in X,
(c) Closed if $f(A)$ is an intuitionistic fuzzy closed set in Y for each intuitionistic fuzzy closed set A in X,
(d) A homomorphism if f is bijective, continuous, and open.

Note that an IFTS (X, τ) is actually defined in the sense of Chang, and the constant functions between such spaces are not necessarily continuous. For example, let $X = [0, 1]$ and $f : (X, \tau) \to (X, \tau')$ be the constant function $f(x) = \frac{1}{2}$, where

$$\tau = \{0_{\widetilde{X}}, 1_{\widetilde{X}}\},\ \tau' = \{0_{\widetilde{X}}, 1_{\widetilde{X}}, A\},$$

and the intuitionistic fuzzy set $A =< x, \mu_A, \gamma_A >$ is defined by;

$$\mu_A(x) = \begin{cases} 0 & x \in [0, \frac{1}{2}), \\ \frac{1}{2} & x \in [\frac{1}{2}, 1] \end{cases} \quad \text{and} \quad \gamma_{\mathrm{A}}(\mathrm{x}) = \begin{cases} 1 & x \in [0, \frac{1}{2}), \\ \frac{1}{2} & x \in [\frac{1}{2}, 1] \end{cases}.$$

Now, the function f is not continuous, since A is an intuitionistic fuzzy open set in τ', but $f^{-1}(A)$ is not an intuitionistic fuzzy open set in τ. To avoid such unpleasants, we may use the definition of intuitionistic fuzzy topological spaces in the sense of Lowen. It means that, an intuitionistic fuzzy topological space is a pair (X, τ), where (X, τ) is an IFTS and moreover, all constant IFS of the form $C_{\alpha,\beta} = \{< x, \alpha, \beta >: x \in X\}$, where $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$, belongs to τ.

3. Intuitionistic fuzzy inductive dimension

The concept of the boundary of sets is essential in the definition of inductive dimensions of topological spaces, see [11]. Here, we present a new definition of the boundary of an intuitionistic fuzzy set, that allows to characterize intuitionistic fuzzy clopen sets as the sets with empty boundaries. We also introduce the concept of inductive dimension for intuitionistic fuzzy topological spaces.

Let X be an IFTS And A be an IFS in X. The intuitionistic fuzzy boundary of A (IBd(A), for short) is defined by $\mathrm{IBd(A)} = \mathrm{cl(A)} \wedge \mathrm{cl(A^c)}$ (Manimaran et al. [16]). The following example shows that, according to this definition, the intuitionistic fuzzy boundary of an intuitionistic fuzzy clopen set is not necessarily empty.

Example 3.1. Let (X, τ) be an intuitionistic fuzzy topological space, where $X = [0, 1]$, $\tau = \{0_{\widetilde{X}}, 1_{\widetilde{X}}, A, B, A \wedge B, A \vee B\}$ such that $A = \{< x, \mu_A, \gamma_A >: x \in X\}$, $B = \{< x, \mu_B, \gamma_B >: x \in X\}$ are defined by;

$$\mu_A(x) = \begin{cases} 0 & x \in [0, \frac{1}{3}], \\ \frac{3}{2}x - \frac{1}{2} & x \in [\frac{1}{3}, 1] \end{cases} \quad \gamma_A(x) = \begin{cases} 1 & x \in [0, \frac{1}{3}], \\ -\frac{3}{2}(x-1) & x \in [\frac{1}{3}, 1] \end{cases}$$
$$\mu_B(x) = \begin{cases} 1 & x \in [0, \frac{1}{3}], \\ -\frac{3}{2}(x-1) & x \in [\frac{1}{3}, 1] \end{cases} \quad \gamma_B(x) = \begin{cases} 0 & x \in [0, \frac{1}{3}], \\ \frac{3}{2}x - \frac{1}{2} & x \in [\frac{1}{3}, 1] \end{cases}.$$

Then, $\mathrm{IBd(A)} = \mathrm{cl(A)} \wedge \mathrm{cl(A^c)} = \mathrm{A} \wedge \mathrm{B}$. Note that A is an intuitionistic fuzzy clopen set and $\mathrm{IBd(A)} \neq 0_{\widetilde{\mathrm{X}}}$.

Now, we are going to propose an alternative definition of the boundary of an intuitionistic fuzzy set. We will call it intuitionistic fuzzy frontier of an intuitionistic fuzzy set.

Definition 3.2. Let A be an intuitionistic fuzzy set in an intuitionistic fuzzy topological space (X, τ). The intuitionistic fuzzy frontier of A, denoted by $\mathrm{IFr}(A)$, is defined as the infimum of all intuitionistic fuzzy closed sets B in X with the property $B(x) \geq \mathrm{cl}(A)(x)$ for all $x \in X$, for which $\mathrm{cl}(A)(x) - \mathrm{int}(A)(x) > 0$.

Proposition 3.3. *Let A and B be intuitionistic fuzzy sets in X and $x \in X$. Then;*

(a) $\mathrm{IFr}(0_{\tilde{X}}) = 0_{\tilde{X}}$,
(b) $\mathrm{IFr}(1_{\tilde{X}}) = 0_{\tilde{X}}$,
(c) $\mathrm{IFr}(A)$ *is an intuitionistic fuzzy closed set of X,*
(d) $\mathrm{IFr}(A) \leq \mathrm{cl}(A)$,
(e) *If* $\mathrm{cl}(A)(x) - \mathrm{int}(A)(x) > 0$ *then* $\mathrm{IFr}(A)(x) = \mathrm{cl}(A)(x)$,
(f) $\mathrm{IFr}(A) \geq \mathrm{cl}(A) - \mathrm{int}(A)$,
(g) $\mathrm{cl}(A) = \mathrm{int}(A) \vee \mathrm{IFr}(A) = A \vee \mathrm{IFr}(A)$,
(h) $\mathrm{IFr}(\mathrm{IFr}(A)) \leq \mathrm{IFr}(A)$,
(k) $A \vee B \vee \mathrm{IFr}(A \vee B) = (A \vee \mathrm{IFr}(A)) \wedge (B \vee \mathrm{IFr}(B))$.

Proof. (a), (b) and (c) follow directly from Definition3.2.

(d) $\mathrm{cl}(A)$ is an intuitionistic fuzzy closed set and $\mathrm{cl}(A)(x) \geq \mathrm{cl}(A)(x)$ for all $x \in X$ so, in particular, for those such that
$$\mathrm{cl}(A)(x) - \mathrm{int}(A)(x) > 0.$$
Hence, $\mathrm{IFr}(A) \leq \mathrm{cl}(A)$.

(e) It is a direct consequence of Definition3.2 and (d).

(f) There are two possibilities:
(1) If $\mathrm{cl}(A)(x) - \mathrm{int}(A)(x) > 0$, then
$\mathrm{IFr}(A)(x) = \mathrm{cl}(A)(x) \geq \mathrm{cl}(A)(x) - \mathrm{int}(A)(x)$.
(2) If $\mathrm{cl}(A)(x) = \mathrm{int}(A)(x)$, then
$\mathrm{IFr}(A)(x) \geq \mathrm{cl}(A)(x) - \mathrm{int}(A)(x) = 0$,

(g) Obviously, $\mathrm{cl}(A) \geq \mathrm{int}(A) \vee \mathrm{IFr}(A)$. If $\mathrm{cl}(A)(x) - \mathrm{int}(A)(x) > 0$, then $\mathrm{cl}(A)(x) = \mathrm{IFr}(A)(x)$. If $\mathrm{cl}(A)(x) - \mathrm{int}(A)(x) = 0$, then $\mathrm{cl}(A)(x) = \mathrm{int}(A)(x)$. Thus,
$$\mathrm{cl}(A) = \mathrm{int}(A) \vee \mathrm{IFr}(A) \leq A \vee \mathrm{IFr}(A) \leq \mathrm{cl}(A).$$

(h) $\mathrm{IFr}(\mathrm{IFr}(A)) \leq \mathrm{cl}(\mathrm{IFr}(A)) = \mathrm{IFr}(A)$,

(k)
$$\begin{aligned} A \vee B \vee \mathrm{IFr}(A \vee B) &= \mathrm{cl}(A \vee B) \\ &= \mathrm{cl}(A) \wedge \mathrm{cl}(B) \\ &= (A \vee \mathrm{IFr}(A)) \wedge (B \vee \mathrm{IFr}(B)). \end{aligned}$$

□

Corollary 3.4. *Let A be an intuitionistic fuzzy set in X. Then;*
(a) *If* $\operatorname{int}(A) = 0_{\widetilde{X}}$, *then* $\operatorname{IFr}(A) = \operatorname{cl}(A)$,
(b) $\operatorname{IFr}(A^c) \geq \operatorname{cl}(A) - \operatorname{int}(A)$,
(c) *A is an intuitionistic fuzzy closed set of X if and only if* $\operatorname{IFr}(A) \leq A$.

Theorem 3.5. *Let A be an intuitionistic fuzzy set in X. Then A is intuitionistic fuzzy clopen if and only if* $\operatorname{IFr}(A) = 0_{\widetilde{X}}$.

Proof. If A is intuitionistic fuzzy clopen set, then $\operatorname{cl}(A)(x) = \operatorname{int}(A)(x)$ for all $x \in X$. Thus, $\operatorname{IFr}(A) = 0_{\widetilde{X}}$. Conversely, if $\operatorname{IFr}(A) = 0_{\widetilde{X}}$, then inequality $\operatorname{cl}(A)(x) - \operatorname{int}(A)(x) > 0$ does not hold, for any $x \in X$. Hence, $\operatorname{cl}(A)(x) - \operatorname{int}(A)(x) = 0$, for each $x \in X$. Thus, A is intuitionistic fuzzy clopen set. □

Theorem 3.6. *Let A be an intuitionistic fuzzy closed set of X. Then,* $\operatorname{IFr}(A) = \operatorname{IFr}(\operatorname{IFr}(A))$.

Proof. It suffices to prove that $\operatorname{IFr}(A) \leq \operatorname{IFr}(\operatorname{IFr}(A))$. Choose an intuitionistic fuzzy closed set B such that for each $x \in X$ with

$$\operatorname{IFr}(A)(x) - \operatorname{int}(\operatorname{IFr}(A))(x) > 0,$$

we have $B(x) \geq \operatorname{IFr}(A)(x)$. It is easy to see that for any $x \in X$ with $A(x) - \operatorname{int}(A(x)) > 0$, we have $B(x) \geq A(x)$, as desired. □

Corollary 3.7. *For every intuitionistic fuzzy set A in X,*

$$\operatorname{IFr}(\operatorname{IFr}(A)) = \operatorname{IFr}(\operatorname{IFr}(\operatorname{IFr}(A))).$$

Theorem 3.8. *Let Y be a fuzzy subspace of X and A be an intuitionistic fuzzy set in X. Then,* $\operatorname{IFr}(A|_Y) \leq \operatorname{IFr}(A)|_Y$.

Proof. First note that $\operatorname{cl}(A|_Y) \leq \operatorname{cl}(A)|_Y$ and $\operatorname{int}(A|_Y) \geq \operatorname{int}(A)|_Y$. Let B be an intuitionistic fuzzy closed set such that for all $x \in X$ with $\operatorname{cl}(A)(x) - \operatorname{int}(A)(x) > 0$, we have $B(x) \geq \operatorname{cl}(A)(x)$. Thus, for all $y \in Y$ with $\operatorname{cl}(A|_Y)(y) - \operatorname{int}(A|_Y)(y) > 0$, we have $(B|_Y)(y) \geq \operatorname{cl}(A|_Y)(y)$. In fact, whenever $0 < \operatorname{cl}(A|_Y)(y) - \operatorname{int}(A|_Y)(y) \leq \operatorname{cl}(A)(y) - \operatorname{int}(A)(y)$ we have $B(y) \geq \operatorname{cl}(A)(y) \geq \operatorname{cl}(A|_Y)(y)$. Hence, $\operatorname{IFr}(A|_Y) \leq \operatorname{IFr}(A)|_Y$. □

Example 3.9. Let τ be an intuitionistic fuzzy topology on $X = [0, 1]$ with subbase

$$\{C_{\alpha,\beta} : \ \alpha, \beta \in [0, 1], \ \alpha + \beta \leq 1\} \cup \{A\},$$

where $A = \{< x, \mu_A(x), \gamma_A(x) >: x \in X\}$ is a IFS on X defined by

$$\mu_A(x) = \begin{cases} 0 & x \in [0, \frac{1}{2}], \\ \frac{2}{3} & x \in (\frac{1}{2}, 1] \end{cases} \quad \text{and} \quad \gamma_A(x) = \begin{cases} 1 & x \in [0, \frac{1}{2}], \\ \frac{1}{3} & x \in (\frac{1}{2}, 1] \end{cases} .$$

Clearly, any non-constant intuitionistic fuzzy open set $O = \{< x, \mu_O, \gamma_O >: x \in X\}$ is defined by

$$\mu_O(x) = \begin{cases} a & x \in [0, \frac{1}{2}], \\ b & x \in (\frac{1}{2}, 1] \end{cases} \quad \text{and} \quad \gamma_{\mathrm{O}}(\mathrm{x}) = \begin{cases} a' & x \in [0, \frac{1}{2}], \\ b' & x \in (\frac{1}{2}, 1] \end{cases},$$

where $0 \leq a < b \leq \frac{2}{3}$ and $1 \geq a' > b' \geq \frac{1}{3}$. Analogously, the non-constsnt intuitionistic fuzzy closed sets are

$$C = \{< x, \mu_C(x), \gamma_C(x) >: x \in X\},$$

$$\mu_C(x) = \begin{cases} c & x \in [0, \frac{1}{2}], \\ d & x \in (\frac{1}{2}, 1] \end{cases} \quad \text{and} \quad \gamma_{\mathrm{C}}(\mathrm{x}) = \begin{cases} c' & x \in [0, \frac{1}{2}], \\ d' & x \in (\frac{1}{2}, 1] \end{cases},$$

where $1 \geq c > d \geq \frac{1}{3}$, and $0 \leq c' < d' \leq \frac{2}{3}$. It is easy to see that $\mathrm{cl}(A) = C_{\frac{2}{3}, \frac{1}{3}}$, $\mathrm{int}(A) = A$, and $\mathrm{IFr}(A) = \{< x, \mu(x), \gamma(x) >: x \in X\}$, where

$$\mu(x) = \begin{cases} \frac{2}{3} & x \in [0, \frac{1}{2}], \\ \frac{1}{3} & x \in (\frac{1}{2}, 1] \end{cases} \quad \text{and} \quad \gamma(\mathrm{x}) = \begin{cases} \frac{1}{3} & x \in [0, \frac{1}{2}], \\ \frac{2}{3} & x \in (\frac{1}{2}, 1] \end{cases}.$$

Definition 3.10. Let (X, τ) be an intuitionistic fuzzy topological space. The inductive dimension of X , denoted by $I - \mathrm{ind\ X}$, consists in the following conditions;

(i) $I - \mathrm{ind\ X} = -1$ if and only if $X = \emptyset$,

(ii) $I - \mathrm{ind\ X} \leq \mathrm{n}$ if for each intuitionistic fuzzy point $x_{(\alpha,\beta)}$, and each $A \in \tau$ satisfying $x_{(\alpha,\beta)} \in A$ there exists $B \in \tau$ such that $x_{(\alpha,\beta)} \in B \leq A$, and $I - \mathrm{ind\ IFr(B)} \leq \mathrm{n} - 1$.

(iii) $I - \mathrm{ind\ X} = \mathrm{n}$ if $I - \mathrm{ind\ X} \leq \mathrm{n}$, and the inequality

$$I - \mathrm{ind\ X} \leq \mathrm{n} - 1,$$

does not hold.

(iv) $I - \mathrm{ind\ X} = \infty$ if there is no $n \in \mathbb{N}$ such that $I - \mathrm{ind\ X} \leq \mathrm{n}$.

Applying induction with respect to $I - \mathrm{ind}\ X$, one can easily verify that the small inductive dimension is a topological invariant. Using Theorem 3.5 and above definition for $n = 0$, we may reformulate the following particular case.

Definition 3.11. An intuitionistic fuzzy topological space X is zero-dimensional, $I - \mathrm{ind}\ X = 0$, if for every intuitionistic fuzzy point $x_{(\alpha,\beta)}$ in X and every intuitionistic fuzzy open set A containing $x_{(\alpha,\beta)}$, there exists an intuitionistic fuzzy clopen set B in X such that

$$x_{(\alpha,\beta)} \in B \leq A.$$

Example 3.12. Let $X = [0, 1]$ and δ be an intuitionistic fuzzy topology on X defined in Example 3.9. There exists no intuitionistic fuzzy clopen set B such that $B \leq A$, because the constant intuitionistic fuzzy sets are the only intuitionistic fuzzy clopen sets. Thus, $I - \text{ind}\, X \neq 0$.

Now, we may begin to establish a series of classic desired properties of an inductive dimension, which certainly provide many facilities for more theoretical and applied researches in future. We just bring a sample of them.

Theorem 3.13. *For every fuzzy subspace Y of intuitionistic fuzzy topological space X, we have $I - \text{ind}\, Y \leq I - \text{ind}\, X$.*

Proof. The theorem is obvious if $I - \text{ind}\, X = \infty$, so one can suppose that $I - \text{ind}\, X < \infty$. We shall apply induction with respect to $I - \text{ind}\, X$. Clearly, the inequality holds if $I - \text{ind}\, X = -1$. Assume that the theorem is proved for $n - 1$. Let $I - \text{ind}\, X = n$. Consider an intuitionistic fuzzy point $x_{(\alpha,\beta)} \in Y$, and an intuitionistic fuzzy open set A containing $x_{(\alpha,\beta)}$. There exists an intuitionistic fuzzy open set A_1 of X such that $A = A_1|_Y$. Since $I - \text{ind}\, X \leq n$, there exists an intuitionistic fuzzy open set B_1 of X such that $x_{(\alpha,\beta)} \in B_1 \leq A_1$, and $I - \text{ind}\, \mathrm{IFr(B_1)} \leq \mathrm{n - 1}$. Now, putting $B = B_1|_Y$, we have

$$\mathrm{IFr}(B_1|_Y) \leq \mathrm{IFr}(B_1)|_Y,$$

then $I - \text{ind}\, \mathrm{IFr(B_1|_Y)} \leq \mathrm{n - 1}$. Thus, $I - \text{ind}\, Y \leq n = I - \text{ind}\, X$. □

Corollary 3.14. *Every non-empty subspace of a zero-dimensional intuitionistic fuzzy topological space X is zero-dimensional.*

Let (X_i, τ_i) be an intuitionistic fuzzy topological space for each $i \in J$, and let $X = \prod_{i \in J} X_i$. The ith projection mapping is defined for each $i \in J$ as follows:

$$\pi_i : X \to X_i, \quad \pi_i((x_j)_{j \in J}) = x_i.$$

The set $X = \prod_{i \in J} X_i$ with the intuitionistic fuzzy topology generated by the family $S = \{\pi_i^{-1}(s_i) : \ i \in J, \ s_i \in \tau_i\}$ is called the product of the intuitionistic fuzzy topological spaces $\{(X_i, \tau_i) : \ i \in J\}$ [10]. The following theorem can be readily formulated for the product of two fuzzy spaces.

Theorem 3.15. *Let (X_1, τ_1) and (X_2, τ_2) be intuitionistic fuzzy topological spaces. The intuitionistic fuzzy product space $X_1 \times X_2$ is zero-dimensional if and only if the intuitionistic fuzzy topological spaces X_1 and X_2 are zero-dimensional.*

Proof. It is easy to see that the spaces X_1 and X_2 are homomorphic to a subspace of $X_1 \times X_2$, so that if $X_1 \times X_2$ is zero-dimensional, then the intuitionistic fuzzy topological spaces X_1 and X_2 are zero-dimensional, by Corollary 3.14. To prove the reverse implication, it is enough to note that the family $\beta = \{B_1 \times B_2 : \ B_1 \in \beta_1, \ B_2 \in \beta_2\}$ forms a base consisting of intuitionistic fuzzy clopen sets for $X_1 \times X_2$, where β_1 and β_2 are bases consisting of intuitionistic fuzzy clopen sets for X_1 and X_2, respectively. □

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AN INDUCTIVE FUZZY DIMENSION

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در این مقاله بر اساس تعریف جدید ارائه شده برای مرز یک مجموعه فازی شهودی، یک ساختار استقرائی برای بعد فضاهای فازی شهودی که اولین بار توسط چوکر عنوان شده است معرفی می‌شود. این تعریف جدید، مجموعه‌های فازی شهودی بسته و باز فضا را به عنوان مجموعه‌هایی با مرز تهی مشخصه‌سازی می‌کند. در ادامه خواص و همچنین کاربردهایی از مرز، مورد مطالعه قرار می‌گیرد.

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