

GENERALIZED GORENSTEIN DIMENSION OVER GROUP RINGS

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ABSTRACT. Let (R, \mathfrak{m}) be a commutative noetherian local ring and Γ be a finite group. It is proved that if R admits a dualizing module, then the group ring $R\Gamma$ has a dualizing bimodule, as well. Moreover, it is shown that a finitely generated $R\Gamma$ -module M has generalized Gorenstein dimension zero if and only if it has generalized Gorenstein dimension zero as an R -module.

1. INTRODUCTION

Semi-dualizing modules arise naturally in the investigations of various duality theories in commutative algebras. One instance of this, is Grothendieck and Hartshorne's local duality wherein a canonical module, or more generally a dualizing complex, is employed to study local cohomology [13]. Another instance is Auslander and Bridger's methodical study of duality properties with respect to a rank one free module that gives rise to the Gorenstein dimension [3]. Foxby [10], Golod [12] and Vasconcelos [16] independently initiated the study of semi-dualizing modules (under different names) over commutative noetherian local rings. This notion then has been the subject of several expositions; see, for example, [8, 11, 17]. A finitely generated R -module C is said to be *semi-dualizing*, provided that $R \cong \text{Hom}_R(C, C)$ and $\text{Ext}_R^i(C, C) = 0$, for any $i > 0$. A rank one free module and a canonical module, are examples of semi-dualizing modules. The definition of semi-dualizing modules has been extended in various settings by

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many authors. In particular, Araya, Takahashi and Yoshino [1] have extended this definition to a pair of non-commutative, but noetherian rings, while White [18] extended to the non-noetherian, but commutative settings. This definition is defined for a pair of associative rings by Holm and White in [14]. Associated to a semi-dualizing R -module C , of most importance is the notion of G_C -dimension of a finitely generated module, which is studied by several authors; see [12, 8, 18]. As this invariant is a generalization of Gorenstein dimension (which is defined by Auslander and Bridger [3]), hence it is called a *generalized Gorenstein dimension*.

In this paper, we study generalized Gorenstein dimension of finitely generated modules over noetherian algebras. To be precise, assume that (R, \mathfrak{m}) is a commutative noetherian local ring and Λ is a noetherian R -algebra. Assume that ω is a semi-dualizing Λ -bimodule; see Definition 2.1. Among others, it is proved that if ω is dualizing, in the sense that it has finite injective dimension as a left and as a right Λ -module, then any finitely generated left Λ -module M has finite generalized Gorenstein dimension; see Proposition 2.9. This result, enables us to infer that any finitely generated left Λ -module admits a right $G_\omega(\Lambda)$ -approximation, where $G_\omega(\Lambda)$ stands for the class of all left Λ -modules with generalized Gorenstein dimension zero; see Corollary 2.11. Section 3 is devoted to study the generalized Gorenstein dimension over a group algebra $R\Gamma$, where Γ is a finite group. At first, it will turn out that if R admits a dualizing module ω , then $R\Gamma \otimes_R \omega$ is a dualizing bimodule for $R\Gamma$; see Proposition 3.2. The main result of this paper asserts that if M is a finitely generated (left) $R\Gamma$ -module, then M has generalized Gorenstein dimension zero if and only if it has generalized Gorenstein dimension zero as an R -module; see Theorem 3.5.

Throughout this paper, (R, \mathfrak{m}) denotes a commutative noetherian local ring and Λ is a noetherian R -algebra. All modules will be considered are finitely generated left modules. Also, right Λ -modules will be identified with left modules over the *opposite algebra* Λ^{op} .

2. GENERALIZED GORENSTEIN DIMENSION OF MODULES

In this section, by defining the notion of semi-dualizing bimodule for the noetherian algebra Λ , we investigate the generalized Gorenstein dimension of finitely generated Λ -modules. It is proved that if Λ admits a dualizing bimodule, then generalized Gorenstein dimension of each Λ -module is finite.

Definition 2.1. A finitely generated Λ -bimodule ω is called *semi-dualizing* for Λ , if:

- (1) The homothety morphisms $\Lambda \rightarrow \text{Hom}_{\Lambda^{\text{op}}}(\omega, \omega)$ and $\Lambda^{\text{op}} \rightarrow \text{Hom}_{\Lambda}(\omega, \omega)$ are bijections,
- (2) $\text{Ext}_{\Lambda}^i(\omega, \omega) = 0 = \text{Ext}_{\Lambda^{\text{op}}}^i(\omega, \omega)$, for all $i > 0$.

Definition 2.2. Let ω be a semi-dualizing bimodule for Λ . Recall from [4] that a Λ -module M is said to have *generalized Gorenstein dimension zero* (with respect to ω), denoted by $G_{\omega} - \dim_{\Lambda} M = 0$, if the following conditions are satisfied: (1) M is ω -reflexive, that is, the canonical Λ -homomorphism $M \rightarrow \text{Hom}_{\Lambda^{\text{op}}}(M^*, \omega)$ is an isomorphism and (2) $\text{Ext}_{\Lambda}^i(M, \omega) = 0 = \text{Ext}_{\Lambda^{\text{op}}}^i(M^*, \omega)$, for all $i > 0$, whereas $(-)^* = \text{Hom}_{\Lambda}(-, \omega)$.

It follows from the definition that any finitely generated projective Λ -module and also any module belonging to $\text{add}\omega$ have generalized Gorenstein dimension zero.

Here, by $\text{add}\omega$ we mean the subcategory of all direct summands of finite direct sums of copies of ω . We use $G_{\omega}(\Lambda)$ to denote the full subcategory of $\text{mod}\Lambda$ consisting of all modules with generalized Gorenstein dimension zero. It is easy to see that the class of all modules in $G_{\omega}(\Lambda)$ is closed under direct summands and finite direct sums. Moreover, according to [15, Lemma 2.4(1)], $G_{\omega}(\Lambda)$ is projectively resolving; that is, it contains projective modules, and in any short exact sequence of Λ -modules; $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ with $M'' \in G_{\omega}(\Lambda)$, we have $M \in G_{\omega}(\Lambda)$ if and only if $M' \in G_{\omega}(\Lambda)$.

Definition 2.3. Let M be a non-zero Λ -module. We write $G_{\omega} - \dim_{\Lambda} M = 0$, if it belongs to $G_{\omega}(\Lambda)$. It is clear that ω and Λ always belong to $G_{\omega}(\Lambda)$. We say that G_{ω} -dimension of M is $n \geq 0$, if n is the least integer for which there exists an exact sequence of Λ -modules;

$$0 \rightarrow G_n \rightarrow \cdots \rightarrow G_0 \rightarrow M \rightarrow 0,$$

where $G_i \in G_{\omega}(\Lambda)$, for any i . If there is no such a resolution of finite length, we set $G_{\omega} - \dim_{\Lambda} M = \infty$. By convention, $G_{\omega} - \dim_{\Lambda} 0 = -\infty$.

The following result will be used later.

Lemma 2.4. *Let M be a Λ -module. Then M is in $G_{\omega}(\Lambda)$ if and only if there exists an exact sequence of Λ -modules*

$$\mathbf{P}_{\bullet} : \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow \omega_{-1} \rightarrow \omega_{-2} \rightarrow \cdots,$$

where $\omega_i \in \text{add}\omega$ and P_j 's are projective Λ -modules, for each i, j , such that \mathbf{P}_{\bullet} remains exact after applying the functor $\text{Hom}_{\Lambda}(-, \omega)$, and $M = \text{Ker}(\omega_{-1} \rightarrow \omega_{-2})$.

Proof. Assume that $G_\omega - \dim_\Lambda M = 0$. Consider the following (finitely generated) projective resolution of the Λ^{op} -module $\text{Hom}_\Lambda(M, \omega)$,

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \text{Hom}_\Lambda(M, \omega) \longrightarrow 0.$$

As $\text{Ext}_{\Lambda^{\text{op}}}^i(M^*, \omega) = 0$ for any $i > 0$, applying the functor $\text{Hom}_{\Lambda^{\text{op}}}(-, \omega)$ on the above sequence, yields the following exact sequence of Λ -modules;

$$0 \rightarrow M \rightarrow \text{Hom}_{\Lambda^{\text{op}}}(P_0, \omega) \rightarrow \text{Hom}_{\Lambda^{\text{op}}}(P_1, \omega) \longrightarrow \text{Hom}_{\Lambda^{\text{op}}}(P_2, \omega) \rightarrow \cdots .$$

Therefore, by splicing this acyclic complex with the projective resolution of the Λ -module M , one obtains the desired acyclic complex. We should note that, it can be easily seen that this complex is $\text{Hom}_\Lambda(-, \omega)$ -exact. Conversely, assume that we are given an acyclic and $\text{Hom}_\Lambda(-, \omega)$ -exact complex \mathbf{P}_\bullet satisfying the mentioned conditions, and M is its zeroth syzygy. We want to prove that $G_\omega - \dim_\Lambda M = 0$. Let us first show that M is ω -reflexive. Since \mathbf{P}_\bullet is $\text{Hom}_\Lambda(-, \omega)$ -exact, hence $\text{Hom}_\Lambda(\mathbf{P}_\bullet, \omega) = \mathbf{P}_\bullet^*$ is acyclic and again reflexivity of each ω_i implies that $\text{Hom}_{\Lambda^{\text{op}}}(\mathbf{P}_\bullet^*, \omega) \cong \mathbf{P}_\bullet$. In particular, the complex $\text{Hom}_{\Lambda^{\text{op}}}(\mathbf{P}_\bullet^*, \omega)$ is exact; hence $\text{Hom}_{\Lambda^{\text{op}}}(M^*, \omega) \cong M$. Applying the functor $\text{Hom}_\Lambda(-, \omega)$ on the short exact sequence of Λ -modules; $0 \longrightarrow L \longrightarrow P_0 \longrightarrow M \longrightarrow 0$, yields the following exact sequence:

$$0 \longrightarrow \text{Hom}_\Lambda(M, \omega) \longrightarrow \text{Hom}_\Lambda(P_0, \omega) \longrightarrow \text{Hom}_\Lambda(L, \omega) \longrightarrow 0.$$

Thus, in view of the fact $H^i(\text{Hom}_\Lambda(\mathbf{P}_\bullet, \omega)) = 0$, we deduce that $\text{Ext}_\Lambda^i(M, \omega) = 0$, for all $i > 0$. Furthermore, by using the same argument, one deduces that $\text{Ext}_{\Lambda^{\text{op}}}^i(M^*, \omega) = 0$, for all $i > 0$. So the proof is complete. \square

2.5. For a subcategory \mathcal{X} of $\text{mod } \Lambda$, we let $\widehat{\mathcal{X}}$ denote the category whose objects are the modules M for which there exists an exact sequence of Λ -modules; $0 \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0$, where $X_i \in \mathcal{X}$ for all $0 \leq i \leq n$.

Proposition 2.6. *Let M be a Λ -module of finite G_ω -dimension. Then the following conditions are satisfied:*

- (1) *There exists a short exact sequence of Λ -modules; $0 \longrightarrow L \longrightarrow G \longrightarrow M \longrightarrow 0$, whenever $L \in \widehat{\text{add } \omega}$ and $G_\omega - \dim_\Lambda G = 0$.*
- (2) *There exists an exact sequence of Λ -modules; $0 \longrightarrow M \longrightarrow \omega' \longrightarrow G' \longrightarrow 0$, whenever $\omega' \in \widehat{\text{add } \omega}$ and $G_\omega - \dim_\Lambda G' = 0$.*

Proof. (1). This follows from the argument given in the proof of [15, Proposition 3.4].

(2). In view of part (1), there exists an exact sequence $0 \longrightarrow \omega_1 \longrightarrow G \longrightarrow M \longrightarrow 0$, where $\omega_1 \in \widehat{\text{add } \omega}$ and $G_\omega - \dim_\Lambda G = 0$. According

to Lemma 2.4, there exists an Λ -monomorphism $G \rightarrow \omega_0$ in which $\omega_0 \in \widehat{\text{add}}\omega$. Consider the following push-out diagram along the maps $G \rightarrow M$ and $G \rightarrow \omega_0$;

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & \widehat{\omega}_1 & \rightarrow & G & \rightarrow & M \rightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \rightarrow & \widehat{\omega}_1 & \rightarrow & \omega_0 & \rightarrow & L \rightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & G' & = & G' \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

Now, the exact sequence $0 \rightarrow M \rightarrow L \rightarrow G' \rightarrow 0$ would be the desired sequence, and the proof is complete. \square

As a consequence of the above proposition, we are able to state the following result.

Corollary 2.7. *Let I be an injective Λ -module. If I has finite G_ω -dimension, then $I \in \widehat{\text{add}}\omega$.*

Proof. According to the above proposition, there exists an exact sequence of Λ -modules; $0 \rightarrow I \rightarrow L \rightarrow G \rightarrow 0$, in which $L \in \widehat{\text{add}}\omega$ and $G \in G_\omega(\Lambda)$. Now, I being injective yields that this sequence is indeed split. So the result follows. \square

Definition 2.8. Let ω be a semi-dualizing bimodule for Λ . Following Enochs et al. [9], we say that ω is a dualizing bimodule, if it has finite injective dimension as both Λ and Λ^{op} -module.

Proposition 2.9. *Let ω be a dualizing bimodule for Λ . Then for any Λ -module M , $G_\omega - \dim_\Lambda M < \infty$.*

Proof. Assume that $\text{id}_\Lambda \omega \leq n$, for some positive integer n . Take the following exact sequence of Λ -modules;

$$0 \rightarrow \Omega^n M \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_0 \rightarrow M \rightarrow 0,$$

in which for any i , P_i is projective. As $P_i \in G_\omega(\Lambda)$, we only need to show that $\Omega^n M \in G_\omega(\Lambda)$. Since $\text{id}_\Lambda \omega \leq n$, one may obtain that $\text{Ext}_R^i(\Omega^n M, \omega) \cong \text{Ext}_R^{i+n}(M, \omega) = 0$, for any $i > 0$. So, applying the functor $\text{Hom}_\Lambda(-, \omega)$ to the projective resolution of $\Omega^n M$; $\dots \rightarrow Q_1 \rightarrow Q_0 \rightarrow \Omega^n M \rightarrow 0$, gives rise to the following exact sequence of Λ^{op} -modules;

$$0 \rightarrow \text{Hom}_\Lambda(\Omega^n M, \omega) \rightarrow \text{Hom}_\Lambda(Q_0, \omega) \rightarrow \text{Hom}_\Lambda(Q_1, \omega) \rightarrow \dots$$

As $\text{id}_{\Lambda^{\text{op}}}\omega < \infty$, by applying the functor $\text{Hom}_{\Lambda^{\text{op}}}(-, \omega)$ to the latter sequence, one obtains the following exact sequence of Λ -modules:

$$\cdots \rightarrow \text{Hom}_{\Lambda^{\text{op}}}(\text{Hom}_{\Lambda}(Q_0, \omega), \omega) \rightarrow \text{Hom}_{\Lambda^{\text{op}}}(\text{Hom}_{\Lambda}(\Omega^n M, \omega), \omega) \rightarrow 0,$$

implying the Λ -isomorphism $M \cong \text{Hom}_{\Lambda^{\text{op}}}(\text{Hom}_{\Lambda}(M, \omega), \omega)$. Moreover, it is clear that $\text{Ext}_{\Lambda^{\text{op}}}^i(\text{Hom}_{\Lambda}(\Omega^n M, \omega), \omega) = 0$, for all $i > 0$. Thus, $\Omega^n M \in \mathbf{G}_{\omega}(\Lambda)$, as needed. \square

2.10. Let M be a Λ -module. A Λ -homomorphism $f : G \rightarrow M$ with G is in $\mathbf{G}_{\omega}(\Lambda)$ is said to be a right $\mathbf{G}_{\omega}(\Lambda)$ -approximation, if the map $\text{Hom}_{\Lambda}(G', f) : \text{Hom}_{\Lambda}(G', G) \rightarrow \text{Hom}_{\Lambda}(G', M)$ is surjective, for any Λ -module $G' \in \mathbf{G}_{\omega}(\Lambda)$.

Corollary 2.11. *Let ω be a dualizing bimodule for Λ . Then any Λ -module admits a right $\mathbf{G}_{\omega}(\Lambda)$ -approximation.*

Proof. Assume that M is an arbitrary Λ -module. By virtue of Proposition 2.6 (1), there exists a short exact sequence of Λ -modules; $0 \rightarrow L \rightarrow G \rightarrow M \rightarrow 0$, whenever $L \in \widehat{\text{add}}\omega$ and $\mathbf{G}_{\omega}\text{-dim}_{\Lambda} G = 0$. Suppose that $N \in \mathbf{G}_{\omega}(\Lambda)$ is arbitrary. As $L \in \widehat{\text{add}}\omega$, we have $\text{Ext}_{\Lambda}^1(N, L) = 0$, implying that $\text{Hom}_{\Lambda}(N, G) \rightarrow \text{Hom}_{\Lambda}(N, M)$ is an epimorphism. This means that $G \rightarrow M$ is a left $\mathbf{G}_{\omega}(\Lambda)$ -approximation of M . So the proof is complete. \square

3. GENERALIZED GORENSTEIN DIMENSION OVER GROUP RINGS

This section is devoted to study the generalized Gorenstein dimension of modules over group rings. First, we show that having a dualizing bimodule ascends from R to $R\Gamma$, whenever R is a commutative noetherian local ring and Γ is a finite group. Our main result in this section indicates that, if M is a finitely generated $R\Gamma$ -module, then generalized Gorenstein dimensions of M over R and $R\Gamma$, are identical. We should point out that the notion of Gorenstein dimension of modules over group rings has been studied first in [2, 5].

3.1. One should observe that, since Γ is a finite group, hence $R\Gamma$ is a noetherian R -algebra. We also point out that, as $R\Gamma$ is isomorphic with the opposite ring $(R\Gamma)^{\text{op}}$, the distinction between left and right modules is redundant.

Proposition 3.2. *Let ω be a dualizing R -module and let Γ be a finite group. Then $R\Gamma \otimes_R \omega$ is a dualizing $R\Gamma$ -bimodule.*

Proof. Since $R\Gamma$ is a faithfully flat R -module, for any $i \geq 0$, one may have the following isomorphism:

$$\text{Ext}_{R\Gamma}^i(R\Gamma \otimes_R \omega, R\Gamma \otimes_R \omega) \cong R\Gamma \otimes_R \text{Ext}_R^i(\omega, \omega).$$

By the hypothesis, $\text{Ext}_R^i(\omega, \omega) = 0$ for any $i > 0$; implying that the right hand side of the above isomorphism will be zero. Consequently, using again the faithfully flatness of $R\Gamma$ over R , we conclude that

$$\text{Hom}_{R\Gamma}(R\Gamma \otimes_R \omega, R\Gamma \otimes_R \omega) \cong R\Gamma \otimes_R \text{Hom}_R(\omega, \omega) \cong R\Gamma \otimes_R R \cong R\Gamma.$$

So, $R\Gamma \otimes_R \omega$ is a semi-dualizing $R\Gamma$ -bimodule. Moreover, assume that $0 \rightarrow \omega \rightarrow I^0 \rightarrow \dots \rightarrow I^t \rightarrow 0$ is a (minimal) injective resolution of ω . Since $R\Gamma$ is a free R -module, applying the functor $R\Gamma \otimes_R -$ to this sequence gives rise to the following exact sequence of $R\Gamma$ -modules:

$$0 \rightarrow R\Gamma \otimes_R \omega \rightarrow R\Gamma \otimes_R I^0 \rightarrow \dots \rightarrow R\Gamma \otimes_R I^t \rightarrow 0.$$

Hence, as Γ is a finite group, by making use of [6, Proposition III.5.9], we infer that $R\Gamma \otimes_R I^i \cong \text{Hom}_R(R\Gamma, I^i)$ for any $0 \leq i \leq t$, and so $R\Gamma \otimes_R I^i$ is an injective $R\Gamma$ -module. This, in turn, means that $R\Gamma \otimes_R \omega$ has finite injective dimension. Thus, $R\Gamma \otimes_R \omega$ is a dualizing $R\Gamma$ -bimodule, as needed. \square

Example 3.3. (1) Let (R, \mathfrak{m}) be a complete Cohen-Macaulay local ring. According to [7, Corollary 3.3.8], R admits a canonical module, say ω . Clearly, ω is a dualizing R -module in our sense. So, by Proposition 3.2, $R\Gamma \otimes_R \omega$ is a dualizing $R\Gamma$ -bimodule.

(2) Assume that (R, \mathfrak{m}) is a Gorenstein local ring. So, R is a dualizing R -module. Thus, by the above proposition, $R\Gamma \otimes_R R \cong R\Gamma$ is a dualizing $R\Gamma$ -bimodule. In particular, $R\Gamma$ is a (not necessarily commutative) Gorenstein ring.

Lemma 3.4. *Let M be an R -module and Γ be a finite group. If $G_\omega - \dim_R M = 0$, then $G_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} R\Gamma \otimes_R M = 0$.*

Proof. Since $G_\omega - \dim_R M = 0$, in view of Lemma 2.4, there exists an acyclic complex of R -modules;

$$\mathbf{P}_\bullet : \dots \rightarrow P_1 \rightarrow P_0 \rightarrow \omega_{-1} \rightarrow \omega_{-2} \rightarrow \dots,$$

such that $M = \text{Coker}(P_1 \rightarrow P_0)$ and the the functor $\text{Hom}_R(-, \omega)$ leaves \mathbf{P}_\bullet exact. According to the fact that $R\Gamma$ is a (finitely generated) free R -module, one may obtain the following acyclic complex of $R\Gamma$ -modules;

$$\mathbf{P}_\bullet : \dots \rightarrow R\Gamma \otimes_R P_1 \rightarrow R\Gamma \otimes_R P_0 \rightarrow R\Gamma \otimes_R \omega_{-1} \rightarrow R\Gamma \otimes_R \omega_{-2} \rightarrow \dots,$$

in which $R\Gamma \otimes_R P_i$ is projective and $R\Gamma \otimes_R \omega_i \in \text{add}(R\Gamma \otimes_R \omega)$. Moreover, we have the following isomorphism of $R\Gamma$ -modules;

$$\text{Hom}_{R\Gamma}(R\Gamma \otimes_R \mathbf{P}_\bullet, R\Gamma \otimes_R \omega) \cong R\Gamma \otimes_R \text{Hom}_R(\mathbf{P}_\bullet, \omega).$$

Since, by the hypothesis, $\text{Hom}_R(\mathbf{P}_\bullet, \omega)$ is acyclic, the same is true for $R\Gamma \otimes_R \text{Hom}_R(\mathbf{P}_\bullet, \omega)$, implying that the right hand side of the above isomorphism is acyclic. So, the proof is complete. \square

Theorem 3.5. *Let Γ be a finite group and M be an $R\Gamma$ -module. Then $G_\omega - \dim_R M = 0$ if and only if $G_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} M = 0$.*

Proof. Let us first prove the ‘sufficiency’. Since $G_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} M = 0$, in view of Lemma 2.4, there exists an exact sequence of Λ -modules

$$\mathbf{P}_\bullet : \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \omega_{-1} \longrightarrow \omega_{-2} \longrightarrow \cdots,$$

where each ω_i belongs to $\text{add}\omega$ and all P_j s are projective Λ -module, such that \mathbf{P}_\bullet remains exact after applying the functor $\text{Hom}_\Lambda(-, \omega)$, and $M = \text{Ker}(\omega_{-1} \longrightarrow \omega_{-2})$. As $R\Gamma$ is a finitely generated free R -module, we get that any projective $R\Gamma$ -module is also projective as an R -module, and any module in $\text{add}(R\Gamma \otimes_R \omega)$ also belongs to $\text{add}(\omega)$. So, it remains to show that the complex $\text{Hom}_R(\mathbf{P}_\bullet, \omega)$ is exact. To see this, consider the following isomorphisms;

$$\begin{aligned} \text{Hom}_R(\mathbf{P}_\bullet, \omega) &\cong \text{Hom}_R(R\Gamma \otimes_{R\Gamma} \mathbf{P}_\bullet, \omega) \\ &\cong \text{Hom}_{R\Gamma}(\mathbf{P}_\bullet, \text{Hom}_R(R\Gamma, \omega)) \\ &\cong \text{Hom}_{R\Gamma}(\mathbf{P}_\bullet, R\Gamma \otimes_R \omega). \end{aligned}$$

We should point out that the first isomorphism holds trivially, the second one is the adjointness of Hom and \otimes , and the validity of the last isomorphism comes from [6, Proposition III.5.9]. By our assumption, the last complex is acyclic, implying that the same is true for the complex $\text{Hom}_R(\mathbf{P}_\bullet, \omega)$, as needed. Next, we want to prove the ‘necessity’. Assume that $G_\omega - \dim_R M = 0$. In view of Lemma 2.4, there exists a short exact sequence of R -modules $0 \longrightarrow M \longrightarrow \omega_{-1} \longrightarrow L \longrightarrow 0$, in which $G_\omega - \dim_R L = 0$ and $\omega_{-1} \in \text{add}(\omega)$. Consider the following commutative diagram of $R\Gamma$ -modules with exact rows;

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & R\Gamma \otimes_R M & \longrightarrow & L' \longrightarrow 0 \\ & & \text{id} \downarrow & & \theta \downarrow & & \varphi \downarrow \\ 0 & \longrightarrow & M & \longrightarrow & R\Gamma \otimes_R \omega_{-1} & \longrightarrow & L'' \longrightarrow 0. \end{array}$$

Clearly, φ is a monomorphism and $\text{Coker}\varphi = R\Gamma \otimes_R L$. It is known that the exact sequence $0 \longrightarrow M \longrightarrow R\Gamma \otimes_R M \longrightarrow L' \longrightarrow 0$ splits over R . Thus, $L' \cong \bigoplus_{i=1}^{t-1} M$ as R -modules, where $t = |\Gamma|$, the order of Γ . Consequently, the hypothesis made on M implies that $G_\omega - \dim_R L' = 0$. On the other hand, since $G_\omega - \dim_R L = 0$, it is not hard to see that $G_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma}(R\Gamma \otimes_R L) = 0$ and so $G_\omega - \dim_R(R\Gamma \otimes_R L) = 0$, thanks to the fact that $R\Gamma \otimes_R L \cong \bigoplus_{i=1}^t L$, as R -modules. We claim that $G_\omega - \dim_R L'' = 0$. The exact sequence

$$0 \longrightarrow R\Gamma \otimes_R M \longrightarrow R\Gamma \otimes_R \omega_{-1} \longrightarrow R\Gamma \otimes_R L \longrightarrow 0,$$

splits over R as follows:

$$0 \longrightarrow \bigoplus_{i=1}^t M \longrightarrow \bigoplus_{i=1}^t \omega_{-1} \longrightarrow \bigoplus_{i=1}^t L \longrightarrow 0. \quad (1)$$

Similarly, the short exact sequence $0 \longrightarrow L' \longrightarrow L'' \longrightarrow R\Gamma \otimes_R L \longrightarrow 0$, which splits over R , can be written as follows, when viewed as an exact sequence of R -modules;

$$0 \longrightarrow \bigoplus_{i=1}^{t-1} M \longrightarrow L'' \longrightarrow \bigoplus_{i=1}^t L \longrightarrow 0. \quad (2)$$

Considering the exact sequences (1) and (2), we conclude that $L'' \cong \bigoplus_{i=1}^{t-1} \omega_{-1} \oplus L$, as R -modules. Hence, $G_\omega - \dim_R L'' = 0$. In fact, we obtain the short exact sequence $0 \longrightarrow M \longrightarrow R\Gamma \otimes_R \omega_{-1} \longrightarrow L'' \longrightarrow 0$, in which $G_\omega - \dim_R L'' = 0$. By repeating this manner, one gets the following exact sequence of $R\Gamma$ -modules

$$\omega_\bullet : 0 \longrightarrow R\Gamma \otimes_R \omega_{-1} \longrightarrow R\Gamma \otimes_R \omega_{-2} \longrightarrow \cdots,$$

where $\omega_i \in \text{add}(\omega)$ for all i , and every cosyzygy of ω_\bullet belongs to $G_\omega(R)$. Now, we show that ω_\bullet is $\text{Hom}_{R\Gamma}(-, R\Gamma \otimes_R \omega)$ -exact. Consider the following isomorphisms:

$$\text{Hom}_{R\Gamma}(\omega_\bullet, R\Gamma \otimes_R \omega) \cong \text{Hom}_R(R\Gamma \otimes_{R\Gamma} \omega_\bullet, \omega) \cong \text{Hom}_R(\omega_\bullet, \omega).$$

By making use of the fact that every cosyzygy of ω_\bullet belongs to $G_\omega(R)$, we conclude that the right hand side is exact, which gives the claim. Hence, in order to complete the proof, it suffices to find a left resolution of modules in $\text{add}((R\Gamma \otimes_R \omega) \oplus R\Gamma)$ for M , which is $\text{Hom}_{R\Gamma}(-, R\Gamma \otimes_R \omega)$ -exact. Take the following projective resolution of $R\Gamma$ -module M ;

$$\mathbf{P}_\bullet : \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where each P_i is finitely generated. By replacing \mathbf{P}_\bullet with ω_\bullet in the above isomorphisms and invoking the fact that $G_\omega - \dim_R M = 0$, one deduces that $\text{Hom}_{R\Gamma}(-, R\Gamma \otimes_R \omega)$ leaves the sequence \mathbf{P}_\bullet exact. Now, splicing ω_\bullet and \mathbf{P}_\bullet and Lemma 2.4 completes the proof. \square

The above theorem immediately gives rise to the following result.

Corollary 3.6. *Let R be a commutative ring with a semi-dualizing module ω . Let Γ be a finite group and M be an $R\Gamma$ -module. Then $G_\omega - \dim_R M = G_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} M$.*

Proof. We first show that $G_\omega - \dim_R M \leq G_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} M$. If $G_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} M = \infty$, there is nothing to prove. So, assume that $G_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} M$ is finite, say t . Hence, there exists an exact sequence of $R\Gamma$ -modules; $0 \longrightarrow G_t \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$, whenever $G_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} G_i = 0$ for any i . According to Theorem 3.5, $G_\omega - \dim_R G_i = 0$, implying that $G_\omega - \dim_R M \leq t$. Next, we want to show the reverse inequality. To that end, we may assume that $G_\omega -$

$\dim_R M = n$, for some non-negative integer n . Take an exact sequence of $R\Gamma$ -modules; $0 \rightarrow \Omega^n M \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$, where each P_i is projective. Since $R\Gamma$ is a free R -module, any P_i is projective over R . Hence, by the hypothesis, we get $G_\omega - \dim_R \Omega^n M = 0$. Using again of Theorem 3.5, gives rise to the equality $G_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} \Omega^n M = 0$. Consequently, $G_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} M \leq n$, and the proof is complete. \square

Let Γ be a finite group and ω be a semi-dualizing R -module. We denote by \mathcal{X} (resp., \mathcal{Y}) the subcategory of $\mathbf{mod} R$ (resp., $\mathbf{mod} R\Gamma$) consisting of all modules of finite G_ω -dimension (resp., $G_{R\Gamma \otimes_R \omega}$ -dimension).

Proposition 3.7. *With the above notations, one has the equality*

$$\sup\{G_\omega - \dim_R M \mid M \in \mathcal{X}\} = \sup\{G_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} M \mid M \in \mathcal{Y}\}.$$

Proof. Let us first show the inequality

$$\sup\{G_\omega - \dim_R M \mid M \in \mathcal{X}\} \leq \sup\{G_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} M \mid M \in \mathcal{Y}\}.$$

If $\sup\{G_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} M \mid M \in \mathcal{Y}\} = \infty$, there is nothing to prove. So, assume that $\sup\{G_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} M \mid M \in \mathcal{Y}\}$ exists finite, say t . Taking an arbitrary object $X \in \mathcal{X}$, there is an exact sequence of R -modules; $0 \rightarrow G_n \rightarrow \cdots \rightarrow G_0 \rightarrow X \rightarrow 0$ in which for any i , $G_\omega - \dim_R G_i = 0$. Apply the functor $R\Gamma \otimes_R -$ to this, in order to obtain the exact sequence of $R\Gamma$ -modules; $0 \rightarrow R\Gamma \otimes_R G_n \rightarrow \cdots \rightarrow R\Gamma \otimes_R G_0 \rightarrow R\Gamma \otimes_R X \rightarrow 0$. In view of Lemma 3.4, $G_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} R\Gamma \otimes_R G_i = 0$, implying that $R\Gamma \otimes_R X \in \mathcal{Y}$. Hence, our assumption yields that $G_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} R\Gamma \otimes_R X \leq t$, and so by making use of Corollary 3.6, we infer that $G_\omega - \dim_R R\Gamma \otimes_R X \leq t$ as well. Now, X being an R -direct summand of $R\Gamma \otimes_R X$ implies that $G_\omega - \dim_R X \leq t$. Consequently, $\sup\{G_\omega - \dim_R M \mid M \in \mathcal{X}\} \leq t$. Next, we will show that $\sup\{G_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} M \mid M \in \mathcal{Y}\} \leq \sup\{G_\omega - \dim_R M \mid M \in \mathcal{X}\}$. To do this, we may assume that $\sup\{G_\omega - \dim_R M \mid M \in \mathcal{X}\} = s < \infty$. Take an arbitrary $R\Gamma$ -module M with finite $G_{R\Gamma \otimes_R \omega}$ -dimension. According to Corollary 3.6, we have that $G_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} M = G_\omega - \dim_R M \leq s$. This, in particular, gives the desired result. \square

Example 3.8. Let (R, \mathfrak{m}) be a complete Cohen-Macaulay local ring with canonical module ω . According to Proposition 3.2, $R\Gamma \otimes_R \omega$ is a dualizing $R\Gamma$ -bimodule. Now, by making use of Proposition 2.9 in conjunction with Corollary 3.6, one may deduce that for any finitely generated $R\Gamma$ -module M , $G_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} M \leq n$, where $n = \text{id}_R \omega$.

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REFERENCES

1. T. Araya, R. Takahashi and Y. Yoshino, Homological invariants associated to semi-dualizing bimodules, *J. Math. Kyoto Univ.* **45**(2) (2005), 287–306.
2. J. Asadollahi, A. Bahlekeh, and Sh. Salarian, On the hierarchy of cohomological dimensions of groups, *J. Pure Appl. Algebra* **213** (2009), 1795–1803.
3. M. Auslander and M. Bridger, *Stable module theory*, American Mathematical Soc., 1969.
4. M. Auslander and I. Reiten, Cohen-Macaulay and Gorenstein artin algebras. *Representation theory of finite groups and finite-dimensional algebras*, (1991), 221–245.
5. A. Bahlekeh, F. Dembegiotti and O. Talelli, Gorenstein dimension and proper actions, *Bull. London Math. Soc.* **41** (2009), 859–871.
6. K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, Vol. 37, Berlin, Heidelberg, New York, Springer 1982.
7. W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, *Cambridge Stud. Adv. Math.*, Vol. 39, Cambridge Univ. Press, 1993.
8. L. W. Christensen, Semi-dualizing complexes and their Auslander categories, *Trans. Amer. Math. Soc.* **353** (2001), 1839–1883.
9. E. E. Enochs, O. M. G. Jenda and J. A. López-Ramos, Dualizing modules and n -perfect rings, *Proc. Edin. Math. Soc.* **48** (2005), 75–90.
10. H. B. Foxby, Gorenstein modules and related modules, *Math. Scand.* **31** (1972), 267–284.
11. A. A. Gerko, On the structure of the set of semi-dualizing complexes *Illinois J. Math.* **48** (2004), 965–976.
12. E. S. Golod, G-dimension and generalized perfect ideals, *Trudy Mat. Inst. Steklov.* **165** (1984), 62–66.
13. R. Hartshorne, Local cohomology, A seminar given by A. Grothendieck, Harvard University, Fall, 1961, Lecture Notes in Mathematics, Springer-Verlag, Berlin, **41** 1967.
14. H. Holm and D. White, Foxby equivalences over associative rings, *J. Math. Kyoto Univ.* **47** (2007), 781–808.
15. Z. Huang, Selforthogonal modules with finite injective dimension III, *Algebra Represent Theor.* **12**(2) (2009), 371–384.
16. W. V. Vasconcelos, *Divisor theory in module categories*, North-Holland Publishing Co., Amsterdam, 1974.
17. S. Wagstaff, Semi-dualizing modules and divisor class group, *Illinois J. Math.* **51**(1) (2007), 255–285.
18. D. White, Gorenstein projective dimension with respect to a semidualizing module, *J. Comm. Algebra* **2** (2010), 111–137.

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GENERALIZED GORENSTEIN DIMENSION OVER GROUP RINGS

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فرض کنید (R, \mathfrak{m}) یک حلقه‌ی جابجایی نوتری و موضعی بوده و Γ یک گروه متناهی باشد. ثابت می‌کنیم که اگر R دارای مدول دوگانی باشد، آنگاه حلقه گروه $R\Gamma$ نیز دارای دومدول دوگانی می‌باشد. به‌علاوه، ثابت می‌کنیم که $R\Gamma$ -مدول با تولید متناهی M دارای بعد گرنشتاین تعمیم یافته‌ی صفر است، اگر و تنها اگر بعد گرنشتاین تعمیم یافته‌ی M به عنوان R -مدول، برابر با صفر باشد. کلمات کلیدی: دومدول‌های نیم‌دوگانی، بعد گرنشتاین تعمیم یافته، حلقه گروه‌ها.