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REES SHORT EXACT SEQUENCES OF S-POSETS

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ABSTRACT. In this paper, the notion of Rees short exact sequence for S-posets is introduced, and we investigate the conditions for which these sequences are left or right split. Unlike the case for S-acts, being right split does not imply left split. Furthermore, we present equivalent conditions of a right S-poset P for the functor $\mathrm{Hom}(P, -)$ to be exact.

1. INTRODUCTION

A monoid S is said to be a *pomonoid* if it is a poset whose partial order $\leq$ is compatible with the binary operation of S. A *right S-poset* A_S, is a right S-act A equipped with a partial order $\leq$ and in addition, for all $s, t \in S$ and $a, b \in A$, if $s \leq t$ then $as \leq at$, and if $a \leq b$ then $as \leq bs$. An *S-subposet* of a right S-poset A, is a subset of A that is closed under the S-action. The one element S-poset is denoted by $\Theta = \{\theta\}$. Moreover, S-morphisms are the functions that preserve both the action and the order. An *S-morphism* $\iota : A \longrightarrow B$ is a *regular monomorphism* if and only if it is an *order-embedding*, i.e.,

$$a \leq a' \Leftrightarrow \iota(a) \leq \iota(a'),$$

for all $a, a' \in A$. An *S-isomorphism* is an S-morphism which is both regular monomorphism and epimorphism. A surjective order embedding of posets is called an *order isomorphism*.

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As in the unordered case, the coproduct in S-posets is simply the disjoint union, with S-action and order given componentwise, and as usual the coproduct of a family $\{A_i, i \in I\}$ will be denoted by $\coprod_{i\in I} A_i$.

An S-poset P is called *projective* if for any S-epimorphism $\pi : A \longrightarrow B$ and any S-morphism $f : P \longrightarrow B$, there exists an S-morphism $\phi : P \longrightarrow A$ such that $\pi\phi = f$, i.e., $Hom(P, -)$ preserves epimorphisms. One can find the following characterization for projective S-posets in [5].

Lemma 1.1. *An S-poset P is projective if and only if $P = \coprod_{i\in I} e_i S$ where $e_i^2 = e_i \in S,\ i \in I$.*

Short exact sequences of modules have been investigated in many papers. In [3], projective S-acts and exact sequences in S-acts are introduced. Thereby, in [4] Rees short exact sequence of S-acts is studied. In this paper, we introduce Rees short exact sequences of S-posets. In Section 2, we study general properties of Rees short exact sequences. In Section 3, we give some conditions on a Rees short exact sequence to be left or right split. Projectivity of S-posets was investigated in some papers such as [2] and [5]. Now, a question is that whether $Hom(P, -)$ can be an exact functor if P is a projective S-poset. In Section 4, we obtain a characterization for $Hom(P, -)$ to be exact, and give a negative answer to this question.

Let A be a right S-poset. An S-poset congruence θ on A is a right S-act congruence with the property that the S-act A/θ can be made into an S-poset in such a way that the natural map $A \longrightarrow A/\theta$ is an S-poset map. For an S-act congruence θ on A, we write $a \leq_\theta a'$, if the so-called θ-chain

$$a \leq a_1\theta b_1 \leq a_2\theta b_2 \leq\ \dots\ \leq a_n\theta b_n \leq a',$$

from a to a' exists in A, where $a_i, b_i \in A,\ 1 \leq i \leq n$. It can be shown that an S-act congruence θ on a right S-poset A is an S-poset congruence if and only if $a\theta a'$ whenever $a \leq_\theta a' \leq_\theta a$. Let $H \subseteq A \times A$. The relation $\nu(H)$ is called the *S-poset congruence on A_S induced by H*, the following method of constructing $\nu(H)$ by means of an auxiliary relation $\alpha(H)$ was given in [1].

Indeed, $a \leq_{\alpha(H)} b$ if and only if $a \leq b$ or there exist $n \geq 1, (c_i, d_i) \in H, s_i \in S, 1 \leq i \leq n$ such that

$$a \leq c_1 s_1\ d_1 s_1 \leq c_2 s_2\ \dots\ d_n s_n \leq b.$$

The relation $\nu(H)$ given by $a\ \nu(H)\ b$ if and only if $a \leq_{\alpha(H)} b \leq_{\alpha(H)} a$.

A subposet B of A is called *convex* if $B = [B]$, where

$$[B] = \{a \in A \mid \exists\ b, b' \in B,\ b \leq a \leq b'\}.$$

Let K be a proper convex right ideal of a pomonoid S. We define now the congruence ρ_K on S such that $s\rho_K t$ if $s = t$ or $s, t \in K$. The quotient S/ρ_K is called the Rees factor and denoted by S/K. Then, $[s]_{\rho_K} \leq [t]_{\rho_K}$ if and only if $s \leq t$, or there exists $k,\ k' \in K$ such that $s \leq k$ and $k' \leq t$. In general, for any convex S-subposet B of A, the quotient A/ρ_B is the Rees factor, and it is usually denoted by A/B.

2. Rees short exact sequences

In this section, we introduce Rees short exact sequences of S-posets, and study some general properties of them. First, we need some preliminaries.

Let $f : A \longrightarrow B$ be an S-epimorphism. The subkernel of an S-poset morphism f is defined by

$$\overrightarrow{ker} f := \{(a, a') \in A \times A : f(a) \leq f(a')\}.$$

Then, $\nu(\overrightarrow{ker} f) = ker f := \{(a, a') \in A \times A : f(a) = f(a')\}$, and in $A_S/ker f$,

$$[a]_{kerf} \leq [a']_{kerf} \quad \text{if and only if} \quad f(a) \leq f(a').$$

Moreover, the mapping $\overline{f} : A_S/ker f \longrightarrow B_S$ defined by $\overline{f}([a]_{kerf}) = f(a)$ for $a \in A$ is an S-isomorphism. For more information, see [1]. From now onwards we denote the subkernel of f briefly by $\mathcal{K}_f$. Obviously, f is a regular monomorphism if and only if

$$\mathcal{K}_f = \xi_A = \{(a, a') \in A \times A|\ a \leq a'\}.$$

Let $f : A \longrightarrow B$ be an S-morphism. We have $f(A) = \{f(a) :\ a \in A\}$ and $f(A)$ is an S-subposet of B_S. Let $\theta_f = (f(A) \times f(A)) \cup \Delta_B$, where $\Delta_B = \{(b, b) :\ \ b \in B\}$. One can verify that θ_f is a Rees S-act congruence on B_S. We have $b \leq_{\theta_f} b'$ if there exists a θ-chain

$$b \leq b_1\theta c_1 \leq b_2\theta c_2\ ...\ \leq b_n\theta c_n \leq b',$$

where $b_i, c_i \in B,\ 1 \leq i \leq n$. It is easy to prove $b \leq_{\theta_f} b'$ if and only if $b \leq b'$ or $b \leq f(a),\ f(a') \leq b'$ for $a, a' \in A$. We define

$$\mathcal{I}_f = \{(b, b') \in B \times B :\ b \leq_{\theta_f} b'\},$$

and $\mathcal{I}m_f = \mathcal{I}_f \cap (\mathcal{I}_f)^{op} = ([f(A)] \times [f(A)]) \cup \Delta_B$, where $[f(A)]$ is the convex closure of $f(A)$. Clearly, $\mathcal{I}m_f$ is an S-poset congruence, and $[b]_{\mathcal{I}m_f} \leq [b']_{\mathcal{I}m_f}$ if and only if $(b, b') \in \mathcal{I}_f$.

Definition 2.1. *Suppose that A, B, C are posets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ are order preserving maps. Then the sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact at B *if* $\mathcal{I}_f = \mathcal{K}_g$*, i.e.,* $\leq_{\mathcal{I}m_f} = \leq_{kerg}$ *and it is called a short exact sequence. If* g *is surjective,* f *is a regular monomorphism, and* $\mathcal{I}_f = \mathcal{K}_g$*.*

The following lemma will be useful in the sequel.

Lemma 2.2. *Suppose that* A, B, C, L, M, N *are posets, and* f, g, h, r *are order preserving maps. Let the following diagram be commutative:*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ L & \xrightarrow{h} & M & \xrightarrow{r} & N \end{array}$$

where α, β, γ *are order isomorphisms. Then the followings hold:*

(i) *The upper row is exact at* B *if and only if the lower row is exact at* M*.*
(ii) *The upper row is a short exact sequence if and only if the lower row is a short exact sequence.*

Proof. (i). Suppose that the upper row is exact at B. So, $\mathcal{I}_f = \mathcal{K}_g$. We will show that $\mathcal{I}_h = \mathcal{K}_r$. Suppose that $(m, m') \in \mathcal{I}_h$ and $m \nleq m'$. Then, $m \leq h(l)$, $h(l') \leq m'$ for $l, l' \in L$. So, $l = \alpha(a), l' = \alpha(a')$ for $a, a' \in A$. Since $h\alpha = \beta f$, $m \leq \beta(f(a))$. We have $m = \beta(b)$ for $b \in B$, β is an order embedding, which implies that $b \leq f(a)$. Similarly, $m' = \beta(b')$ and $f(a') \leq b'$. Thus, $(b, b') \in \mathcal{I}_f$, and since $\mathcal{I}_f = \mathcal{K}_g$, $g(b) \leq g(b')$. Therefore,

$$r(m) = r\beta(b) = \gamma g(b) \leq \gamma g(b') = r\beta(b') = r(m'),$$

and $(m, m') \in \mathcal{K}_r$. Hence, $\mathcal{I}_h \subseteq \mathcal{K}_r$. Conversely, Suppose that $r(m) \leq r(m')$ and $m \nleq m'$. Let $m = \beta(b)$ and $m' = \beta(b')$ for $b, b' \in B$. Then,

$$\gamma g(b) = r\beta(b) = r(m) \leq r(m') = r\beta(b') = \gamma g(b'),$$

and so $g(b) \leq g(b')$. Since $\mathcal{I}_f = \mathcal{K}_g$, $(b, b') \in \mathcal{I}_f$. Thus, there exist $a, a' \in A$ such that $b \leq f(a), f(a') \leq b'$. So $\beta(b) \leq \beta(f(a))$, $\beta(f(a')) \leq \beta(b')$, and then $m \leq \beta(f(a)) = h(\alpha(a))$ and $h(\alpha(a')) = \beta(f(a')) \leq m'$. Therefore, $(m, m') \in \mathcal{I}_h$, as required.

Similarly, we can show that, if the lower row of the diagram is exact at M, then the upper row is also exact at B.

(ii). Suppose that the upper row is a short exact sequence. By (i), the lower row of the diagram is exact at M. First, we show that h is a regular monomorphism. Suppose that $h(l) \leq h(l')$ for $l, l' \in L$. Since α is surjective, there exist $a, a' \in A$ such that $l = \alpha(a)$, $l' = \alpha(a')$. So, $\beta f(a) = h\alpha(a) = h(l) \leq h(l') = h\alpha(a') = \beta f(a')$. Now, since β and f are regular monomorphisms, we get $a \leq a'$ and h is a regular monomorphism. Moreover, since the diagram is commutative, the surjectivity of γ is easily follows. $\square$

Definition 2.3. *Let A, B, C be S-posets. Then the sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

is called a Rees short exact sequence if it is a short exact sequence and f, g are also S-morphisms.

Example 2.4. *(i). Let K be a proper convex right ideal of a pomonoid S. Then*

$$K \xrightarrow{\iota_K} S \xrightarrow{\pi} S/K,$$

is a Rees short exact sequence.

(ii). Let K be a proper right ideal of a pomonoid S. Then

$$K \xrightarrow{\iota_K} S \xrightarrow{\pi} S/[K],$$

is a Rees short exact sequence.

(iii). Let B be a proper S-subposet of an S-poset A_S. Then

$$B \xrightarrow{\iota_B} A \xrightarrow{\pi} A/[B],$$

is a Rees short exact sequence.

(iv). Let A, B be S-posets. Then the sequence

$$A \xrightarrow{\iota_A} A \coprod B \coprod \Theta \xrightarrow{\rho} B \coprod \Theta,$$

is a Rees short exact sequence, where $\rho(c) = c$ if $c \in B$, otherwise $\rho(c) = \theta$. Clearly, $\mathcal{I}_{\iota_A} = \mathcal{K}_\rho = (A \times A) \cup \xi_{A \coprod B \coprod \Theta}$.

(v). Let A, B be S-posets and B contains a zero 0. Consider the sequence

$$A \xrightarrow{\iota} A \prod B \xrightarrow{\pi} B,$$

where $\iota(a) = (a, 0)$, $\pi(a, b) = b$. In general, this sequence is not exact. Indeed

$$\mathcal{I}_\iota = \{((a, b), (a', b'))|\ a, a' \in A, b \leq 0 \leq b'\} \cup \xi_{A \prod B},$$

and $\mathcal{K}_\pi = \{((a, b), (a', b'))|\ a, a' \in A, b \leq b'\}$. Therefore, it is easily checked that the sequence is a Rees short exact sequence if and only if $A = \Theta$ or $B = \Theta$.

In the following lemma, we give an evident result about Rees short exact sequences.

Lemma 2.5. *Let A, B, C be S-posets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be S-morphisms. Then the followings hold:*

(i) *If $\mathcal{I}_f = \mathcal{K}_g$, then $\mathcal{I}m_f = ker g$.*

(ii) *The sequence* $B \xrightarrow{g} C \xrightarrow{\pi} \Theta$ *is exact at* C *if and only if* g *is an epimorphism.*

(iii) *If* A *contains a zero, then* f *is a regular monomorphism if and only if the sequence* $\Theta \xrightarrow{\iota} A \xrightarrow{f} B$ *is exact at* B.

Lemma 2.5 is not valid for S-posets. For instance, let $S = \{0, 1, s, s^2, \ldots\}$ be a free word monoid equipped with the order $0 < 1 < s < s^2 < \ldots$ and $K = sS = \{0, s, s^2, \ldots\}$. So, $[K] = S$. Let $L = s^2S \subseteq M = sS$, then $M/[L] = \Theta$. Define $\alpha : K \longrightarrow L$ and $\beta : S \longrightarrow M$ by $\alpha(s^i) = s^{i+1}$ and $\beta(1) = s^2$, also $h : L \longrightarrow M$ by $h(s^i) = s^{i+1}$. Then, we have the following commutative diagram:

$$\begin{array}{ccccc} K & \xrightarrow{\iota_K} & S & \xrightarrow{\pi} & \Theta \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\iota} \\ L & \xrightarrow{h} & M & \xrightarrow{\pi} & \Theta. \end{array}$$

As we mentioned, the upper and lower rows are Rees short exact sequences. It is clear that α and ι are epimorphisms but not β.

Suppose that A, B are S-posets, and $f : A \longrightarrow B$ is an S-morphism. Clearly, $\mathcal{K}_f$ is an S-subposet of $A \prod A$. Now, we close this section with a slight type of Snake Lemma for S-posets.

Lemma 2.6. *Suppose that* A, B, C, L, M, N *are* S*-posets, and* $f, g, h, r, \alpha, \beta, \gamma$ *are* S*-morphisms. Let the following diagram be commutative:*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ L & \xrightarrow{h} & M & \xrightarrow{r} & N \end{array}$$

where the upper and lower rows are exact at B *and* M*, respectively, and for each* $b_1 \leq b_2 \in B$ *there exists* $a \in A$ *such that* $b_1 \leq f(a) \leq b_2$*. Then, there exists a sequence* $\mathcal{K}_\alpha \longrightarrow \mathcal{K}_\beta \longrightarrow \mathcal{K}_\gamma$ *which is exact at* $\mathcal{K}_\beta$.

Proof. Define $\sigma : \mathcal{K}_\alpha \longrightarrow \mathcal{K}_\beta$ and $\tau : \mathcal{K}_\beta \longrightarrow \mathcal{K}_\gamma$ by $\sigma(a_1, a_2) = (f(a_1), f(a_2))$ and $\tau(b_1, b_2) = (g(b_1), g(b_2))$. It can be easily checked that σ and τ are S-morphisms. We will show that $\mathcal{K}_\tau = \mathcal{I}_\sigma$. Suppose that $((b_1, b'_1), (b_2, b'_2)) \in \mathcal{K}_\tau$. So

$$\tau(b_1, b'_1) \leq \tau(b_2, b'_2), \quad \text{i.e.,} \quad (g(b_1), g(b'_1)) \leq (g(b_2), g(b'_2)).$$

Hence, (b_1, b_2), $(b'_1, b'_2) \in \mathcal{K}_g = \mathcal{I}_f$. We have the following four cases:

(i) If $b_1 \leq b_2$, $b'_1 \leq b'_2$, obviously $((b_1, b'_1), (b_2, b'_2)) \in \mathcal{I}_\sigma$.

(ii) If $b_1 \leq f(a_1),\ f(a_2) \leq b_2,\ b'_1 \leq f(a'_1),\ f(a'_2) \leq b'_2$ for some $a_1, a_2, a'_1, a'_2 \in A$, then

$$(b_1, b'_1) \leq \sigma(a_1, a'_1),\ \sigma(a_2, a'_2) \leq (b_2, b'_2),$$

and therefore $((b_1, b'_1), (b_2, b'_2)) \in \mathcal{I}_\sigma$.

(iii) If $b_1 \leq f(a_1),\ f(a_2) \leq b_2,\ b'_1 \leq b'_2$ for some $a_1, a_2 \in A$, by assumption there exists $a' \in A$ such that $b'_1 \leq f(a') \leq b'_2$. Then, $(b_1, b'_1) \leq \sigma(a_1, a'),\ \sigma(a_2, a') \leq (b_2, b'_2)$, and therefore $((b_1, b'_1), (b_2, b'_2)) \in \mathcal{I}_\sigma$.

(iv) If $b_1 \leq b_2,\ b'_1 \leq f(a'_1),\ f(a'_2) \leq b'_2$ for some $a'_1, a'_2 \in A$, the result can be obtained by a similar argument as in the previous part.

Therefore, $\mathcal{K}_\tau \subseteq \mathcal{I}_\sigma$. Similarly, one can shows that $\mathcal{I}_\sigma \subseteq \mathcal{K}_\tau$. □

3. Split Rees short exact sequences

In this section, the conditions under which a Rees short exact sequence is right or left split are given. Moreover, we give examples which illustrate that right split and left split does not imply each other.

Definition 3.1. *Let A, B, C be S-posets. The Rees short exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is called left (resp., right) split if there exists an S-morphism $f' : B \longrightarrow A$ (resp., $g' : C \longrightarrow B$) such that $f'f = 1_A$ (resp., $gg' = 1_C$), where 1_A is the identity map on A.*

As an example of this definition, let A, B be S-posets. Then, the sequence

$$A \coprod \Theta \xrightarrow{\iota_{A\coprod\Theta}} A \coprod B \coprod \Theta \xrightarrow{\rho_{B\coprod\Theta}} B \coprod \Theta,$$

is right and left split by

$$\rho_{A\coprod\Theta} : A \coprod B \coprod \Theta \longrightarrow A \coprod \Theta,$$

and

$$\iota_{B\coprod\Theta} : B \coprod \Theta \longrightarrow A \coprod B \coprod \Theta,$$

with $\rho_{A\coprod\Theta}\iota_{A\coprod\Theta} = 1_{A\coprod\Theta}$ and $\rho_{B\coprod\Theta}\iota_{B\coprod\Theta} = 1_{B\coprod\Theta}$.

As a direct consequence of Definition 3.1, we have:

Corollary 3.2. *Let E, A, B, P be right S-posets. Then*

(i) *If E is a regular injective S-poset, then the Rees short exact sequence $E \xrightarrow{f} A \xrightarrow{g} B$ is left split.*

(ii) *If P is a projective S-poset, then the Rees short exact sequence $A \xrightarrow{f} B \xrightarrow{g} P$ is right split.*

The following definition is useful in investigating right split sequences.

Definition 3.3. *Suppose that B is a right S-poset and contains a zero. An S-subposet A of B is said to be a minimal 0-direct summand of B if there exists an S-subposet T of B such that $B = A \cup T$, and for each $a \in A,\ t \in T$ $(a \leq t \Leftrightarrow 0 \leq t)$ and $(t \leq a \Leftrightarrow t \leq 0)$. We denote it by $B = A \acute{\oplus} T$.*

If $B = A\acute{\oplus}T$, clearly $A \cap T = \{0\}$

Theorem 3.4. *Let A, B, C be S-posets that each contains a zero, and*

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

be a Rees short exact sequence. Then, it is right split if and only if $[f(A)]$ is a minimal 0-direct summand of B .

Proof. Necessity. First we show that $b \in [f(A)]$ if and only if $g(b) = 0$. If $b \in [f(A)]$, since $0 \in f(A)$, we have $(b, 0) \in \mathcal{I}m_f$. So, $(b, 0) \in kerg$, and then $g(b) = g(0) = 0$. Conversely, if $g(b) = 0$, then $(b, 0) \in kerg = \mathcal{I}m_f$. So, $b \in [f(A)]$.

Suppose that the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is right split. So, there exists $g' : C \longrightarrow B$ such that $gg' = 1_C$. We show that $B = [f(A)]\acute{\oplus}g'(g(B))$. Let $b \in B$. Take $g(b) = c$. Then,

$$g(g'(c)) = g(g'(g(b))) = g(b), \quad \text{and} \quad (g'(c), b) \in kerg = \mathcal{I}m_f.$$

So either $b \in [f(A)]$ or $b = g'(c) = g'(g(b)) \in g'(g(B))$. Then, $B = [f(A)] \cup g'(g(B))$. Assume that $b \in [f(A)]$, $b' \in g'(g(B))$ and $b \leq b'$. So, $b' = g'g(b'')$ for some $b'' \in B$. Since $g(b) = 0$, $0 = g(b) \leq g(g'g(b'')) = g(b'')$. Then, $0 = g'(0) \leq g'g(b'') = b'$. If $b' \leq b$, the result follows similarly.

Sufficiency. Let $B = [f(A)]\acute{\oplus}T$ for an S-subposet T of B. First, we show that $g|_T : T \longrightarrow C$ is an S-isomorphism. Since g is an epimorphism, for each $c \in C$ there exists $b \in B$ such that $g(b) = c$. If $c \neq 0$, then $b \notin [f(A)]$ and so $b \in T$. Hence, $g|_T$ is an epimorphism. Suppose that $g(t) \leq g(t')$ for $t,\ t' \in T$. So, $(t, t') \in \mathcal{K}_g = \mathcal{I}_f$, and $t \leq t'$ or $t \leq f(a'),\ f(a') \leq t'$. Then, $t \leq t'$ or in view of Definition 3.3, $t \leq 0 \leq t'$. So, $g|_T$ is a regular monomorphism. Now, let

$$g' = (g|_T)^{-1} : C \overset{(g|_T)^{-1}}{\longrightarrow} T \overset{\iota}{\longrightarrow} B.$$

It is clear that g' is an S-morphism. Moreover, let $c \in C$ and $g(t) = c$ for $t \in T$, then $gg'(c) = g\iota(g|_T)^{-1}(c) = g\iota(t) = g(t) = c$. Therefore, $gg' = 1_C$, and the sequence is right split. $\square$

Theorem 3.5. *Let A, B, C be S-posets that each contains a zero, and*

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

be a Rees short exact sequence. Then, it is left split if and only if there exists an S-morphism $\sigma : B \longrightarrow A\prod C$ such that $Im(\sigma f) = A\prod\{0\}$ and $\sigma f : A \longrightarrow A\prod\{0\}$ exists an S-isomorphism.

Proof. Suppose that the sequence is left split. Then, by definition, there is an S-morphism $f' : B \longrightarrow A$ such that $f'f = 1_A$. Let $\sigma : B \longrightarrow A\prod C$ be defined by $\sigma(b) = (f'(b), g(b))$. It is obvious that σ is an S-morphism such that $Im(\sigma f) = A\prod\{0\}$ and $\sigma f : A \longrightarrow A\prod\{0\}$ is an S-isomorphism. Conversely, suppose that there exists an S-morphism $\sigma : B \longrightarrow A\prod C$ such that $Im(\sigma f) = A\prod\{0\}$ and $\sigma f : A \longrightarrow A\prod\{0\}$ is an S-isomorphism. Let

$$f' = (\sigma f)^{-1}\pi\sigma : B \xrightarrow{\sigma} A\prod C \xrightarrow{\pi} A\prod\{0\} \xrightarrow{(\sigma f)^{-1}} A,$$

where $\pi(a, c) = (a, 0)$ for each $a \in A,\ c \in C$. It is clear that f' is an S-morphism and $f'f = 1_A$, as required. □

Example 2.3 of [4] illustrates that, for S-acts a left split Rees short exact sequence is not necessarily right split. In this example, if we consider the order as the trivial order this statement is also valid in the category of S-posets. In the category of S-acts, right splitness implies left splitness. But the following example illustrates that, for S-posets, right splitness of a Rees short exact sequence does not imply left splitness.

Example 3.6. *Let $S = \{0, 1, s, s^2, \ldots\}$ be a free word monoid equipped with the order $0 < 1 < s < s^2 < \ldots$. Take $K = sS = \{0, s, s^2, \ldots\}$. So, K is a right ideal of S but not convex, and $[K] = S$. Then*

$$K \xrightarrow{\iota_K} S \xrightarrow{\pi} S/[K]$$

is a Rees short exact sequence. Clearly, it is right split but it is not left split. Otherwise, there exists $\rho : S \longrightarrow K$ such that $\rho\iota_K = 1_K$. Then $\rho(1) \neq 0$ and if $\rho(1) = s^i$ for some $i \geq 1$, then $\rho(s) = s^{i+1}$. So $\rho\iota_K(s) = \rho(s) = s^{i+1} \neq s$, a contradiction.

4. The Exactness of $Hom(P, -)$

In this section, we show that $Hom(P, -)$ is an exact functor if and only if $P = eS$, for some $e^2 = e \in S$. So, in general, the functor $Hom(P, -)$ is not exact for an arbitrary projective S-poset P. Let S and T be pomonoids, $A_S,\ P_S$ right S-posets, and $_TP$ be also a left T-poset. It is easily proved that $Hom(_TP_S, A_S)$ is a right T-poset where, for any $f \in Hom(_TP_S, A_S)$ and $t \in T$, we have $(ft)(c) = f(tc)$, and $f \leq g$ if and only if $f(c) \leq g(c)$ for each $c \in P$. Let $f : A_S \to B_S$ be an S-morphism. Take $f^* := Hom(P, f) : Hom(_TP_S, A_S) \longrightarrow$

$Hom({}_TP_S, B_S)$ by $f^*(\psi)(c) = f(\psi(c))$ for every $c \in P$ and $\psi \in Hom({}_TP_S, A_S)$. Then, f^* is a T-morphism.

Lemma 4.1. *The functor $Hom(P, -)$ preserves regular monomorphisms.*

Proof. Let A_S, B_S be right S-posets and $f : A \longrightarrow B$ a regular monomorphism. We want to show that $f^* : Hom(P, A) \longrightarrow Hom(P, B)$ is a regular monomorphism. Suppose that $f^*(\alpha) \leq f^*(\beta)$ for $\alpha, \beta \in Hom(P, A)$. Then $f\alpha \leq f\beta$, and for each $c \in P$ we have $f(\alpha(c)) \leq f(\beta(c))$. Since f is a regular monomorphism, $\alpha(c) \leq \beta(c)$. This means that $\alpha \leq \beta$, and we are done. □

Let A be an S-poset and $e \in S$ an idempotent. We define $ae.es = aese$ for any $es \in eS$ and $ae \in Ae$. Clearly, Ae is a right eS-poset. The following lemmas are useful to reach the main result.

Lemma 4.2. *Let A, B and C be S-posets and $A \xrightarrow{f} B \xrightarrow{g} C$ a Rees short exact sequence. Then, for any idempotent $e \in S$, the sequence $Ae \xrightarrow{ef} Be \xrightarrow{eg} Ce$ is also a Rees short exact sequence as eS-posets, where $ef = f|_{Ae}$ and $eg = g|_{Be}$ are eS-morphisms.*

Proof. It is easy to prove that fe and ge are eS-morphisms. Since f is a regular monomorphism, clearly ef is also a regular monomorphism. Moreover, for any $ce \in Ce$, since g is an epimorphism, there exists $b \in B$ such that $g(b) = ce$. Then, $eg(be) = (eg(b))e = ce$, and hence, eg is an epimorphism. Now, we show that $\mathcal{K}_{eg} = \mathcal{I}_{ef}$. Let $(be, b'e) \in \mathcal{I}_{ef}$ with $be \nleq b'e$. Since $(be, b'e) \in \mathcal{I}_f$ and $\mathcal{K}_g = \mathcal{I}_f$, we have $eg(be) = g(be) \leq g(b'e) = eg(b'e)$, and hence $(be, b'e) \in \mathcal{K}_{eg}$. Conversely, suppose that $(be, b'e) \in \mathcal{K}_{eg}$ with $be \nleq b'e$. Since $\mathcal{K}_g = \mathcal{I}_f$, $(be, b'e) \in \mathcal{I}_f$. Therefore, there exist $a, a' \in A$ such that $be \leq f(a)$, $f(a') \leq b'e$. This implies that $be \leq f(ae)$, $f(a'e) \leq b'e$, and then $(be, b'e) \in \mathcal{I}_{ef}$. Therefore, $\mathcal{K}_{eg} = \mathcal{I}_{ef}$. □

Lemma 4.3. *Let B be an S-poset. Then, $\sigma_B : Hom(eS, B) \longrightarrow Be$ defined by $\sigma_B(f) = f(e)$ is an eS-isomorphism.*

Proof. As we mentioned earlier, $Hom(eS, B)$ and Be are right eS-posets. It can be easily checked that σ_B is an eS-morphism. Let $\sigma_B(f) \leq \sigma_B(g)$ for f, $g \in Hom(eS, B)$. Then, $f(e) \leq g(e)$, and so for each $s \in S$, we have $f(es) \leq g(es)$. Hence, $f \leq g$ and σ_B is a regular monomorphism. Moreover, for any $be \in Be$, we define $f : eS \longrightarrow B$ by $f(es) = bes$. Then, $f \in Hom(eS, B)$ and $\sigma_B(f) = f(e) = be$. This shows that σ_B is an epimorphism and the result follows. □

Lemma 4.4. *Let B be an S-poset. Then $\alpha_B : Hom(\coprod_{i\in I} e_iS, B) \longrightarrow \prod_{i\in I} Be_i$ defined by $\alpha_B(f) = (f(e_i))_{i\in I}$ is an order isomorphism.*

Proof. Similar to the argument of the previous lemma, one can prove that α_B is a regular monomorphism. For any $(b_ie_i)_{i\in I} \in \prod_{i\in I} Be_i$, we have $f_i : e_iS \longrightarrow B$ with $\sigma_B(f_i) = f(e_i) = b_ie_i$. Take $f = \coprod_{i\in I} f_i \in Hom(\coprod_{i\in I} e_iS, B)$. So $\alpha_B(f) = (b_ie_i)_{i\in I}$, and it is surjective. □

A functor $\tau : S - Posets \longrightarrow Posets$ is called *exact* if $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at B implies that $\tau(A) \xrightarrow{\tau(f)} \tau(B) \xrightarrow{\tau(g)} \tau(C)$ is exact at $\tau(B)$. Using Lemma 2.5, an exact functor preserves epimorphisms. By Lemma 4.1, one can show that the functor $Hom(P, -)$ preserves regular monomorphisms. So it is deduced that the functor $Hom(P, -)$ is exact if we could imply that the sequence $Hom(P, A) \xrightarrow{f^*} Hom(P, B) \xrightarrow{g^*} Hom(P, C)$ is a short exact sequence for each Rees short exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$.

Now, a characterization for the functor $Hom(P, -)$ to be exact is given.

Theorem 4.5. *Let S be a pomonoid and P be an S-poset. Then, $Hom(P, -)$ is an exact functor if and only if $P = eS$, for some idempotent $e \in S$.*

Proof. Necessity. Suppose that $Hom(P, -)$ is an exact functor. Then P is projective and by Lemma 1.1, $P = \coprod_{i\in I} e_iS$ for some idempotent $e_i \in S$. It suffices to prove that $|I| = 1$. Let B be an S-poset and $A = S \coprod S$. Then, we have the Rees short exact sequence $A \xrightarrow{\iota} A \coprod B \xrightarrow{\pi} B \coprod \Theta$, where $\iota(a) = a$ fore each $a \in A$, and $\pi(c) = c$ if $c \in B$ otherwise $\pi(c) = \theta$. By Lemma 4.4, $\alpha_A : Hom(\coprod_{i\in I} e_iS, A) \longrightarrow \prod_{i\in I} Ae_i$ is an order isomorphism. Also, we have the following commutative diagram:

$$\begin{array}{ccccc} Hom(\coprod_{i\in I} e_iS, A) & \xrightarrow{f^*} & Hom(\coprod_{i\in I} e_iS, A\coprod B) & \xrightarrow{g^*} & Hom(\coprod_{i\in I} e_iS, B\coprod \Theta) \\ \downarrow{\alpha_A} & & \downarrow{\alpha_{A\coprod B}} & & \downarrow{\alpha_{B\coprod \Theta}} \\ \prod_{i\in I} Ae_i & \xrightarrow{\iota'} & \prod_{i\in I}(Ae_i \coprod Be_i) & \xrightarrow{g'} & \prod_{i\in I}(Be_i \coprod \Theta) \end{array}$$

where ι' is the inclusion map and $g'((c_ie_i)_{i\in I}) = (g(c_ie_i))_{i\in I}$. By assumption, the upper row is exact at $Hom(\coprod_{i\in I} e_iS, A\coprod B)$. So, by Lemma 2.2, the lower row in the diagram is exact at $\prod_{i\in I}(Ae_i \coprod Be_i)$. Suppose that $|I| > 1$. Since $|Ae_i| > 1$ for any $i \in I$, we can take $a_1e_j,\ a_2e_j \in Ae_j$ with $a_1e_j \neq a_2e_j$ for some $j \in I$. Denote $c_1 = (x_ie_i)_{i\in I}$ and $c_2 = (y_ie_i)_{i\in I}$, where $x_j = a_1$, $y_j = a_2$, and $x_i = y_i \in B$ if $i \neq j$

for all $i \in I$. It is obvious that $(c_1, c_2) \in \mathcal{K}_{g'}$ but $(c_1, c_2) \notin \mathcal{I}_{\iota'}$, a contradiction.

Sufficiency. Let e be an idempotent of a pomonoid S and $P = eS$. Suppose that $A \xrightarrow{f} B \xrightarrow{g} C$ is a Rees short exact sequence. By Lemma 4.3, the sequence $Ae \xrightarrow{ef} Be \xrightarrow{eg} Ce$ is also a Rees short exact sequence of eS-posets. Moreover, we have the following commutative diagram:

$$\begin{array}{ccccc} Hom(P, A) & \xrightarrow{f^*} & Hom(P, B) & \xrightarrow{g^*} & Hom(P, C) \\ \downarrow{\scriptstyle \sigma_A} & & \downarrow{\scriptstyle \sigma_B} & & \downarrow{\scriptstyle \sigma_C} \\ Ae & \xrightarrow{\quad ef \quad} & Be & \xrightarrow{\quad eg \quad} & Ce. \end{array}$$

By Lemma 2.2, the upper row in the diagram is exact, and so the functor $Hom(P, -)$ is exact. □

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REES SHORT EXACT SEQUENCES OF S-POSETS

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دنباله‌های کامل کوتاه ریس در رسته‌ی S-سیستم‌های جزئی مرتب

رقیه خسروی
دانشگاه فسا

در این مقاله مفهوم دنباله‌های کامل کوتاه ریس در رسته‌ی S-سیستم‌های جزئی مرتب معرفی شده است. شرایطی نیز که تحت آن این دنباله‌ها تجزیه راست یا تجزیه چپ باشند بررسی می‌کنیم. برخلاف رسته‌ی S-سیستم‌ها، برای S-سیستم‌های جزئی مرتب تجزیه راست بودن دنباله‌ها تجزیه چپ را نتیجه نمی‌دهد. بعلاوه برای یک S-سیستم جزئی مرتب راست P شرایط معادل با کامل بودن تابعگر $\mathrm{Hom}(P, -)$ ارائه می‌دهیم.

کلمات کلیدی:تکواره جزئی مرتب، S-سیستم جزئی مرتب، دنباله کامل کوتاه ریس .