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ON THE SPECTRUM OF DERANGEMENT GRAPHS OF ORDER A PRODUCT OF THREE PRIMES

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Abstract. A permutation with no fixed points is called a derangement. The subset $\mathcal{D}$ of a permutation group is derangement if all elements of $\mathcal{D}$ are derangement. Let G be a permutation group, a derangement graph is one with vertex set G and derangement set $\mathcal{D}$ as connecting set. In this paper, we determine the spectrum of derangement graphs of order a product of three primes.

1. Introduction

Let G be a permutation group, we say that $S \subseteq G$ is intersecting if there exists at least an integer $i \in \{1, \cdots, n\}$ such that for two permutations $\alpha, \beta \in S$, we have $\alpha(i) = \beta(i)$. A derangement is a permutation with no fixed points. The subset $\mathcal{D}$ of a permutation group is derangement set if their elements are derangements. Suppose G is a permutation group and $\mathcal{D} \subseteq G$ is a derangement set. The derangement graph $X_G = C(G, \mathcal{D})$ has G as it's vertices and two vertices are adjacent if and only if they do not intersect. Since $\mathcal{D}$ is a union of conjugacy classes, X_G is a normal Cayley graph.

Here, our notation is standard and mainly taken from [2, 8, 9]. In the next section, we introduce some basic definitions which will be used in the continuing of this paper and in Section 3, we determine the eigenvalues of derangement graphs of order a product of three primes.

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2. Definitions and Preliminaries

Let X be a graph with vertex set $V(X) = \{v_1, v_2, \cdots, v_n\}$, the adjacency matrix $A = A(X)$ of graph X is the square symmetric matrix $[a_{ij}]$ such that $a_{ij} = 1$ if v_i and v_j are adjacent and 0, otherwise $(1 \leq i, j \leq n)$. The characteristic polynomial of graph X with adjacency matrix A is defined as $\phi(X, \lambda) = det(\lambda I - A)$. The eigenvalues of X are all roots of $\phi(X, \lambda)$ and the spectrum of X is the multiset $\{[\lambda_1]^{t_1}, \cdots, [\lambda_r]^{t_r}\}$, where λ_i's$(1 \leq i \leq r)$ are eigenvalues of X with multiplicity t_i's, see [3, 4].

Let $\mathbb{F}$ be a field and consider the representation $\rho : G \to GL(n, \mathbb{F})$ with $\rho(g) = [g]_\beta$, for some basis β. The character $\chi_\rho : G \to \mathbb{C}$ afforded by ρ is defined as $\chi_\rho(g) = tr([g]_\beta)$. The character χ corresponded to an irreducible representation is called the irreducible character and χ is linear if $\chi(1) = 1$. The set of all irreducible characters of group G is denoted by $Irr(G)$.

For the finite group G, the subset S is symmetric if $1 \notin S$ and $S = S^{-1}$. The Cayley graph $X = C(G, S)$ on G with respect to S has the vertex set $V(X) = G$ and edge set $E(X) = \{(g, sg)|g \in G, s \in S\}$.

Theorem 2.1. [10] *Let S be a symmetric subset of abelian group G. Then the eigenvalues of the adjacency matrix of $X = C(G, S)$ are given by*

$$\lambda_\varphi = \sum_{s \in S} \varphi(s),$$

where $\varphi \in Irr(G)$.

The following corollary is implicitly contained in [5, 10].

Corollary 2.2. *Let G be a finite group with a normal symmetric subset S. Let A be the adjacency matrix of Cayley graph $X = C(G, S)$. Then all eigenvalues of A are*

$$[\lambda_\chi]^{\chi(1)^2}, \ \chi \in Irr(G)$$

where $\lambda_\chi = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s)$.

For the subset S of G let S by $S^g = g^{-1}Sg$, where $g \in G$. A transitive permutation group $G \leq Sym(n)$ is called a Frobenius group if G contains a subgroup $H \neq \{e\}$, where $H \cap H^g = \{e\}$, for all $g \in G \backslash H$ and the Frobenius kernel of G is $K = (G \setminus \cup_{g \in G} H^g) \cup \{e\}$. It is not difficult to see that the derangement elements of G are non-identity elements of K.

Theorem 2.3. [12] (*Frobenius Theorem*) *Suppose H is a proper non-identity subgroup of G such that for all $g \in G \setminus H$,*

$$H \cap H^g = \{e\}. \tag{2.1}$$

Let $K = G \setminus \cup_{g\in G}(H \setminus \{e\})^g$, then

$$K \lhd G,\ G = KH \text{ and } H \cap K = \{e\}.$$

Theorem 2.4. [2] *Let $G_1, G_2, \ldots, G_k$ be finite groups and $G = G_1 \times \cdots \times G_k$. Then*

$$X_G = \overline{\overline{X_{G_1}} \times \cdots \times \overline{X_{G_k}}},$$

where $\overline{X_G}$ denotes to the complement of graph X_G.

3. Main results

The aim of this section is to compute the spectrum of derangement graph of order a product of three primes. We denote a complete graph on n vertices by K_n. The spectrum of this graph is $\{[-1]^{n-1}, [n-1]^1\}$ and if $G = \cup_{1\le i\le t} K_n$, then $spec(G) = \{[-1]^{t(n-1)}, [n-1]^t\}$, see [3].

Theorem 3.1. [2] *Let $G = KH \le Sym(n)$ be a Frobenius group with the kernel K. Then the derangement graph X_G is a disjoint union of $|H|$ copies of K_n.*

A non-abelian group of order pq has the following presentation (p is a prime number and $q|p-1$):

$$F_{p,q} = \langle x, y :\ x^p = y^q = e, y^{-1}xy = x^r\rangle, \tag{3.1}$$

where $r^q = 1(mod\, p)$, see [9]. One can see that this group is a Frobenius group.

3.1. Groups of order pqr.

Let $p > q > r$ be three distinct prime numbers. In [6] the structures of all groups of order pqr were verified as follows:

- $G_1 = \mathbb{Z}_{pqr}$,
- $G_2 = F_{p,qr}(qr|p-1)$,
- $G_3 = \mathbb{Z}_r \times F_{p,q}(q|p-1)$,
- $G_4 = \mathbb{Z}_p \times F_{q,r}(r|q-1)$,
- $G_5 = \mathbb{Z}_q \times F_{p,r}(r|p-1)$,
- $G_{5+d} = \langle a, b, c : a^p = b^q = c^r = e, ab = ba, c^{-1}bc = b^u, c^{-1}ac = a^{v^d}\rangle$, where $r|p-1, q-1, q|p-1, o(u) = r$ in $\mathbb{Z}_q^*$ and $o(v) = r$ in $\mathbb{Z}_p^*$ $(1 \le d \le r-1)$.

Theorem 3.2. *Groups G_2 and G_{5+d} $(1 \le d \le r-1)$ are Frobenius.*

Proof. Let a, b be generators of group G_2 and consider two subgroups $K = \langle a \rangle$ and $H = \langle b \rangle$ of G_2, where $o(a) = p$ and $o(b) = qr$. Then, K is a normal subgroup of G_2 and hence $G_2 = KH$ is a Frobenius group.

Let $G = G_6$ $(d = 1)$ with subgroups $K = \langle x, y \rangle$ and $H = \langle z \rangle$, where x, y, z are generators of G. Let $g = x^i y^j z^k \in G \setminus H$ and suppose on the contrary that $H \cap H^g \neq \{e\}$. So there exist $1 \leq l, t \leq r - 1$ such that $z^{-k} y^{-j} x^{-i} z^t x^i y^j z^k = z^l$. Hence, $z^{-k+t} x^{-v^t i + i} y^{j - u^t j} z^k = z^l$ and so $z^t x^{v^k (i - v^t i)} y^{u^k (j - u^t j)} = z^l$. This yields that

$$\begin{cases} i - v^t i \equiv 0 \ (mod\ p) \\ j - u^t j \equiv 0 \ (mod\ q) \\ t = l \end{cases},$$

and so $i = j = 0$, which means that $g \in H$, a contradiction. Hence, $H \cap H^g = \{e\}$ and Theorem 2.3 implies that G is a Frobenius group. By a similar method, we can prove that G_{5+d} $(2 \leq d \leq r - 1)$ is a Frobenius group. □

Theorem 3.3. *Suppose G is a group of order pqr and $\mathcal{D}$ is a derangement. Then the spectrum of derangement graph $X_G = C(G, \mathcal{D})$ is*

(i) $Spec(X_{G_1}) = \{[-1]^{pqr-1}, [pqr - 1]\}$,

(ii) $Spec(X_{G_2}) = \{[-1]^{qr(p-1)}, [p - 1]^{qr}\}$,

(iii) $Spec(X_G) = \{[-1]^{pqr-1}, [pqr - 1]\}$, *where* $G \in \{G_3, G_4, G_5\}$,

(iv) $Spec(X_{G_{5+d}})(1 \leq d \leq r - 1) = \{[-1]^{r(pq-1)}, [pq - 1]^r\}$.

Proof. We can consider the following cases:

(i) Since G_1 is cyclic, the derangement graph X_{G_1} is a complete graph and its spectrum is as desired.
(ii) By Theorem 3.2, G_2 is a Frobenius group and by Theorem 3.1, X_{G_2} is a disjoint union of qr copies of K_p.
(iii) By using Theorem 2.4, the derangement graph X_G, where $G \in \{G_3, G_4, G_5\}$ is a complete graph and its spectrum is as given.
(iv) Let $G = G_{5+d}$ $(1 \leq d \leq r - 1)$, by Theorem 3.2, X_G is a union of r copies of K_{pq} and this completes the proof.

□

3.2. **Groups of order p^2q.** According to [11] the structures of groups of order p^2q, where $p < q$ are as follows:

- $L_1 = \mathbb{Z}_{p^2q}$,
- $L_2 = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$,
- $L_3 = \mathbb{Z}_p \times F_{q,p}$ $(p|q - 1)$,

- $L_4 = F_{q,p^2}$ $(p^2|q-1)$,
- $L_5 = \langle a, b : a^{p^2} = b^q = e, a^{-1}ba = b^u,\ u^p \equiv 1\ (mod\ q)\rangle$ $(p^2|q-1)$.

Also, the structures of groups of order p^2q, where $p > q$ are as follows:

- $Q_1 = \mathbb{Z}_{p^2q}$,
- $Q_2 = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$,
- $Q_3 = \mathbb{Z}_p \times F_{p,q}$ $(q|p-1)$,
- $Q_4 = \langle a, b : a^q = b^{p^2} = 1, a^{-1}ba = b^{\alpha},\ \alpha^q \equiv 1\ (mod\ p^2)\rangle$ $(q|p-1)$,
- $Q_5 = \langle a, b, c \mid a^q = b^p = c^p = 1, a^{-1}ba = c, a^{-1}ca = b^{-1}c^{2\alpha}, bc = cb, (\alpha+\sqrt{\alpha^2-1})^q = 1\ (mod\ p)\ \rangle$, $(q|p+1)$, α^2-1 is not perfect square.
- $Q_{5+i} = \langle a, b, c \mid a^q = b^p = c^p = 1, a^{-1}ba = b^{\beta}, a^{-1}ca = c^{\beta^i}, bc = cb,\ \beta^q \equiv 1\ (mod\ p)\rangle$ $(q|p-1)$, $B = \{1, 2, 3, \ldots, \frac{q-1}{2}$ *and* $q-1\}$.

Theorem 3.4. *All groups L_4, Q_4, Q_5 and Q_{5+i} $(i \in B)$ are Frobenius.*

Proof. With respect to the presentation of $L_4 = F_{q,p^2}$, it has two generators x, y and consider two subgroups $K = \langle x\rangle$ and $H = \langle y\rangle$, where $o(x) = q$ and $o(y) = p^2$. It is not difficult to see that $L_4 = KH$ and thus it is Frobenius group. Let $Q = Q_4$, $H = \langle a\rangle$ and $K = \langle b\rangle$. Similar to the last case, Q_4 is Frobenius group. Now consider the group Q_5 with two subgroups $K = \langle b, c\rangle$ and $H = \langle a\rangle$. We conclude that Q_5 is a Frobenius group. By a similar method, $Q_{5+i} (i \in B)$ is a Frobenius group. □

Theorem 3.5. *The spectrum of derangement graph $X_G = C(G, \mathcal{D})$, where G is a group of order p^2q is as follows:*

(i) $Spec(X_L) = \{[-1]^{p^2q-1}, [p^2q-1]\}$, *where* $L \in \{L_1, L_2, L_3\}$,

(ii) $Spec(X_{L_4}) = \{[-1]^{p^2(q-1)}, [q-1]^{p^2}\}$,

(iii) $Spec(X_{L_5}) = \{[-1]^{p^2q-1}, [p^2q-1]\}$,

(iv) $Spec(X_Q) = \{[-1]^{p^2q-1}, [p^2q-1]\}$, *where* $Q \in \{Q_1, Q_2, Q_3\}$,

(v) $Spec(X_Q) = \{[-1]^{q(p^2-1)}, [p^2-1]^q\}$, *where* $Q \in \{Q_4, Q_5, Q_{5+i}\}$ $(i \in B)$.

Proof. In [7], it is proved that the derangement graph X_{L_5} is isomorphic with a complete graph, and so its spectrum is as given. The proofs of the rest cases are similar to the proof of Theorem 3.3. □

3.3. **Groups of order p^3.** Let $p \geq 2$ be a prime number. Then there are three abelian groups of order p^3, namely $\mathbb{Z}_{p^3}$, $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$, $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, and two non-abelian groups with the following presentations:

$$K_1 = \langle a, b | a^p = b^{p^2} = e, a^{-1}ba = b^{p+1} \rangle,$$
$$K_2 = \langle a, b, c | a^p = b^p = c^p = e, [a, b] = c, [a, c] = [b, c] = e \rangle.$$

One can see that the derangement graph X_G of an abelian group G of order p^3 is isomorphic with the complete graph K_{p^3} and thus

$$Spec(X_G) = \{[-1]^{p^3-1}, [p^3 - 1]\}.$$

Theorem 3.6. [9] *Let $K_1 = \{a^r b^s z^t : 0 \leq r, s, t \leq p-1\}$ be a non-abelian group of order p^3. Write $\epsilon = e^{2\pi i/p}$. Then the irreducible characters of K_1 are*

$$\chi_{u,v} \ (0 \leq u \leq p-1, 0 \leq v \leq p-1)$$

and

$$\varphi_u \ (1 \leq u \leq p-1)$$

where for all r, s, t, we have $\chi_{u,v}(a^r b^s z^t) = \epsilon^{ru+sv}$ and

$$\varphi_u(a^r b^s z^t) = \begin{cases} p\epsilon^{ut} & r = s = 0 \\ 0 & otherwise \end{cases}.$$

Theorem 3.7. *Two graphs X_{K_1} and X_{K_2} are co-spectral with the following spectrum:*

$$\{[-p^2 + p - 1]^{p-1}, [-1]^{p^{2(p-1)}}, [p-1]^{p(p-1)}, [p^3 - p^2 + p - 1]\}.$$

Proof. In [7, Theorem 3.13], it is proved that the derangement set of group K_1 is $\mathcal{D}_{K_1} = K_1 - \{e, a^i b^j\}$, where $1 \leq i \leq p-1$ and $j = tp$ $(0 \leq t \leq p-1)$. This yields that X_{K_1} is a regular graph of order $p^3 - p^2 + p - 1$.

Let $\chi_{u,v}$'s be all irreducible characters of K_1 as given in Theorem 3.6 and let $\mathcal{A} = \{1, \ldots, p^2 - 1\} - \{kp \mid 1 \leq k \leq p-1\}$. The degree of $\chi_{u,v}(0 \leq u \leq p-1,\ 0 \leq v \leq p-1)$ is 1 which yields that the multiplicity of $\lambda_{\chi_{u,v}}$ is 1. By Corollary 2.2, we can conclude that

$$\lambda_{\chi_{u,v}} = \sum_{i=1}^{p^2-1} \chi_{u,v}(b^i) + \sum_{i=1}^{p-1} \sum_{j \in \mathcal{A}} \chi_{u,v}(a^i b^j).$$

It is clear that $\lambda_{\chi_{0,0}} = |S|$ and the second eigenvalue of X_{K_1} is

$$\lambda_{\chi_{u,0}} = p^2 - 1 + (p^2 - p) \sum_{i=1}^{p-1} \varepsilon^i = p - 1 \ (1 \leq u \leq p-1).$$

We also have

$$\lambda_{\chi_{u,v}} = \sum_{i=1}^{p^2-1} \varepsilon^i + p \sum_{i=1}^{p-1} -\varepsilon^i = p-1 \ (1 \leq u \leq p-1, \ v \neq 0).$$

On the other hand, $\lambda_{\chi_{0,v}} = \sum_{i=1}^{p^2-1} \varepsilon^i + p(p-1) \sum_{i=1}^{p-1} \varepsilon^i = -p^2+p-1$ $(v \neq 0)$ is the smallest eigenvalue of X_{K_1}. Now let φ_u be an irreducible character of K_1 as given in Theorem 3.6. If $1 \leq k \leq p-1$, then $\varphi_u(b^{kp}) \neq 0$. The degree of these characters is p and so

$$\lambda_{\varphi_u} = \sum_{i=1}^{p-1} \varepsilon^i = -1.$$

Since $\varphi_u(id)^2 = p^2$, the multiplicity of eigenvalue -1 is $p^2(p-1)$. Now, we compute the eigenvalues of derangement X_{K_2}. By [7, Theorem 3.17], the derangement set of group K_2 is $\mathcal{D}_{K_2} = K_2 - \{e, a^i c^k\}$, where $1 \leq i \leq p-1, \ 0 \leq k \leq p-1$. Hence X_{K_2} is a regular graph of degree $p^3 - p^2 + p - 1$. Suppose $\chi_{u,v}$ is an irreducible character of K_2 as given in Theorem 3.6. The multiplicity of $\lambda_{\chi_{u,v}}$ is 1 and we have

$$\lambda_{\chi_{u,v}} = \sum_{i,j=1}^{p-1} \sum_{t=0}^{p-1} \chi_{u,v}(a^i b^j c^t) + \sum_{j=1}^{p-1} \sum_{t=0}^{p-1} \chi_{u,v}(b^j c^t) + \sum_{t=1}^{p-1} \chi_{u,v}(c^t).$$

One can see $\lambda_{\chi_{0,0}} = |S|$ and thus the second eigenvalue of X_{K_2} is

$$\lambda_{\chi_{u,0}} = (p^2 - p) \sum_{i=1}^{p-1} \varepsilon^i + p^2 - 1 = p - 1 \ (1 \leq u \leq p-1).$$

Also we have

$$\lambda_{\chi_{u,v}} = p \sum_{i=1}^{p-1} \varepsilon^i + p \sum_{i=1}^{p-1} -\varepsilon^i + p - 1 = p - 1 \ (1 \leq u \leq p-1, \ v \neq 0).$$

On the other hand, the smallest eigenvalue of X_{K_2} is

$$\lambda_{\chi_{0,v}} = p^2 \sum_{i=1}^{p-1} \varepsilon^i + (p-1) = -p^2 + p - 1 \ (v \neq 0).$$

Now, let φ be an irreducible character of K_2 in the Theorem 3.6. It is clear that $\varphi_u(c^t) = p\varepsilon^{ut}$. Similar to the last case, the degree of these characters is p. Hence, $\lambda_{\varphi_u} = \Sigma_{i=1}^{p-1} \varepsilon^i = -1$ and the multiplicity of eigenvalue -1 is $p^2(p-1)$.

□

Remark 3.8. If G is a graph with adjacency matrix A, then by $G \otimes J_n$, we denote a graph with adjacency matrix $A \otimes J_n$, and by $G \circledast J_n$ we denote a graph with adjacency matrix $(A+I) \otimes J_n - I$, see [13]. If G has v vertices with spectrum $\{[s]^g, [-1]^m, [r]^f, [k]\}$, where m is greater than or equal zero, then $G \circledast J_n$ is a graph on vn vertices with the following spectrum:

$$\{[sn+n-1]^g, [-1]^{m+vn-v}, [rn+n-1]^f, [kn+n-1]\}.$$

A strongly regular graph is a k-regular graph on n vertices with parameters (n, k, λ, μ) such that every two adjacent vertices have λ common neighbours and every two non-adjacent vertices have μ common neighbours. We can prove that two graphs X_{K_1} and X_{K_2} are isomorphic with $G \circledast J_n$, where G is a $(p^2, p^2-2, p^2-2, p^2-p)$-strongly regular graph with the following spectrum:

$$\{[-p]^{p-1}, [0]^{p(p-1)}, [p^2-p]\}.$$

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ON THE SPECTRUM OF DERENGEMENT GRAPHS OF ORDER A PRODUCT OF THREE PRIMES

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در مورد طیف گراف‌های پریش از مرتبه حاصلضرب سه عدد اول

مجتبی قربانی و مینا رجبی-پارسا
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پریش، یک جایگشتی است که هیچ نقطه ثابتی ندارد. زیرمجموعه D از یک گروه جایگشتی را پریش می‌نامیم، هرگاه همه اعضای D پریش باشند. فرض کنید G یک گروه جایگشتی باشد. یک گراف پریش، گرافی است با مجموعه راس‌های G که در آن دو راس مجاورند اگر و تنها اگر با هم اشتراک نداشته باشند. در این مقاله، طیف گراف‌های پریش از مرتبه حاصلضرب سه عدد اول را مطالعه می‌کنیم.

کلمات کلیدی: گروه‌های جایگشتی، مقادیر ویژه گراف، گروه فروبنیوس.