

## ON $\alpha$ -SEMI-SHORT MODULES

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ABSTRACT. We introduce and study the concept of  $\alpha$ -semi short modules. Using this concept we extend some of the basic results of  $\alpha$ -short modules to  $\alpha$ -semi short modules. We observe that if  $M$  is an  $\alpha$ -semi short module then the dual perfect dimension of  $M$  is  $\alpha$  or  $\alpha + 1$ .

### 1. INTRODUCTION

Lemonnier [26] has introduced the concept of deviation (resp., codeviation) of an arbitrary poset, which in particular, when applied to the lattice of all submodules of a module  $M_R$  give the concept of Krull dimension, see [17], [16] and [28] (resp., the concept of dual Krull dimension of  $M$ . The dual Krull dimension in [14], [13], [15], [19], [20], [21], [22], [8], [11], [9], [10], and [24] is called Noetherian dimension and in [7] is called N-dimension. This dimension is called Krull dimension in [29]. The name of dual Krull dimension is also used by some authors, see [2], [4] and [1]). The Noetherian dimension of an  $R$ -module  $M$  is denoted by  $n\text{-dim } M$  and by  $k\text{-dim } M$  we denote the Krull dimension of  $M$ . We recall that if an  $R$ -module  $M$  has Noetherian dimension and  $\alpha$  is an ordinal number, then  $M$  is called  $\alpha$ -atomic if  $n\text{-dim } M = \alpha$  and  $n\text{-dim } N < \alpha$ , for all proper submodule  $N$  of  $M$ . An  $R$ -module  $M$  is called atomic if it is  $\alpha$ -atomic for some ordinal  $\alpha$  (note, atomic modules are also called conotable, dual critical and N-critical in some other articles; see for example [27], [2], and [7]). The author introduced

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and extensively investigated perfect dimension and dual perfect dimension of an  $R$ -module  $M$ , see [13]. The dual perfect dimension (resp., perfect dimension), which is denoted  $dp\text{-dim } M$  (resp.,  $p\text{-dim } M$ ) is defined to be the codeviation (resp., deviation) of the poset of the finitely generated submodules of  $M$ . It is convenient, when we are dealing with the latter dimensions, to begin our list of ordinals with  $-1$ . We recall that an  $R$ -module  $M$  is called  $\alpha$ -perfect atomic, where  $\alpha$  is an ordinal, if  $dp\text{-dim } M = \alpha$  and  $dp\text{-dim } N < \alpha$  for any proper finitely generated submodule  $N$  of  $M$ .  $M$  is said to be perfect-atomic if it is  $\alpha$ -perfect atomic for some  $\alpha$ . Bilhan and Smith have introduced and extensively investigated short modules and almost Noetherian modules, see [6]. Later Davoudian, Karamzadeh and Shirali undertook a systematic study of the concepts of  $\alpha$ -short modules and  $\alpha$ -almost Noetherian modules, see [14]. We recall that an  $R$ -module  $M$  is called an  $\alpha$ -short module, if for each submodule  $N$  of  $M$ , either  $n\text{-dim } N \leq \alpha$  or  $n\text{-dim } \frac{M}{N} \leq \alpha$  and  $\alpha$  is the least ordinal number with this property. We shall call an  $R$ -module  $M$  to be  $\alpha$ -semi short, if for each finitely generated submodule  $N$  of  $M$ , either  $dp\text{-dim } N \leq \alpha$  or  $dp\text{-dim } \frac{M}{N} \leq \alpha$  and  $\alpha$  is the least ordinal number with this property. Using this concept, we show that each  $\alpha$ -semi short module  $M$  has dual perfect dimension and  $\alpha \leq dp\text{-dim } M \leq \alpha + 1$ . We observe that an Artinian serial module  $M$  is  $\alpha$ -short if and only if it is  $\beta$ -semi short, where  $\alpha$  and  $\beta$  are ordinal numbers and  $\beta \leq \alpha \leq \beta + 1$ . We also recall that an  $R$ -module  $M$  is called  $\alpha$ -almost Noetherian, if for each proper submodule  $N$  of  $M$ ,  $n\text{-dim } N < \alpha$  and  $\alpha$  is the least ordinal number with this property, see [14]. We shall call an  $R$ -module  $M$  to be  $\alpha$ -semi Noetherian if for each proper finitely generated submodule  $N$  of  $M$ ,  $dp\text{-dim } N < \alpha$  and  $\alpha$  is the least ordinal number with this property. In section 2 of this paper we investigate some basic properties of  $\alpha$ -semi Noetherian and  $\alpha$ -semi short modules. We show that if  $M$  is an  $\alpha$ -semi short module (resp.,  $\alpha$ -semi Noetherian module), then  $dp\text{-dim } M = \alpha$  or  $dp\text{-dim } M = \alpha + 1$  (resp.,  $dp\text{-dim } M \leq \alpha$ ). In the last section we also investigate some properties of  $\alpha$ -semi Noetherian and  $\alpha$ -semi short modules. Finally, we should emphasize here that the results in sections 2 and 3 are new and are similar to the corresponding results in [14].

## 2. $\alpha$ -SEMI SHORT MODULES AND $\alpha$ -ALMOST SEMI NOETHERIAN MODULES

We recall that an  $R$ -module  $M$  is called  $\alpha$ -almost Noetherian, if for each proper submodule  $N$  of  $M$ ,  $n\text{-dim } N < \alpha$  and  $\alpha$  is the least ordinal

number with this property. In the following definition we consider a related concept.

**Definition 2.1.** An  $R$ -module  $M$  is called  $\alpha$ -semi Noetherian if for each proper finitely generated submodule  $N$  of  $M$ ,  $dp\text{-dim } N < \alpha$  and  $\alpha$  is the least ordinal number with this property.

It is manifest that if  $M$  is an  $\alpha$ -semi Noetherian module, then each submodule and each factor module of  $M$  is  $\beta$ -semi Noetherian for some  $\beta \leq \alpha$  (note, see [13, Lemmas 2.5, 2.10]).

In view of [13, Proposition 2.7], we have the next three trivial, but useful facts.

**Lemma 2.2.** *If  $M$  is an  $\alpha$ -semi Noetherian module, then  $M$  has dual perfect dimension and  $dp\text{-dim } M \leq \alpha$ . In particular,  $dp\text{-dim } M = \alpha$  if and only if  $M$  is  $\alpha$ -perfect atomic.*

**Lemma 2.3.** *If  $M$  is a module with  $dp\text{-dim } M = \alpha$ , then either  $M$  is  $\alpha$ -perfect atomic, in which case it is  $\alpha$ -semi Noetherian, or it is  $\alpha + 1$ -semi Noetherian.*

**Lemma 2.4.** *If  $M$  is an  $\alpha$ -semi Noetherian module, then either  $M$  is  $\alpha$ -perfect atomic or  $\alpha = dp\text{-dim } M + 1$ . In particular, if  $M$  is  $\alpha$ -semi Noetherian module, where  $\alpha$  is a limit ordinal, then  $M$  is  $\alpha$ -perfect atomic.*

**Proposition 2.5.** *An  $R$ -module  $M$  has dual perfect dimension if and only if  $M$  is  $\alpha$ -semi Noetherian for some ordinal  $\alpha$ .*

Next, we give our definition of  $\alpha$ -semi short modules.

**Definition 2.6.** An  $R$ -module  $M$  is called  $\alpha$ -semi short module, if for each finitely generated submodule  $N$  of  $M$ , either  $dp\text{-dim } N \leq \alpha$  or  $dp\text{-dim } \frac{M}{N} \leq \alpha$  and  $\alpha$  is the least ordinal number with this property.

In view of [13, Corollary 2.13], we have the following results.

*Remark 2.7.* If  $M$  is an  $R$ -module with  $dp\text{-dim } M = \alpha$ , then  $M$  is  $\beta$ -semi short for some  $\beta \leq \alpha$ .

*Remark 2.8.* If  $M$  is an  $\alpha$ -semi short module, then each submodule and each factor module of  $M$  is  $\beta$ -semi short for some  $\beta \leq \alpha$ .

We cite the following result from [13, Proposition 2.9].

**Lemma 2.9.** *If  $M$  is an  $R$ -module and for each finitely generated submodule  $N$  of  $M$ , either  $N$  or  $\frac{M}{N}$  has dual perfect dimension, then so does  $M$ .*

The previous lemma and Remark 2.7, immediately yield the next result.

**Corollary 2.10.** *Let  $M$  be an  $\alpha$ -semi short module. Then  $M$  has dual perfect dimension and  $\alpha \leq dp\text{-dim } M$ .*

The following is now immediate.

**Proposition 2.11.** *An  $R$ -module  $M$  has dual perfect dimension if and only if  $M$  is  $\alpha$ -semi short for some ordinal  $\alpha$ .*

**Proposition 2.12.** *If  $M$  is an  $\alpha$ -semi short  $R$ -module, then either  $dp\text{-dim } M = \alpha$  or  $dp\text{-dim } M = \alpha + 1$ .*

*Proof.* Clearly in view of Corollary 2.10, we have  $dp\text{-dim } M \geq \alpha$ . If  $dp\text{-dim } M \neq \alpha$ , then  $dp\text{-dim } M \geq \alpha + 1$ . Now let  $M_1 \subseteq M_2 \subseteq \dots$  be any ascending chain of finitely generated submodules of  $M$ . If there exists some  $k$  such that  $dp\text{-dim } \frac{M}{M_k} \leq \alpha$ , then  $dp\text{-dim } \frac{M_{i+1}}{M_i} \leq dp\text{-dim } \frac{M}{M_i} = dp\text{-dim } \frac{M/M_k}{M_i/M_k} \leq dp\text{-dim } \frac{M}{M_k} \leq \alpha$  for each  $i \geq k$ , see [13, Corollary 2.13]. Otherwise  $dp\text{-dim } M_i \leq \alpha$  ( $M$  is  $\alpha$ -semi short) for each  $i$ , hence  $dp\text{-dim } \frac{M_{i+1}}{M_i} \leq dp\text{-dim } M_{i+1} \leq \alpha$  for each  $i$ . Thus in any case there exists an integer  $k$  such that for each  $i \geq k$ ,  $dp\text{-dim } \frac{M_{i+1}}{M_i} \leq \alpha$ . This shows that  $dp\text{-dim } M \leq \alpha + 1$ , i.e.,  $dp\text{-dim } M = \alpha + 1$ .  $\square$

*Remark 2.13.* An  $R$ -module  $M$  is  $-1$ -semi short if and only if it is simple.

**Proposition 2.14.** *Let  $M$  be an  $R$ -module, with  $dp\text{-dim } M = \alpha$ , where  $\alpha$  is a limit ordinal. Then  $M$  is  $\alpha$ -semi short.*

*Proof.* We know that  $M$  is  $\beta$ -semi short for some  $\beta \leq \alpha$ . If  $\beta < \alpha$ , then by Proposition 2.12,  $dp\text{-dim } M \leq \beta + 1 < \alpha$ . Which is a contradiction. Thus  $M$  is  $\alpha$ -semi short.  $\square$

**Proposition 2.15.** *Let  $M$  be an  $R$ -module and  $dp\text{-dim } M = \alpha = \beta + 1$ . Then  $M$  is either  $\alpha$ -semi short or it is  $\beta$ -semi short.*

*Proof.* We know that  $M$  is  $\gamma$ -semi short for some  $\gamma \leq \alpha$ . If  $\gamma < \beta$ , then by Proposition 2.12, we have  $dp\text{-dim } M \leq \gamma + 1 < \beta + 1$ , which is impossible. Hence we are done.  $\square$

**Proposition 2.16.** *Let  $M$  be an  $\alpha$ -perfect atomic  $R$ -module, where  $\alpha = \beta + 1$ , then  $M$  is a  $\beta$ -semi short module.*

*Proof.* Let  $N$  be a finitely generated submodule of  $M$ . Hence, we have  $dp\text{-dim } N < \alpha$ . This shows that for some  $\beta' \leq \beta$ ,  $M$  is  $\beta'$ -semi short. If  $\beta' < \beta$ , then  $\beta' + 1 \leq \beta < \alpha$ . But  $dp\text{-dim } M \leq \beta' + 1 \leq \beta < \alpha$ , by Proposition 2.12, which is a contradiction. Thus  $\beta' = \beta$  and we are done.  $\square$

The following remark, which is a trivial consequence of the previous fact, shows that the converse of Proposition 2.14, is not true in general.

*Remark 2.17.* Let  $M$  be an  $\alpha + 1$ -perfect atomic  $R$ -module, where  $\alpha$  is a limit ordinal. Then  $M$  is an  $\alpha$ -semi short module but  $dp\text{-dim } M \neq \alpha$ .

**Proposition 2.18.** *Let  $M$  be an  $R$ -module such that  $dp\text{-dim } M = \alpha + 1$ . Then  $M$  is either  $\alpha$ -semi short  $R$ -module or there exists a finitely generated submodule  $N$  of  $M$  such that  $dp\text{-dim } N = dp\text{-dim } \frac{M}{N} = \alpha + 1$ .*

*Proof.* We know that  $M$  is  $\alpha$ -semi short or an  $\alpha + 1$ -semi short  $R$ -module, by Proposition 2.15. Let us assume that  $M$  is not  $\alpha$ -semi short  $R$ -module, hence there exists a finitely generated submodule  $N$  of  $M$  such that  $dp\text{-dim } N \geq \alpha + 1$  and  $dp\text{-dim } \frac{M}{N} \geq \alpha + 1$ . This shows that  $dp\text{-dim } N = \alpha + 1$  and  $dp\text{-dim } \frac{M}{N} = \alpha + 1$  and we are through.  $\square$

**Proposition 2.19.** *Let  $M$  be a non-zero  $\alpha$ -semi short  $R$ -module. Then either  $M$  is  $\beta$ -semi Noetherian for some ordinal  $\beta \leq \alpha + 1$  or there exists a finitely generated submodule  $N$  of  $M$  with  $dp\text{-dim } \frac{M}{N} \leq \alpha$ .*

*Proof.* Suppose that  $M$  is not  $\beta$ -semi Noetherian for any  $\beta \leq \alpha + 1$ . This means that there must exist a finitely generated submodule  $N$  of  $M$  such that  $dp\text{-dim } N \not\leq \alpha$ . Inasmuch as  $M$  is  $\alpha$ -semi short, we infer that  $dp\text{-dim } \frac{M}{N} \leq \alpha$  and we are done.  $\square$

Finally we conclude this section by providing some examples of  $\alpha$ -semi Noetherian (resp.,  $\alpha$ -semi short) modules, where  $\alpha$  is any ordinal. Recall that a left  $R$ -module  $M$ , (note,  $R$  is not necessarily commutative) is called uniserial if its submodules are linearly ordered by inclusion. A serial module is a module that is a direct sum of uniserial modules. First, we recall that given any ordinal  $\alpha$  there exists an Artinian serial module  $M$  such that  $n\text{-dim } M = \alpha$ , see [22, Example 1] and [15, Lemma 2.4]. Thus  $dp\text{-dim } M = \alpha$ , see [13, Corollary 4.4]. Consequently, we may take  $M$  to be an Artinian serial module with  $dp\text{-dim } M = \alpha$ . Hence  $dp\text{-dim } M = \alpha$  and for any ordinal  $\beta \leq \alpha$ , we take  $N$  to be its  $\beta$ -perfect atomic submodule, see [13, Corollary 3.10], then by Lemma 2.3,  $N$  is  $\beta$ -semi Noetherian. We recall that the only  $\alpha$ -semi Noetherian modules, where  $\alpha$  is a limit ordinal, are  $\alpha$ -perfect atomic modules, see Lemma 2.4. Therefore to see an example of  $\alpha$ -semi Noetherian module which is not  $\alpha$ -perfect atomic, the ordinal  $\alpha$  must be a non-limit ordinal. Thus we may take  $M$  to be a non-perfect atomic module with  $dp\text{-dim } M = \beta$ , where  $\alpha = \beta + 1$ , hence it follows trivially that  $M$  is an  $\alpha$ -semi Noetherian. As for examples of  $\alpha$ -semi short modules, one can similarly use the facts that there are Artinian serial modules  $M$  with Noetherian dimension equal to  $\alpha$ , see [22, 15].

In view of [13, Corollary 4.4], we infer that  $dp\text{-dim } M = \alpha$ . By [13, Corollary 3.10], for each  $\beta \leq \alpha$  there are  $\beta$ -perfect atomic submodules of  $M$  and then apply Propositions 2.14, 2.15, 2.16, to give various examples of  $\alpha$ -semi short modules (for example, by Proposition 2.16, every  $\alpha + 1$ -perfect atomic module is  $\alpha$ -semi short).

### 3. PROPERTIES OF $\alpha$ -SEMI SHORT MODULES AND $\alpha$ -SEMI NOETHERIAN MODULES

In this section some properties of  $\alpha$ -semi short modules over an arbitrary ring  $R$  are investigated.

In the following two propositions we investigate the connection between  $\alpha$ -short modules and  $\alpha$ -semi short modules, where  $M$  is an Artinian serial module.

**Proposition 3.1.** *Let  $M$  be an Artinian serial  $R$ -module. If  $M$  is a  $\beta$ -semi short module, then  $M$  is  $\alpha$ -short for some  $\alpha \leq \beta + 1$ .*

*Proof.* In view of Proposition 2.12, we get  $dp\text{-dim } M \leq \beta + 1$ . Thus by [13, Corollary 4.4], we have  $n\text{-dim } M \leq \beta + 1$ . This shows that  $M$  is an  $\alpha$ -short module for some  $\alpha \leq \beta + 1$ , see [14, Remark 1.2].  $\square$

**Proposition 3.2.** *If  $M$  is an  $\alpha$ -short  $R$ -module, then it is  $\beta$ -semi short for some  $\beta \leq \alpha$ .*

*Proof.* Let  $N$  be a finitely generated submodule of  $M$ , then  $n\text{-dim } N \leq \alpha$  or  $n\text{-dim } \frac{M}{N} \leq \alpha$  (note,  $M$  is  $\alpha$ -short). In view of [13, Lemma 2.3], we infer that  $dp\text{-dim } N \leq \alpha$  or  $dp\text{-dim } \frac{M}{N} \leq \alpha$ . This implies that  $M$  is  $\beta$ -semi short for some  $\beta \leq \alpha$ .  $\square$

In view of Propositions 3.1 and 3.2, we have the following corollary.

**Corollary 3.3.** *Let  $M$  be an Artinian serial  $R$ -module and  $\alpha$  and  $\beta$  are ordinal numbers. Then  $M$  is  $\beta$ -semi short if and only if it is  $\alpha$ -short, where  $\beta \leq \alpha \leq \beta + 1$ .*

The next example shows that in the previous corollary all the cases for  $\alpha$  can occur.

**Example 3.4.** Let  $\mathbb{Z}$  be the ring of integers. Then the  $\mathbb{Z}$ -module  $\mathbb{Z}_{p^\infty}$  is both 0-short and 0-semi short. And the  $\mathbb{Z}$ -module  $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$  is 1-short but it is 0-semi short.

In view of Corollary 3.3, we have the following result.

**Corollary 3.5.** *If  $M$  is an  $\alpha$ -short module, where  $\alpha$  is a limit ordinal number, then  $M$  is  $\alpha$ -semi short.*

**Proposition 3.6.** *Let  $R$  be a ring and  $M$  be a nonzero  $\alpha$ -semi short module, which is not a perfect atomic module, then  $M$  contains a finitely generated submodule  $L$  such that  $dp\text{-dim } \frac{M}{L} \leq \alpha$ .*

*Proof.* Since  $M$  is not perfect atomic, we infer that there exists a finitely generated submodule  $L \subsetneq M$ , such that  $dp\text{-dim } L = dp\text{-dim } M$ . We know that  $dp\text{-dim } M = \alpha$  or  $dp\text{-dim } M = \alpha + 1$ , by Proposition 2.12. If  $dp\text{-dim } M = \alpha$  it is clear that  $dp\text{-dim } \frac{M}{L} \leq \alpha$ . Hence we may suppose that  $dp\text{-dim } L = dp\text{-dim } M = \alpha + 1$ . Consequently,  $dp\text{-dim } \frac{M}{L} \leq \alpha$  and we are done.  $\square$

**Theorem 3.7.** *Let  $\alpha$  be an ordinal number and  $M$  be an  $R$ -module. If every proper finitely generated submodule of  $M$  is  $\gamma$ -semi short for some ordinal number  $\gamma \leq \alpha$ . Then  $dp\text{-dim } M \leq \alpha + 2$ , in particular,  $M$  is  $\mu$ -semi short for some ordinal  $\mu \leq \alpha + 1$ .*

*Proof.* Let  $N \subsetneq M$  be any finitely generated submodule of  $M$ . Since  $N$  is  $\gamma$ -semi short for some ordinal number  $\gamma \leq \alpha$ , we infer that  $dp\text{-dim } N \leq \gamma + 1 \leq \alpha + 1$ , by Proposition 2.12. This immediately implies that  $dp\text{-dim } M \leq \alpha + 2$ , see [13, Proposition 2.7]. The final part is now evident.  $\square$

The next result is the dual of Theorem 3.7.

**Theorem 3.8.** *Let  $M$  be a nonzero  $R$ -module and  $\alpha$  be an ordinal number. Let for every non-zero finitely generated submodule  $N$  of  $M$ ,  $\frac{M}{N}$  be  $\gamma$ -semi short for some ordinal number  $\gamma \leq \alpha$ . Then  $dp\text{-dim } M \leq \alpha + 1$ , in particular,  $M$  is  $\mu$ -semi short for some ordinal  $\mu \leq \alpha + 1$ .*

*Proof.* Let  $N$  be any non-zero finitely generated submodule of  $M$ , then  $\frac{M}{N}$  is  $\gamma$ -semi short for some ordinal number  $\gamma \leq \alpha$ . In view of Proposition 2.12, we infer that  $dp\text{-dim } \frac{M}{N} \leq \gamma + 1 \leq \alpha + 1$ . Therefore  $dp\text{-dim } M \leq \sup\{dp\text{-dim } \frac{M}{N} : 0 \neq N \subseteq M, N \text{ is f.g.}\} \leq \alpha + 1$ , see [13, Proposition 2.6]. The final part is now evident.  $\square$

The next immediate result is the counterparts of Theorems 3.7, 3.8, for  $\alpha$ -semi Noetherian modules.

**Proposition 3.9.** *Let  $M$  be an  $R$ -module and  $\alpha$  be an ordinal number. If each proper finitely generated submodule  $N$  of  $M$  (resp., for each non-zero finitely generated submodule  $N$  of  $M$ ,  $\frac{M}{N}$ ) is  $\gamma$ -semi Noetherian with  $\gamma \leq \alpha$ , then  $M$  is a  $\mu$ -semi Noetherian module with  $\mu \leq \alpha + 1$  and  $dp\text{-dim } M \leq \alpha + 1$  (resp., with  $\mu \leq \alpha + 1$  and  $dp\text{-dim } M \leq \alpha$ ).*

**Proposition 3.10.** *Let  $R$  be a semiprime right Goldie ring. Then the right  $R$ -module  $R$  is  $\alpha$ -semi short if and only if  $dp\text{-dim } R = \alpha$ .*

*Proof.* Let  $R$  be  $\alpha$ -semi short as an  $R$ -module. We are to show that  $dp\text{-dim } R = \alpha$ . If for each essential right ideal  $E$  of  $R$ ,  $dp\text{-dim } \frac{R}{E} \leq \alpha$  then  $dp\text{-dim } R = \sup\{dp\text{-dim } \frac{R}{E} : E \subseteq_e R\} \leq \alpha$ , see [13, Proposition 2.15]. Since  $R$  is  $\alpha$ -semi short we have  $dp\text{-dim } R = \alpha$ , by Proposition 2.12. Now suppose that there exists an essential right ideal  $E'$  of  $R$  such that  $dp\text{-dim } \frac{R}{E'} \not\leq \alpha$ . But  $R$  is a right Goldie ring, hence there exists a regular element  $c$  in  $E'$ , which implies that  $dp\text{-dim } \frac{R}{cR} \not\leq \alpha$ , see [13, Lemma 2.10]. Thus  $dp\text{-dim } R = dp\text{-dim } cR \leq \alpha$ , see [13, Lemma 2.5]. Consequently, we must have  $dp\text{-dim } R = \alpha$ , by Proposition 2.12. Conversely, by Remark 2.7,  $R$  is  $\beta$ -semi short for some  $\beta \leq \alpha$ . But, by the first part of the proof, we must have  $dp\text{-dim } R = \beta$ , i.e.,  $\beta = \alpha$ , and we are through.  $\square$

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ON  $\alpha$ -SEMI SHORT MODULES

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درباره‌ی مدول‌های  $\alpha$  - شبه کوتاه

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در این مقاله مفهوم مدول‌های  $\alpha$  - شبه کوتاه معرفی شده است. با استفاده از این مفهوم، برخی از نتایج مدول‌های  $\alpha$  - کوتاه به مدول‌های  $\alpha$  - شبه کوتاه تعمیم داده شده است. نشان می‌دهیم اگر  $M$  یک مدول  $\alpha$  - کوتاه باشد، آنگاه دوگان بعد تام دارد و دوگان بعد تام آن  $\alpha + 1$  یا  $\alpha$  است.

کلمات کلیدی: مدول‌های  $\alpha$  - کوتاه، مدول‌های  $\alpha$  - تقریباً نوتری، مدول‌های  $\alpha$  - شبه کوتاه، بعد نوتری، دوگان بعد تام، بعد کرول.