

ON GRADED INJECTIVE DIMENSION

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ABSTRACT. There are remarkable relations between the graded homological dimensions and the ordinary homological dimensions. In this paper, we study the injective dimension of a complex of graded modules and derive its some properties. In particular, we define the $*$ dualizing complex for a graded ring and investigate its consequences.

1. INTRODUCTION

Let R be a Noetherian \mathbb{Z} -graded ring. In [5] and [6], Fossum and Fossum-Foxby have studied the graded homological dimension of graded modules and compare them with classical homological dimensions. They showed that for a graded R -module M , one has

$$*id_R M \leq id_R M \leq *id_R M + 1,$$

where $id_R M$ (resp., $*id_R M$) denotes the injective dimension of M in the category of R -modules (resp., category of graded R -modules). It is natural to ask how these inequalities hold for the injective dimension of a complex of graded modules and homogeneous homomorphisms. Section 2 of this paper is devoted to review some hyper-homological algebra for the derived category of the graded ring R . In Section 3, we define the $*$ injective dimension of complexes of graded modules and homogeneous homomorphisms, and derive its some properties. Among other results, we prove the generalization of the dual version of Auslander-Buchbaum equality, which implies the known inequality

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$*\text{id}_R X \leq \text{id}_R X \leq *\text{id}_R X + 1$ for a complex of graded modules and homogeneous homomorphisms X . Throughout this paper, R is commutative and all complexes are chain complexes, that is their indexes increase to left. For more details on graded rings and modules, see [2] and [6].

2. DERIVED CATEGORY OF COMPLEXES OF GRADED MODULES

Let X be a complex of R -modules and R -homomorphisms. The *supremum* and the *infimum* of a complex X , denoted by $\text{sup}(X)$ and $\text{inf}(X)$, are defined by the supremum and the infimum of the set $\{i \in \mathbb{Z} \mid H_i(X) \neq 0\}$. For an integer m , $\Sigma^m X$ denotes the complex X shifted m degrees to the left; it is given by

$$(\Sigma^m X)_\ell = X_{\ell-m} \text{ and } \partial_\ell^{\Sigma^m X} = (-1)^m \partial_{\ell-m}^X,$$

for $\ell \in \mathbb{Z}$.

The symbol $\mathcal{D}(R)$ denotes the *derived category* of R -complexes. The full subcategories $\mathcal{D}_\square(R)$, $\mathcal{D}_{\square}(R)$, $\mathcal{D}_{\square}(R)$ and $\mathcal{D}_0(R)$ of $\mathcal{D}(R)$ consist of R -complexes X while $H_\ell(X) = 0$, for respectively $\ell \gg 0$, $\ell \ll 0$, $|\ell| \gg 0$ and $\ell \neq 0$. Homology isomorphisms are marked by the sign \simeq . The right derived functor of the homomorphism functor of R -complexes and the left derived functor of the tensor product of R -complexes are denoted by $\mathbf{R}\text{Hom}_R(-, -)$ and $- \otimes_R^{\mathbf{L}} -$, respectively.

Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ and $N = \bigoplus_{n \in \mathbb{Z}} N_n$ be two graded R -modules. The $*$ Hom functor is defined by $*\text{Hom}_R(M, N) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_i(M, N)$, such that $\text{Hom}_i(M, N)$ is a \mathbb{Z} -submodule of $\text{Hom}_R(M, N)$ consisting of all $\varphi : M \rightarrow N$ such that $\varphi(M_n) \subseteq N_{n+i}$ for all $n \in \mathbb{Z}$. In general, $*\text{Hom}_R(M, N) \neq \text{Hom}_R(M, N)$, but equality holds if M is finitely generated, see [6, Lemma 4.2]. Also, the tensor product $M \otimes_R N$ of M and N is a graded module with $(M \otimes_R N)_n$ is generated (as a \mathbb{Z} -module) by elements $m \otimes n$ with $m \in M_i$ and $n \in N_j$ where $i + j = n$.

Let $\{M_\alpha\}_{\alpha \in I}$ be a family of graded R -modules. Then, $\bigoplus_\alpha M_\alpha$ becomes a graded R -module with $(\bigoplus_\alpha M_\alpha)_n = \bigoplus_\alpha (M_\alpha)_n$, for all $n \in \mathbb{Z}$, see [6, Page 289]. Recall that the direct products exist in the category of graded modules. Then the direct product is denoted by $*\prod_\alpha M_\alpha$ and $(*\prod_\alpha M_\alpha)_n = \prod_\alpha (M_\alpha)_n$ for all $n \in \mathbb{Z}$, see [6, Page 289]. In this case, there are the following bijections [6, Page 289]

$$\begin{aligned} *\text{Hom}_R\left(\bigoplus_\alpha M_\alpha, -\right) &\xrightarrow{\cong} *\prod_\alpha *\text{Hom}_R(M_\alpha, -), \\ *\text{Hom}_R\left(-, *\prod_\alpha M_\alpha\right) &\xrightarrow{\cong} *\prod_\alpha *\text{Hom}_R(-, M_\alpha). \end{aligned}$$

Likewise, direct limits exist in the category of graded modules with

$$(*\varinjlim M_\alpha)_n = \varinjlim (M_\alpha)_n$$

for all $n \in \mathbb{Z}$, see [6, Page 289].

The symbol $*\mathcal{C}(R)$ denotes the category of complexes of graded R -modules and homogeneous differentials. Remember that the category of graded modules is an abelian category. The derived category of $*\mathcal{C}(R)$ will be denoted by $*\mathcal{D}(R)$, (see [10]). Analogously we have $*\mathcal{C}_\square(R)$, $*\mathcal{C}_\square(R)$, $*\mathcal{C}_\square(R)$ and $*\mathcal{C}_0(R)$ (resp. $*\mathcal{D}_\square(R)$, $*\mathcal{D}_\square(R)$, $*\mathcal{D}_\square(R)$ and $*\mathcal{D}_0(R)$) which are the full subcategories of $*\mathcal{C}(R)$ (resp. $*\mathcal{D}(R)$).

For R -complexes X and Y of graded modules, with homogeneous differentials ∂^X and ∂^Y the *homomorphism complex* $*\text{Hom}_R(X, Y)$ is defined as:

$$*\text{Hom}_R(X, Y)_\ell = * \prod_{p \in \mathbb{Z}} *\text{Hom}_R(X_p, Y_{p+\ell})$$

and when $\psi = (\psi_p)_{p \in \mathbb{Z}}$ belongs to $*\text{Hom}_R(X, Y)_\ell$, then the family $\partial_\ell^{*\text{Hom}_R(X, Y)}(\psi)$ in $*\text{Hom}_R(X, Y)_{\ell-1}$ has p -th component

$$\partial_\ell^{*\text{Hom}_R(X, Y)}(\psi)_p = \partial_{p+\ell}^Y \psi_p - (-1)^\ell \psi_{p-1} \partial_p^X.$$

When $X \in *\mathcal{C}_\square^f(R)$ and $Y \in *\mathcal{C}_\square(R)$ all the products

$$* \prod_{p \in \mathbb{Z}} *\text{Hom}_R(X_p, Y_{p+\ell})$$

are finite. Note that for each $p \in \mathbb{Z}$, X_p is finitely generated R -module, thus $*\text{Hom}_R(X_p, Y_{p+\ell}) = \text{Hom}_R(X_p, Y_{p+\ell})$, see [6, Lemma 4.2]. Therefore

$$* \prod_{p \in \mathbb{Z}} *\text{Hom}_R(X_p, Y_{p+\ell}) = \prod_{p \in \mathbb{Z}} \text{Hom}_R(X_p, Y_{p+\ell}),$$

for every $\ell \in \mathbb{Z}$. Therefore $*\text{Hom}_R(X, Y) = \text{Hom}_R(X, Y)$.

Also the *tensor product complex* $X \otimes_R Y$ is defined as:

$$(X \otimes_R Y)_\ell = \bigoplus_{p \in \mathbb{Z}} (X_p \otimes_R Y_{\ell-p})$$

and the ℓ -th differential $\partial_\ell^{X \otimes_R Y}$ is given on a generator $x_p \otimes y_{\ell-p}$ in $(X \otimes_R Y)_\ell$, where x_p and $y_{\ell-p}$ are homogeneous elements, by

$$\partial_\ell^{X \otimes_R Y}(x_p \otimes y_{\ell-p}) = \partial_p^X(x_p) \otimes y_{\ell-p} + (-1)^p x_p \otimes \partial_{\ell-p}^Y(y_{\ell-p}).$$

If X and Y are R -complexes of graded modules, then $*\text{Hom}_R(X, -)$, $*\text{Hom}_R(-, Y)$, and $X \otimes_R -$ are graded functors on $*\mathcal{C}(R)$.

Note that any object of $*\mathcal{C}_\square(R)$ has an **injective resolution* by [10, Page 47], and any object of $*\mathcal{C}_\square(R)$ has an **projective resolution* by [10, Page 48]. The right derived functor of the $*\text{Hom}$ functor in the

category of graded complexes is denoted by $\mathbf{R}^* \text{Hom}_R(-, -)$ and set ${}^* \text{Ext}_R^i(-, -) = H_{-i}(\mathbf{R}^* \text{Hom}_R(-, -))$. It is easily seen that if R is a Noetherian \mathbb{Z} -graded ring and $X \in {}^* \mathcal{C}_{\square}^f(R)$ and $Y \in {}^* \mathcal{C}_{\square}(R)$ then $\mathbf{R}^* \text{Hom}_R(X, Y) = \mathbf{R} \text{Hom}_R(X, Y)$. Also the left derived functor of $- \otimes_R -$ in the category of graded complexes is denoted by $- \otimes_R^{\mathbf{L}^*} -$. Since * projective graded R -modules coincide with projective R -modules by [6, Proposition 3.1] we easily see that $- \otimes_R^{\mathbf{L}^*} -$ coincides with the ordinary left derived functor of $- \otimes_R -$ in the category of complexes. So we use $- \otimes_R^{\mathbf{L}} -$ instead of $- \otimes_R^{\mathbf{L}^*} -$.

We recall the definition of the *depth* and *width* of complexes. Let \mathfrak{a} be an ideal in a ring R and X a complex of graded R -modules. The \mathfrak{a} -depth and \mathfrak{a} -width of X over R are defined respectively by

$$\begin{aligned} \text{depth}(\mathfrak{a}, X) &:= - \sup \mathbf{R} \text{Hom}_R(R/\mathfrak{a}, X), \\ \text{width}(\mathfrak{a}, X) &:= \inf(R/\mathfrak{a} \otimes_R^{\mathbf{L}} X). \end{aligned}$$

For a local ring (R, \mathfrak{m}) set $\text{depth}_R X := \text{depth}(\mathfrak{m}, X)$; $\text{width}_R X := \text{width}(\mathfrak{m}, X)$. Let (R, \mathfrak{m}) be a * local graded ring and X be a complex of graded R -modules. By [2, Proposition 1.5.15(c)], $- \otimes_R R_{\mathfrak{m}}$ is a faithfully exact functor on the category of graded R -modules. Then we have

$$\begin{aligned} \text{width}(\mathfrak{m}, X) &= \inf \{i | H_i(R/\mathfrak{m} \otimes_R^{\mathbf{L}} X) \neq 0\} \\ &= \inf \{i | H_i(R/\mathfrak{m} \otimes_R^{\mathbf{L}} X) \otimes_R R_{\mathfrak{m}} \neq 0\} \\ &= \inf \{i | H_i(R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}}^{\mathbf{L}} X_{\mathfrak{m}}) \neq 0\} \\ &= \text{width}(\mathfrak{m}R_{\mathfrak{m}}, X_{\mathfrak{m}}) = \text{width}_{R_{\mathfrak{m}}} X_{\mathfrak{m}}. \end{aligned}$$

Likewise we have $\text{depth}(\mathfrak{m}, X) = \text{depth}_{R_{\mathfrak{m}}} X_{\mathfrak{m}}$.

3. * INJECTIVE DIMENSION

The injective dimension of a complex X , denoted by $\text{id}_R X$, is defined and studied in [1]. A graded module J is called * injective if it is an injective object in the category of graded modules. In other words, the functor ${}^* \text{Hom}_R(-, J)$ is exact in this category. A long exact sequence of * injective modules is called * injective resolution. The injective dimension of a graded module M in the category of graded modules, is denoted by ${}^* \text{id}_R M$ (cf. [6, 2]). In this section we study the * injective dimension of homologically left bounded complexes of graded modules.

Let $n \in \mathbb{Z}$. A homologically left bounded complex of graded modules X , is said to have * injective dimension at most n , denoted by ${}^* \text{id}_R X \leq n$, if there exists an * injective resolution $X \rightarrow I$, such that $I_i = 0$ for

$i < -n$. If ${}^* \text{id}_R X \leq n$ holds, but ${}^* \text{id}_R X \leq n - 1$ does not, we write ${}^* \text{id}_R X = n$. If ${}^* \text{id}_R X \leq n$ for all $n \in \mathbb{Z}$ we write ${}^* \text{id}_R X = -\infty$. If ${}^* \text{id}_R X \leq n$ for no $n \in \mathbb{Z}$ we write ${}^* \text{id}_R X = \infty$. The following theorem inspired by [1, Theorem 2.4.I and Corollary 2.5.I].

Theorem 3.1. *For $X \in {}^* \mathcal{D}_{\square}(R)$ and $n \in \mathbb{Z}$ the following are equivalent:*

- (1) ${}^* \text{id}_R X \leq n$.
- (2) $n \geq -\sup U - \inf(\mathbf{R}^* \text{Hom}_R(U, X))$ for all $U \in {}^* \mathcal{D}_{\square}(R)$ and $H(U) \neq 0$.
- (3) $n \geq -\inf X$ and ${}^* \text{Ext}_R^{n+1}(R/J, X) = 0$ for every homogeneous ideal J of R .
- (4) $n \geq -\inf X$ and for any (resp. some) * injective resolution I of X , the graded R -module $\text{Ker}(\partial_{-n} : I_{-n} \rightarrow I_{-n-1})$ is * injective.

Moreover the following holds:

$${}^* \text{id}_R X = \sup\{j \in \mathbb{Z} \mid {}^* \text{Ext}_R^j(R/J, X) \neq 0 \text{ for some homogeneous ideal } J\} \\ = \sup\{-\sup(U) - \inf(\mathbf{R}^* \text{Hom}_R(U, X)) \mid U \not\cong 0 \text{ in } {}^* \mathcal{D}_{\square}(R)\}.$$

Proof. (1) \Rightarrow (2) Let $t := \sup U$ and I be an * injective resolution of X , such that, for all $i < -n$, $I_i = 0$. Then we have

$${}^* \text{Ext}_R^i(U, X) \cong H_{-i}({}^* \text{Hom}_R(U, I)).$$

Since ${}^* \text{Hom}_R(U, I)_{-i} = 0$ for $-i < -n - t$, the assertion follows.

(2) \Rightarrow (3) It is trivial that ${}^* \text{Ext}_R^{n+1}(R/J, X) = 0$ for every homogeneous ideal J of R . For the second assertion let $U = R$ in (2). So that $\text{Ext}_R^i(R, X) = {}^* \text{Ext}_R^i(R, X) = 0$ for $i > n$. Now by [1, Lemma 1.9(b)], we have $H_{-i}(X) = 0$ for $-i < -n$. This means that $n \geq -\inf X$.

(3) \Rightarrow (4) By hypothesis of (4) $H_i(I) = 0$ for $i < -n$. Thus the complex

$$\cdots \rightarrow 0 \rightarrow 0 \rightarrow I_{-n} \rightarrow I_{-n-1} \rightarrow \cdots \rightarrow I_i \rightarrow I_{i-1} \rightarrow \cdots$$

gives an * injective resolution of $\text{Ker } \partial_{-n}$. In particular

$${}^* \text{Ext}_R^1(R/J, \text{Ker } \partial_{-n}) = H_{-n-1} {}^* \text{Hom}_R(R/J, I) = {}^* \text{Ext}_R^{n+1}(R/J, X) = 0$$

for every homogeneous ideal J of R . Thus $\text{Ker } \partial_{-n}$ is * injective by [6, Corollary 4.3].

(4) \Rightarrow (1) Let I be any * injective resolution of X . By assumption, the module $\text{Ker } \partial_{-n}$ is * injective. Thus ${}^* \text{id}_R X < -n$ by definition.

The last equalities are easy consequences of (1), ..., (4). □

For a local ring (R, \mathfrak{m}, k) and for an R -complex X and $i \in \mathbb{Z}$ the i th *Bass number* and *Betti number* of X are defined respectively by

$\mu_R^i(X) := \dim_k H_{-i}(\mathbf{R}\text{Hom}_R(k, X))$ and $\beta_i^R(X) := \dim_k H_i(k \otimes_R^{\mathbf{L}} X)$. It is well-known that for $X \in \mathcal{D}_{\square}(R)$ one has (cf. [1, Proposition 5.3.I])

$$\text{id}_R X = \sup\{m \in \mathbb{Z} \mid \exists \mathfrak{p} \in \text{Spec}(R); \mu_{R_{\mathfrak{p}}}^m(X_{\mathfrak{p}}) \neq 0\}.$$

As a graded analogue we have:

Proposition 3.2. *For $X \in {}^* \mathcal{D}_{\square}(R)$ we have the following equality*

$${}^* \text{id}_R X = \sup\{m \in \mathbb{Z} \mid \exists \mathfrak{p} \in {}^* \text{Spec}(R); \mu_{R_{\mathfrak{p}}}^m(X_{\mathfrak{p}}) \neq 0\}.$$

Proof. The argument is the same as proof of [1, Proposition 5.3.I] with some changes. Denote the supremum by i . By Theorem 3.1, we have ${}^* \text{id}_R X \geq i$. Hence the equality holds if $i = \infty$. Thus assume that i is finite. By Theorem 3.1 we have to show that if ${}^* \text{Ext}_R^j(M, X) \neq 0$ for some finitely generated graded R -module M , then $j \leq i$; this implies that ${}^* \text{id}_R X \leq i$. The elements of $\text{Ass}(M)$ are homogeneous prime ideals. Thus there exists a filtration $0 = M_0 \subset M_1 \subset \dots \subset M_t = M$ of graded submodules of M such that for each i we have $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ with $\mathfrak{p}_i \in \text{Supp } M$ and \mathfrak{p}_i is homogeneous. From the long exact sequence of ${}^* \text{Ext}_R^j(-, X) \neq 0$ the set

$$\{\mathfrak{q} \in \text{Spec}(R) \mid \text{there is an } h \geq j \text{ such that } {}^* \text{Ext}_R^h(R/\mathfrak{q}, X) \neq 0\},$$

turns to be non empty. Let \mathfrak{p} be maximal in this set and for a homogeneous $x \in R \setminus \mathfrak{p}$ consider the exact sequence

$$0 \rightarrow R/\mathfrak{p} \xrightarrow{x} R/\mathfrak{p} \rightarrow R/(\mathfrak{p} + Rx) \rightarrow 0.$$

It induces an exact sequence

$$\begin{aligned} & {}^* \text{Ext}_R^h(R/(\mathfrak{p} + Rx), X) \rightarrow {}^* \text{Ext}_R^h(R/\mathfrak{p}, X) \xrightarrow{x} {}^* \text{Ext}_R^h(R/\mathfrak{p}, X) \\ & \rightarrow {}^* \text{Ext}_R^{h+1}(R/(\mathfrak{p} + Rx), X) \end{aligned}$$

in which the left-hand term is trivial because of the maximality of \mathfrak{p} . Thus ${}^* \text{Ext}_R^h(R/\mathfrak{p}, X) \xrightarrow{x} {}^* \text{Ext}_R^h(R/\mathfrak{p}, X)$ is injective for all homogeneous elements $x \in R \setminus \mathfrak{p}$, hence so is the homogeneous localization homomorphism ${}^* \text{Ext}_R^h(R/\mathfrak{p}, X) \rightarrow {}^* \text{Ext}_R^h(R/\mathfrak{p}, X)_{(\mathfrak{p})}$. Thus the free $R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$ -module ${}^* \text{Ext}_R^h(R/\mathfrak{p}, X)_{(\mathfrak{p})}$ is nonzero. Consequently

$$({}^* \text{Ext}_R^h(R/\mathfrak{p}, X)_{(\mathfrak{p})})_{\mathfrak{p}R_{(\mathfrak{p})}} \cong \text{Ext}_{R_{\mathfrak{p}}}^h(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, X_{\mathfrak{p}})$$

is nonzero. This implies that $j \leq h \leq i$. \square

Remark 3.3. (1) A graded module is called * projective if it is a projective object in the category of graded modules. By [6, Proposition 3.1] the * projective graded R -modules coincide with projective R -modules. The projective dimension of a graded module M in the category of

graded modules, is denoted by ${}^*pd_R M$ (cf. [6]). Let $n \in \mathbb{Z}$. A homologically right bounded complex of graded modules X , is said to have * projective dimension at most n , denoted by ${}^*pd_R X \leq n$, if there exists a * projective resolution $P \rightarrow X$, such that $P_i = 0$ for $i > n$. If ${}^*pd_R X \leq n$ holds, but ${}^*pd_R X \leq n-1$ does not, we write ${}^*pd_R X = n$. If ${}^*pd_R X \leq n$ for all $n \in \mathbb{Z}$ we write ${}^*pd_R X = -\infty$. If ${}^*pd_R X \leq n$ for no $n \in \mathbb{Z}$ we write ${}^*pd_R X = \infty$.

(2) For $X \in {}^*\mathcal{D}_{\square}(R)$ by the same method as in [1, Theorem 2.4.P and Corollary 2.5.P] we have

$${}^*pd_R X = \sup\{j \in \mathbb{Z} \mid {}^*\text{Ext}_R^j(X, N) \neq 0 \text{ for some graded } R\text{-module } N\} \\ = \sup\{\inf(U) - \inf(\mathbf{R} \cdot {}^*\text{Hom}_R(X, U)) \mid U \not\cong 0 \text{ in } {}^*\mathcal{D}_{\square}(R)\}.$$

(3) It is easy to see that for $X \in {}^*\mathcal{D}_{\square}(R)$, we have ${}^*pd_R X \leq pd_R X$.

(4) The notions of * flat module and * flat dimension are obtained by replacing ‘projective’ by ‘flat’ in (1). By [6, Proposition 3.2] the * flat graded R -modules coincide with flat R -modules. Therefore for a homologically right bounded complex of graded modules X , we have ${}^*fd_R X \leq {}^*pd_R X$.

The proof of the following proposition is easy so we omit it (see [2, Theorem 1.5.9]). Let J be an ideal of the graded ring R . Then the graded ideal J^* denotes the ideal generated by all homogeneous elements of J . It is well-known that if \mathfrak{p} is a prime ideal of R , then \mathfrak{p}^* is a homogeneous prime ideal of R by [2, Lemma 1.5.6].

Proposition 3.4. *Let $X \in {}^*\mathcal{D}_{\square}(R)$ and \mathfrak{p} is a non-homogeneous prime ideal in R . Then $\mu_{R_{\mathfrak{p}}}^{i+1}(X_{\mathfrak{p}}) = \mu_{R_{\mathfrak{p}^*}}^i(X_{\mathfrak{p}^*})$ and $\beta_i^{R_{\mathfrak{p}}}(X_{\mathfrak{p}}) = \beta_i^{R_{\mathfrak{p}^*}}(X_{\mathfrak{p}^*})$ for any integer $i \geq 0$.*

Corollary 3.5. *Let $X \in {}^*\mathcal{D}_{\square}(R)$ and \mathfrak{p} be a non-homogeneous prime ideal in R . Then*

$$\text{depth } X_{\mathfrak{p}} = \text{depth } X_{\mathfrak{p}^*} + 1.$$

Proof. Using Proposition 3.4, we can assume that both $\text{depth } X_{\mathfrak{p}}$ and $\text{depth } X_{\mathfrak{p}^*}$ are finite. So the equality follows from the fact that over a local ring (R, \mathfrak{m}, k) we have $\text{depth}_R X = \inf\{i \in \mathbb{Z} \mid \mu_R^i(X) \neq 0\}$. \square

Foxby defined the *small support* of a homologically right bounded complex X over a Noetherian ring R , denoted by $\text{supp}_R X$, as the set of prime ideal of R such that $R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})} \otimes_R^L X$ is non-trivial complex (See [7]). It is well known that;

$$\text{supp}_R X = \{\mathfrak{p} \in \text{Spec } R \mid \exists m \in \mathbb{Z} : \beta_m^{R_{\mathfrak{p}}}(X_{\mathfrak{p}}) \neq 0\}.$$

Let ${}^*\text{supp}_R X$ be a subset of $\text{supp}_R X$ consisting of homogeneous prime ideals of $\text{supp}_R X$. Then from Proposition 3.4 we see that $\mathfrak{p} \in \text{supp}_R X$

if and only if $\mathfrak{p}^* \in {}^* \text{supp}_R X$. Also using [12, Lemma 2.3] for $\mathfrak{p} \in \text{supp}_R X$ we have $\text{depth } X_{\mathfrak{p}} < \infty$ if and only if $\text{width}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} < \infty$. Therefore by corollary 3.5 we get;

$$\text{width}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} < \infty \Leftrightarrow \text{width}_{R_{\mathfrak{p}^*}} X_{\mathfrak{p}^*} < \infty.$$

Proposition 3.6. *Let $X \in {}^* \mathcal{D}_{\square}(R)$ and \mathfrak{p} is a non-homogeneous prime ideal in R . Then*

$$\text{width}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} = \text{width}_{R_{\mathfrak{p}^*}} X_{\mathfrak{p}^*}.$$

Proof. We can assume that both $\text{width } X_{\mathfrak{p}}$ and $\text{width } X_{\mathfrak{p}^*}$ are finite numbers, and the argument would be dual to the proof of [2, Theorem 1.5.9]. \square

The ungraded version of the following theorem was proved for modules by Chouinard [3, Corollary 3.1], and extended to complexes by Yassemi [12, Theorem 2.10].

Theorem 3.7. *Let $X \in {}^* \mathcal{D}_{\square}(R)$. If ${}^* \text{id}_R X < \infty$, then*

$${}^* \text{id}_R X = \sup\{\text{depth } R_{\mathfrak{p}} - \text{width } X_{\mathfrak{p}} \mid \mathfrak{p} \in {}^* \text{Spec}(R)\}.$$

Proof. We have the following computations

$$\begin{aligned} {}^* \text{id}_R X &= \sup\{m \in \mathbb{Z} \mid \exists \mathfrak{p} \in {}^* \text{Spec}(R) : \mu_{R_{\mathfrak{p}}}^m(X_{\mathfrak{p}}) \neq 0\} \\ &= \sup\{m \in \mathbb{Z} \mid \exists \mathfrak{p} \in {}^* \text{Spec}(R) : H_m(\mathbf{R} \text{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), X_{\mathfrak{p}})) \neq 0\} \\ &= \sup\{-\inf \mathbf{R} \text{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), X_{\mathfrak{p}}) \mid \mathfrak{p} \in {}^* \text{Spec}(R)\} \\ &= \sup\{\text{depth } R_{\mathfrak{p}} - \text{width}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} \mid \mathfrak{p} \in {}^* \text{Spec}(R)\}. \end{aligned}$$

The first equality holds by Proposition 3.2, and the last one holds by [12, Lemma 2.6(a)], since $\text{id}_R X < \infty$ by Propositions 3.2 and 3.4. \square

The following corollary was already known for graded modules in [6, Corollary 4.12].

Corollary 3.8. *For every $X \in {}^* \mathcal{D}_{\square}(R)$, we have*

$${}^* \text{id}_R X \leq \text{id}_R X \leq {}^* \text{id}_R X + 1.$$

Proof. First of all note that by Proposition 3.4, $\text{id}_R X < \infty$ if and only if ${}^* \text{id}_R X < \infty$. The first inequality is clear by Theorem 3.7 and [12, Theorem 2.10]. For the second one let $\mathfrak{p} \in \text{Spec } R$ be such that $\text{id}_R X = \text{depth } R_{\mathfrak{p}} - \text{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ by [12, Theorem 2.10]. By Corollary 3.5 and Proposition 3.6 we have

$$\text{depth } R_{\mathfrak{p}} - \text{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \text{depth } R_{\mathfrak{p}^*} - \text{width}_{R_{\mathfrak{p}^*}} M_{\mathfrak{p}^*} + 1 \leq {}^* \text{id}_R X + 1,$$

where the second inequality holds by Theorem 3.7. \square

Here we define the $*$ dualizing complex for a graded ring and prove some related results.

Definition 3.9. A $*$ dualizing complex D for a graded ring R is a homologically finite and bounded complex of graded R -modules, such that ${}^* \text{id}_R D < \infty$ and the homothety morphism $\psi : R \rightarrow \mathbf{R}^* \text{Hom}_R(D, D)$ is invertible in ${}^* \mathcal{D}(R)$.

Corollary 3.10. Any $*$ dualizing complex for R is a dualizing complex for R .

The proof of the following lemma is the same as [10, Chapter V, Proposition 3.4].

Lemma 3.11. Let (R, \mathfrak{m}, k) be a $*$ local ring and D is a $*$ dualizing complex of R . Then there exists an integer t such that $H^i(\mathbf{R}^* \text{Hom}_R(k, D)) = 0$ for $i \neq t$ and $H^t(\mathbf{R}^* \text{Hom}_R(k, D)) \cong k$.

Assume that (R, \mathfrak{m}) is a $*$ local ring. A $*$ dualizing complex D is said to be *normalized $*$ dualizing complex* if $t = 0$ in the lemma. It is easy to see that a suitable shift of any $*$ dualizing complex is a normalized one. Also using [10, Chapter V, Proposition 3.4] we see that if D is a normalized $*$ dualizing complex for (R, \mathfrak{m}) , then $D_{\mathfrak{m}}$ is a normalized dualizing complex for $R_{\mathfrak{m}}$.

Lemma 3.12. Let (R, \mathfrak{m}, k) be a $*$ local ring and that D is a normalized $*$ dualizing complex for R . Then there exists a natural functorial isomorphism from the category of graded modules of finite length to itself

$$\phi : H^0(\mathbf{R}^* \text{Hom}_R(-, D)) \rightarrow {}^* \text{Hom}_R(-, {}^* E_R(k)),$$

where ${}^* E_R(k)$ is the $*$ injective envelope of k over R .

Proof. Since D is a normalized $*$ dualizing complex for R , the functor $T := H^0(\mathbf{R}^* \text{Hom}_R(-, D))$ is an additive contravariant exact functor from the category of graded modules of finite length to itself. Let M be a graded R -module and $m \in M$ is a homogeneous element of degree α . Then $\epsilon_m : R(-\alpha) \rightarrow M$ is a homogeneous morphism which sends 1 into m . Thus we have a homogeneous morphism $\psi(M) : T(M) \rightarrow {}^* \text{Hom}_R(M, T(R))$ which sends a homogeneous element $x \in T(M)$ to a morphism $f_x \in {}^* \text{Hom}_R(M, T(R))$ such that $f_x(m) = T(\epsilon_m)(x)$ for every homogeneous element $m \in M$. It is easy to see that it is functorial on M . Thus there exists a natural functorial morphism $\psi : T \rightarrow {}^* \text{Hom}_R(-, T(R))$. Note that if M is a finite graded R -module, using a finite presentation of M , there is an isomorphism ${}^* \lim_{\longleftarrow} {}^* \text{Hom}_R(M, T(R/\mathfrak{m}^n)) \xrightarrow{\cong} {}^* \text{Hom}_R(M, {}^* \lim_{\longleftarrow} T(R/\mathfrak{m}^n))$. Therefore

by the same method of [9, Lemma 4.4 and Propositions 4.5], there is a functorial isomorphism

$$\phi : H^0(\mathbf{R} * \text{Hom}_R(-, D)) \rightarrow * \text{Hom}_R(-, \varinjlim T(R/\mathfrak{m}^n)),$$

from the category of graded modules of finite length to itself. Using the technique of proof of [9, Proposition 4.7] in conjunction with [6, Corollary 4.3], we see that $\varinjlim T(R/\mathfrak{m}^n)$ is an $*$ injective R -module. Since D is a normalized $*$ dualizing complex for R we have

$$* \text{Hom}_R(k, \varinjlim T(R/\mathfrak{m}^n)) \cong H^0(\mathbf{R} * \text{Hom}_R(k, D)) \cong k.$$

Particularly we can embed k to $\varinjlim T(R/\mathfrak{m}^n)$. In order to show that $\varinjlim T(R/\mathfrak{m}^n)$ is an $*$ essential extension of k , let Q be a graded submodule of $\varinjlim T(R/\mathfrak{m}^n)$ such that $k \cap Q = 0$. Then $* \text{Hom}_R(k, Q)$ can be embed in

$$* \text{Hom}_R(k, \varinjlim T(R/\mathfrak{m}^n)) \cong k.$$

Therefore $* \text{Hom}_R(k, Q) = 0$. On the other hand for each $n \in \mathbb{N}$ the set $V(\mathfrak{m})$ includes $\text{Ass}(T(R/\mathfrak{m}^n))$. Now by [11, Proposition 2.1], the fact that each prime ideal of $\text{Ass}(\varinjlim T(R/\mathfrak{m}^n))$ is the annihilator of a homogeneous element [2, Lemma 1.5.6], and the definition of \varinjlim , we have

$$\text{Ass}(\varinjlim T(R/\mathfrak{m}^n)) \subseteq \bigcup_{n \in \mathbb{N}} \text{Ass}(T(R/\mathfrak{m}^n)) \subseteq V(\mathfrak{m}).$$

Consequently Q has support in $V(\mathfrak{m})$, so that $Q = 0$. Therefore $\varinjlim T(R/\mathfrak{m}^n) \cong * E_R(k)$. \square

Let \mathfrak{a} be an ideal of R . The right derived *local cohomology functor* with support in \mathfrak{a} is denoted by $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$. Its right adjoint, $\mathbf{L}\Lambda^{\mathfrak{a}}(-)$, is the left derived *local homology functor* with support in \mathfrak{a} (see [8] for detail).

Finally, we have the following proposition, its proof uses Lemma 3.12 and the argument is similar to [10, Chapter V, Proposition 6.1].

Proposition 3.13. *Let (R, \mathfrak{m}, k) be a $*$ local ring and that D be a normalized $*$ dualizing complex for R . Then $\mathbf{R}\Gamma_{\mathfrak{m}}(D) \simeq * E_R(k)$.*

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ON GRADED INJECTIVE DIMENSION

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رابطه‌های قابل ذکری مابین بعدهای همولوژیکی و بعدهای همولوژیکی مدرج وجود دارد. در این مقاله، بعد انژکتیو همبافت از مدول‌های مدرج مورد مطالعه قرار گرفته و خواص آن بررسی می‌شود. به ویژه، همبافت دوگان‌ساز مدرج، برای یک حلقه مدرج را تعریف کرده و نتایج مربوطه را تعمیم می‌دهیم. کلمات کلیدی: حلقه‌های مدرج، مدول‌های مدرج، بعد انژکتیو.