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CLASSICAL 2-ABSORBING SECONDARY SUBMODULES

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Abstract. In this work, we introduce the concept of classical 2-absorbing secondary modules over a commutative ring as a generalization of secondary modules and investigate some basic properties of this class of modules. Let R be a commutative ring with identity. We say that a non-zero submodule N of an R-module M is a classical 2-absorbing secondary submodule of M if whenever $a, b \in R$, K is a submodule of M and $abN \subseteq K$, then $aN \subseteq K$ or $bN \subseteq K$ or $ab \in \sqrt{Ann_R(N)}$. This can be regarded as a dual notion of the 2-absorbing primary submodule.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers. Let N be a submodule of an R-module M. For $r \in R$, $(N :_M r)$ will denote $(N :_M r) = \{m \in M : rm \in N\}$. Clearly, $(N :_M r)$ is a submodule of M containing N.

Let M be an R-module. A proper submodule P of M is called *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [13]. A non-zero submodule S of M is said to be *second* if for each $a \in R$, the homomorphism $S \xrightarrow{a} S$ is either surjective or zero [18]. A proper submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of M, implies

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that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [15].

The notion of 2-absorbing ideals as a generalization of prime ideals was introduced and studied in [7]. A proper ideal I of R is a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. In [8], the authors introduced the concept of 2-absorbing primary ideal which is a generalization of primary ideal. A proper ideal I of R is called a *2-absorbing primary ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

The notion of 2-absorbing ideals was extended to 2-absorbing submodules in [12] and [16]. A proper submodule N of M is called a *2-absorbing submodule* of M if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$.

In [6], the authors introduced the dual notion of 2-absorbing submodules (that is, *2-absorbing* (resp., *strongly 2-absorbing*) *second submodules*) of M, and investigated some properties of these classes of modules. A non-zero submodule N of M is said to be a *2-absorbing second submodule of* M if whenever $a, b \in R$, L is a completely irreducible submodule of M, and $abN \subseteq L$, then $aN \subseteq L$ or $bN \subseteq L$ or $ab \in Ann_R(N)$. A non-zero submodule N of M is said to be a *strongly 2-absorbing second submodule of* M if whenever $a, b \in R$, K is a submodule of M, and $abN \subseteq K$, then $aN \subseteq K$ or $bN \subseteq K$ or $ab \in Ann_R(N)$.

The notion of 2-absorbing primary submodules as a generalization of 2-absorbing primary ideals of rings was introduced and studied in [14]. A proper submodule N of M is said to be a *2-absorbing primary submodule* of M if whenever $a, b \in R$, $m \in M$, and $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in \sqrt{(N :_R M)}$.

The purpose of this paper is to introduce the concept of classical 2-absorbing secondary submodules as a dual notion of 2-absorbing primary submodules and obtain some related results.

2. Main results

We start this section with the following definition.

Definition 2.1. We say that a non-zero submodule N of an R-module M is a *classical 2-absorbing secondary submodule* of M if whenever $a, b \in R$, K is a submodule of M and $abN \subseteq K$, then $aN \subseteq K$ or $bN \subseteq K$ or $ab \in \sqrt{Ann_R(N)}$. This can be regarded as a dual notion of the 2-absorbing primary submodule. By a *classical 2-absorbing secondary module*, we mean a module which is a classical 2-absorbing secondary submodule of itself.

Example 2.2. Clearly every strongly 2-absorbing second submodule is a classical 2-absorbing secondary submodule. But the converse is not true in general. For example, for any prime integer p, let $M = \mathbb{Z}_{p^\infty}$ and $N = \langle 1/p^3 + \mathbb{Z}\rangle$. Then N is a classical 2-absorbing secondary submodule which is not a 2-absorbing second submodule of M.

Example 2.3. Clearly every secondary submodule is a classical 2-absorbing secondary submodule. But the converse is not true in general. For example, let p, q be two prime numbers, $N = \langle 1/p + \mathbb{Z}\rangle$, and $K = \langle 1/q^2 + \mathbb{Z}\rangle$. Then $N \oplus K$ is not a secondary submodule of the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{q^\infty}$. But $N \oplus K$ is a classical 2-absorbing secondary submodule of the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{q^\infty}$.

Theorem 2.4. *Let N be a non-zero submodule of an R-module M. The following statements are equivalent:*

(a) *N is a classical 2-absorbing secondary submodule of M;*
(b) *If $IJN \subseteq K$ for some ideals I, J of R and a submodule K of M, then $IN \subseteq K$ or $JN \subseteq K$ or $IJ \subseteq \sqrt{Ann_R(N)}$;*
(c) *For each $a, b \in R$, we have $abN = aN$ or $abN = bN$ or $ab \in \sqrt{Ann_R(N)}$.*

Proof. $(a) \Rightarrow (b)$. Let N be a classical 2-absorbing secondary submodule of M and let $IJN \subseteq K$ for some ideals I, J of R and a submodule K of M. Suppose $IJ \not\subseteq \sqrt{Ann_R(N)}$. Then for some $a \in I$ and $b \in J$, $ab \notin \sqrt{Ann_R(N)}$. Now since $abN \subseteq K$, $aN \subseteq K$ or $bN \subseteq K$. We show that either $IN \subseteq K$ or $JN \subseteq K$. On contrary, we assume that $IN \not\subseteq K$ and $JN \not\subseteq K$. Then there exist $a_1 \in I$, $b_1 \in J$ such that $a_1N \not\subseteq K$ and $b_1N \not\subseteq K$. Since $a_1b_1N \subseteq K$ and N is a classical 2-absorbing secondary submodule, $a_1b_1 \in \sqrt{Ann_R(N)}$. We have the following three cases:

Case I: Suppose $aN \subseteq K$ but $bN \not\subseteq K$. Since $a_1bN \subseteq K$ and $bN \not\subseteq K$ and $a_1N \not\subseteq K$, we have $a_1b \in \sqrt{Ann_R(N)}$. Now, $(a+a_1)bN \subseteq K$ and $aN \subseteq K$ but $a_1N \not\subseteq K$, therefore $(a + a_1)N \not\subseteq K$. As $(a + a_1)bN \subseteq K$ and $bN \not\subseteq K$, then $(a + a_1)N \not\subseteq K$ implies $(a + a_1)b \in \sqrt{Ann_R(N)}$. Thus $a_1b \in \sqrt{Ann_R(N)}$ implies that $ab \in \sqrt{Ann_R(N)}$, a contradiction.

Case II: Suppose $bN \subseteq K$ but $aN \not\subseteq K$. Then similar to the Case I, we get a contradiction.

Case III: Suppose $aN \subseteq K$ and $bN \subseteq K$. Now $bN \subseteq K$ and $b_1N \not\subseteq K$ imply $(b + b_1)N \not\subseteq K$. Since $a_1(b + b_1)N \subseteq K$ and $(b + b_1)N \not\subseteq K$ and $a_1N \not\subseteq K$, we get $a_1(b + b_1) \in \sqrt{Ann_R(N)}$. Since $a_1b_1 \in \sqrt{Ann_R(N)}$, we have $a_1b \in \sqrt{Ann_R(N)}$. Again, $aN \subseteq K$ and $a_1N \not\subseteq K$ imply $(a + a_1)N \not\subseteq K$. Since $(a + a_1)b_1N \subseteq K$ and

$(a+a_1)N \not\subseteq K$ and $b_1N \not\subseteq K$, we have $(a+a_1)b_1 \in \sqrt{Ann_R(N)}$. Now as $a_1b_1 \in \sqrt{Ann_R(N)}$ we get $ab_1 \in \sqrt{Ann_R(N)}$. Since $(a+a_1)(b+b_1)N \subseteq K$ and $(a+a_1)N \not\subseteq K$ and $(b+b_1)N \not\subseteq K$, we have $(a+a_1)(b+b_1) \in \sqrt{Ann_R(N)}$. Since $ab_1, a_1b, a_1b_1 \in \sqrt{Ann_R(N)}$, we have $ab \in \sqrt{Ann_R(N)}$, a contradiction. Hence $IN \subseteq K$ or $JN \subseteq K$.

$(b) \Rightarrow (c)$. Let $a, b \in R$. Then $abN \subseteq abN$ implies that $aN \subseteq abN$ or $bN \subseteq abN$ or $ab \in \sqrt{Ann_R(N)}$. Thus $abN = aN$ or $abN = bN$ or $ab \in \sqrt{Ann_R(N)}$.

$(c) \Rightarrow (a)$. This is clear. □

Let N be a submodule of an R-module M. Then, part (c) of Theorem 2.4 shows that N is a classical 2-absorbing secondary submodule of M if and only if N is a classical 2-absorbing secondary module.

Afterwards, we frequently use the following basic fact without further comment.

Remark 2.5. Let N and K are two submodules of an R-module M. To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$.

Theorem 2.6. *Let N be a classical 2-absorbing secondary submodule of an R-module M. Then $Ann_R(N)$ is a 2-absorbing primary ideal of R.*

Proof. Let $a, b, c \in R$ and $abc \in Ann_R(N)$. Suppose that $ab \notin Ann_R(N)$ and $bc \notin \sqrt{Ann_R(N)}$. We show that $ac \in \sqrt{Ann_R(N)}$. There exist completely irreducible submodules L_1 and L_2 of M such that $abN \not\subseteq L_1$ and $bcN \not\subseteq L_2$. Since $abcN = 0 \subseteq L_1 \cap L_2$, $acN \subseteq (L_1 \cap L_2 :_M b)$. Thus $baN \subseteq L_1 \cap L_2$ or $cbN \subseteq L_1 \cap L_2$ or $ac \in \sqrt{Ann_R(N)}$. If $baN \subseteq L_1 \cap L_2$ or $cbN \subseteq L_1 \cap L_2$, then $baN \subseteq L_1$ or $cbN \subseteq L_2$ which are contradictions. Therefore, $ac \in \sqrt{Ann_R(N)}$. □

Corollary 2.7. *Let N be a classical 2-absorbing secondary submodule of an R-module M. Then $\sqrt{Ann_R(N)}$ is a 2-absorbing ideal of R.*

Proof. By Theorem 2.6, $Ann_R(N)$ is a 2-absorbing primary ideal of R. Thus, by [8, Theorem 2.2], $\sqrt{Ann_R(N)}$ is a 2-absorbing ideal of R. □

The following example shows that the converse of Theorem 2.6 is not true in general.

Example 2.8. Consider $M = \mathbb{Z}_{pq} \oplus \mathbb{Q}$ as a $\mathbb{Z}$-module, where p, q are two prime integers. Then $Ann_R(M) = 0$ is a 2-absorbing primary ideal of $\mathbb{Z}$. But M is not a classical 2-absorbing secondary $\mathbb{Z}$-module.

M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$, equivalently, for each submodule N of M, we have $N = (0 :_M Ann_R(N))$ [3].

In the following theorem, we characterize classical 2-absorbing secondary submodules of a comultiplication module over a Dedekind domain.

Theorem 2.9. *Let R be a Dedekind domain and M be a comultiplication R-module. If N is a classical 2-absorbing secondary submodule of M, then $N = (0 :_M Ann^n_R(K))$ or $N = (0 :_M Ann^n_R(K_1)Ann^m_R(K_2))$, where K, K_1, K_2 are minimal submodules of M and n, m are positive integers.*

Proof. By Theorem 2.6, for any classical 2-absorbing secondary submodule N of M, we have $Ann_R(N)$ is a 2-absorbing primary ideal of R. By using [8, Theorem 2.11], we have either $Ann_R(N) = I^n$ or $Ann_R(N) = I^n_1 I^m_2$, where I, I_1, I_2 are maximal ideals of R. First assume that $Ann_R(N) = I^n$. If $(0 :_M I) = 0$, then $(0 :_M I^n) = 0$, and so $N = 0$, a contradiction. Now by [4, Theorem 3.2], since I is a maximal ideal of R, we have $(0 :_M I)$ is a minimal submodule of M. This implies that $N = (0 :_M Ann^n_R(K))$, where $K = (0 :_M I)$. Now assume that $Ann_R(N) = I^n_1 I^m_2$. If $(0 :_M I_1) = 0$ and $(0 :_M I_2) = 0$, then we can conclude that $N = 0$, a contradiction. Thus either $(0 :_M I_1) \neq 0$ or $(0 :_M I_2) \neq 0$. Hence, one can see that either $N = (0 :_M Ann^n_R(K_1)Ann^m_R(K_2))$ or $N = (0 :_M Ann^m_R(K_2))$ or $N = (0 :_M Ann^n_R(K_1))$, where $K_1 = (0 :_M I_1)$ and $K_2 = (0 :_M I_2)$ are minimal submodules of M. □

Let M be an R-module. For a submodule N of M the *second radical* (or *second socle*) of N is defined as the sum of all second submodules of M contained in N and it is denoted by $sec(N)$ (or $soc(N)$). In case N does not contain any second submodule, the second radical of N is defined to be (0) (see [11] and [1]).

Theorem 2.10. *Let M be a finitely generated comultiplication R-module. If N is a classical 2-absorbing secondary submodule of M, then $sec(N)$ is a strongly 2-absorbing second submodule of M.*

Proof. Let N be a classical 2-absorbing secondary submodule of M. By Corollary 2.7, $\sqrt{Ann_R(N)}$ is a 2-absorbing ideal of R. By [2, Theorem 2.12], $Ann_R(sec(N)) = \sqrt{Ann_R(N)}$. Therefore, $Ann_R(sec(N))$ is a 2-absorbing ideal of R. Now the result follows from [6, Theorem 3.10]. □

Recall that an R-module M is said to be *sum-irreducible* precisely when it is nonzero and cannot be expressed as the sum of two proper submodules of itself [10, Definition and Exercise 7.2.8].

Theorem 2.11. *Let N be a classical 2-absorbing secondary submodule of an R-module M. Then $aN = a^2N$ for all $a \in R \setminus \sqrt{Ann_R(N)}$. The converse holds, if N is a sum-irreducible submodule of M.*

Proof. Let $a \in R \setminus \sqrt{Ann_R(N)}$. Then $a^2 \in R \setminus \sqrt{Ann_R(N)}$. Thus $aN = a^2N$ by Theorem 2.4 $(a) \Rightarrow (c)$. Conversely, let N be a sum-irreducible submodule of M and $abN \subseteq K$ for some $a, b \in R$ and a submodule K of M. Assume that, $ab \notin \sqrt{Ann_R(N)}$. We show that $aN \subseteq K$ or $bN \subseteq K$. As $ab \notin \sqrt{Ann_R(N)}$, we have $a, b \notin \sqrt{Ann_R(N)}$. Thus $aN = a^2N$ by assumption. Let $x \in N$. Then $ax \in aN = a^2N$. Hence $ax = a^2y$ for some $y \in N$. This implies that $x - ay \in (0 :_N a) \subseteq (K :_N a)$. Thus $x = x - ay + ay \in (K :_N a) + (K :_N b)$. Therefore, $N \subseteq (K :_N a) + (K :_N b)$. Clearly, $(K :_N a) + (K :_N b) \subseteq N$. Thus as N is sum-irreducible, $(K :_N a) = N$ or $(K :_N b) = N$, as needed. □

An R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [9].

Theorem 2.12. *Let N be a submodule of an R-module M. Then we have the following.*

(a) *If N is a classical 2-absorbing secondary submodule of M, then IN is a classical 2-absorbing secondary submodule of M for all ideals I of R with $I \not\subseteq Ann_R(N)$.*
(b) *If M is a multiplication classical 2-absorbing secondary module, then every non-zero submodule of M is a classical 2-absorbing secondary submodule of M.*

Proof. (a) Let I be an ideal of R with $I \not\subseteq Ann_R(N)$. Then IN is a non-zero submodule of M. Let $a, b \in R$, K be a submodule of M, and $abIN \subseteq K$. Then $abN \subseteq (K :_M I)$. Thus $aIN \subseteq K$ or $bIN \subseteq K$ or $ab \in \sqrt{Ann_R(N)} \subseteq \sqrt{Ann_R(IN)}$, as needed.
(b) This follows from part (a). □

Theorem 2.13. *Let $f : M \to \acute{M}$ be a monomorphism of R-modules. Then we have the following.*

(a) *If N is a classical 2-absorbing secondary submodule of M, then $f(N)$ is a classical 2-absorbing secondary submodule of $\acute{M}$.*
(b) *If $\acute{N}$ is a classical 2-absorbing secondary submodule of $f(M)$, then $f^{-1}(\acute{N})$ is a classical 2-absorbing secondary submodule of M.*

Proof. (a) Since $N \neq 0$ and f is a monomorphism, we have $f(N) \neq 0$. Let $a, b \in R$, $\acute{K}$ be a submodule of $\acute{M}$, and $abf(N) \subseteq \acute{K}$. Then $abN \subseteq f^{-1}(\acute{K})$. As N is classical 2-absorbing secondary submodule, $aN \subseteq f^{-1}(\acute{K})$ or $bN \subseteq f^{-1}(\acute{K})$ or $ab \in \sqrt{Ann_R(N)}$. Therefore,

$$af(N) \subseteq f(f^{-1}(\acute{K})) = f(M) \cap \acute{K} \subseteq \acute{K}$$

or

$$bf(N) \subseteq f(f^{-1}(\acute{K})) = f(M) \cap \acute{K} \subseteq \acute{K}$$

or $ab \in \sqrt{Ann_R(f(N))}$, as needed.

(b) If $f^{-1}(\acute{N}) = 0$, then $f(M) \cap \acute{N} = ff^{-1}(\acute{N}) = f(0) = 0$. Thus $\acute{N} = 0$, a contradiction. Therefore, $f^{-1}(\acute{N}) \neq 0$. Now let $a, b \in R$, K be a submodule of M, and $abf^{-1}(\acute{N}) \subseteq K$. Then

$$ab\acute{N} = ab(f(M) \cap \acute{N}) = abff^{-1}(\acute{N}) \subseteq f(K).$$

As $\acute{N}$ is classical 2-absorbing secondary submodule, $a\acute{N} \subseteq f(K)$ or $b\acute{N} \subseteq f(K)$ or $ab \in \sqrt{Ann_R(\acute{N})}$. Hence $af^{-1}(\acute{N}) \subseteq f^{-1}f(K) = K$ or $bf^{-1}(\acute{N}) \subseteq f^{-1}f(K) = K$ or $ab \in \sqrt{Ann_R(f^{-1}(\acute{N}))}$, as desired. □

Theorem 2.14. *Let M be an R-module. If E is an injective R-module and N is a 2-absorbing primary submodule of M such that $Hom_R(M/N, E) \neq 0$, then $Hom_R(M/N, E)$ is a classical 2-absorbing secondary R-module.*

Proof. Let $a, b \in R$. Since N is a 2-absorbing primary submodule of M, we can assume that $(N :_M ab) = (N :_M a)$ or $(N :_M (ab)^n) = M$ for some positive integer n. Since E is an injective R-module, by replacing M with M/N in [5, Theorem 3.13 (a)], we have

$$Hom_R(M/(N :_M a), E) = aHom_R(M/N, E).$$

Therefore,

$$abHom_R(M/N, E) = Hom_R(M/(N :_M ab), E) =$$
$$Hom_R(M/(N :_M a), E) = aHom_R(M/N, E)$$

or

$$(ab)^n Hom_R(M/N, E) = Hom_R(M/(N :_M (ab)^n), E) =$$
$$Hom_R(M/M, E) = 0,$$

as needed □

Example 2.15. Let R be a Noetherian ring and let $E = \oplus_{m \in Max(R)} E(R/m)$. Then for each 2-absorbing primary ideal P of R, $(0 :_E P)$ is a classical 2-absorbing secondary submodule of E.

Proof. By using [17, p. 147], $Hom_R(R/P, E) \neq 0$. Now the result follows from the fact that $(0 :_E P) \cong Hom_R(R/P, E)$ and Theorem 2.14. □

Theorem 2.16. *Let M be a classical 2-absorbing secondary R-module and F be a right exact linear covariant functor over the category of R-modules. Then $F(M)$ is a classical 2-absorbing secondary R-module if $F(M) \neq 0$.*

Proof. This follows from [5, Theorem 3.14] and Theorem 2.4 $(a) \Rightarrow (c)$. □

Corollary 2.17. *Let M be an R-module, S be a multiplicative subset of R and N be a classical 2-absorbing secondary submodule of M. Then $S^{-1}N$ is a classical 2-absorbing secondary submodule of $S^{-1}M$ if $S^{-1}N \neq 0$.*

Proof. This follows from Theorem 2.16. □

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References

1. H. Ansari-Toroghy and F. Farshadifar, On the dual notion of prime submodules, *Algebra Colloq.*, **19** (2012), 1109–1116.
2. H. Ansari-Toroghy and F. Farshadifar, On the dual notion of prime radicals of submodules, *Asian Eur. J. Math.*, **6** (2013), Article ID: 1350024 (11 pages).
3. H. Ansari-Toroghy and F. Farshadifar, The dual notion of multiplication modules, *Taiwanese J. Math.*, **11** (2007), 1189–1201.
4. H. Ansari-Toroghy and F. Farshadifar, On comultiplication modules, *Korean Ann. Math.*, **25** (2008), 57–66.
5. H. Ansari-Toroghy and F. Farshadifar, The dual notion of some generalizations of prime submodules, *Comm. Algebra*, **39** (2011), 2396–2416.
6. H. Ansari-Toroghy and F. Farshadifar, Some generalizations of second submodules, *Palestine J. Math.*, **8** (2019), 1–10.
7. A. Badawi, On 2-absorbing ideals of commutative rings, *Bull. Austral. Math. Soc.*, **75** (2007), 417–429.
8. A. Badawi, U. Tekir and E. Yetkin, On 2-absorbing primary ideals in commutative rings, *Bull. Korean Math. Soc.*, **51** (2014), 1163–1173.
9. A. Barnard, Multiplication modules, *J. Algebra*, **71** (1981), 174–178.
10. M. P. Brodmann and R. Y. Sharp, *Local cohomology an algebraic introduction with geometric applications*, Cambridge Univercity Press, 1998.
11. S. Ceken, M. Alkan and P. F. Smith, The dual notion of the prime radical of a module, *J. Algebra*, **392** (2013), 265–275.
12. A. Y. Darani and F. Soheilnia, 2-absorbing and weakly 2-absorbing submoduels, *Thai J. Math.*, **9** (2011), 577– 584.

13. J. Dauns, Prime submodules, *J. Reine Angew. Math.*, **298** (1978), 156–181.
14. M. K. Dubey and P. Aggarwal, On 2-absorbing primary submodules of modules over commutative ring with unity, *Asian Eur. J. Math.*, **8** (2015), Article ID: 1550064 (12 pages).
15. L. Fuchs, W. Heinzer, and B. Olberding, Commutative ideal theory without finiteness conditions: Irreducibility in the quotient filed, in: *Abelian Groups, Rings, Modules, and Homological Algebra, Lect. Notes Pure Appl. Math.*, **249** (2006), 121–145.
16. Sh. Payrovi and S. Babaei, On 2-absorbing submodules, *Algebra Collq.*, **19** (2012), 913–920.
17. D. W. Sharpe and P. Vamos, *Injective modules*, Cambridge University Press, 1972.
18. S. Yassemi, The dual notion of prime submodules, *Arch. Math.* (Brno) **37** (2001), 273–278.

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در این مقاله ما مفهوم مدول‌های ثانویه ۲-جاذب کلاسیک را روی حلقه‌های جابه‌جایی به عنوان تعمیمی از مدول‌های ثانویه معرفی کرده و خواص اولیه این دسته از مدول‌ها را مورد بحث قرار می‌دهیم. یک زیرمدول N از R-مدول M را زیرمدول ثانویه ۲-جاذب کلاسیک گوییم هرگاه $a, b \in R$، K یک زیرمدول از M و $abN \subseteq K$، آنگاه $aN \subseteq K$ یا $bN \subseteq K$ یا $ab \in \sqrt{Ann_R(N)}$. این مفهوم را می‌توان دوگان زیرمدول‌های اولیه ۲-جاذب در نظر گرفت.

کلمات کلیدی: مدول ثانویه، ایده‌آل ۲-جاذب اولیه، مدول ثانویه ۲-جاذب کلاسیک.