

φ -CONNES MODULE AMENABILITY OF DUAL BANACH ALGEBRAS

A. GHAFFARI*, S. JAVADI AND E. TAMIMI

ABSTRACT. In this paper, we define φ -Connes module amenability of a dual Banach algebra \mathcal{A} , where φ is a bounded module homomorphism from \mathcal{A} to \mathcal{A} that is w_{k^*} -continuous. We are mainly concerned with the study of φ -module normal, virtual diagonals. We show that if S is a weakly cancellative and S is an inverse semigroup with subsemigroup E of idempotents, χ is a bounded module homomorphism from $l^1(S)$ to $l^1(S)$ that is w_{k^*} -continuous and $l^1(S)$ as a Banach module over $l^1(E)$ is χ -Connes module amenable, then it has a χ -module normal, virtual diagonal. In the case $\chi = id$, the converse also holds.

1. INTRODUCTION

Connes amenable dual Banach algebras were introduced by Runde in [19]. In [20], Runde showed that if a Banach algebra is Connes amenable, it has a normal, virtual diagonal. The interest in normal, virtual diagonals arises from the fact that for a von Neumann algebra \mathcal{A} , Connes amenability of \mathcal{A} is completely determined by the existence of a normal, virtual diagonal. As noticed by Runde, the notion of a normal, virtual diagonal adapts naturally to the context of general dual Banach algebras. In [21], it is shown that $M(G)$, the measure algebra of a locally compact group G , is Connes amenable if and only if it has a normal, virtual diagonal.

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*Corresponding author.

In [1], Amini introduced the concept of module amenability for Banach algebras, and proved that when S is an inverse semigroup with subsemigroup E of idempotents, then $l^1(S)$ as a Banach module over $\mathcal{U} = l^1(E)$ is module amenable if and only if S is amenable. Also, in [2], it is shown that $l^1(S)^{**}$ is $l^1(E)$ -module amenable if and only if an appropriate group homomorphic image of S is finite. We may refer the reader e.g. to [1, 2, 3, 4, 5, 16], for extensive treatments of various notions of module amenability.

All of these concepts generalized the earlier concept of amenability for Banach algebras introduced by Johnson [12]. Recently, the authors have introduced the ϕ -version of Connes amenability of dual Banach algebra \mathcal{A} that ϕ is a homomorphism from \mathcal{A} onto \mathbb{C} that lies in \mathcal{A}_* [11]. Let \mathcal{A} be a dual Banach algebra with a compatible action of a Banach algebra \mathcal{U} and φ be a bounded module homomorphism from \mathcal{A} to \mathcal{A} that is w_{k^*} -continuous. In this paper, we introduce the concept of φ -Connes module amenability for \mathcal{A} and give a characterization of φ -Connes module amenability in terms of φ -modul normal virtual diagonals. In particular, we show that if χ is a bounded module homomorphism from $l^1(S)$ to $l^1(S)$ that is w_{k^*} -continuous and $l^1(S)$ as a Banach module over $l^1(E)$ is χ -Connes module amenable, then it has a χ -module normal virtual diagonal. In the case $\chi = id$, the converse also holds, restoring [21, Theorem 1] for the case of measure algebra of a discrete group.

2. MAIN RESULTS

Let \mathcal{A} be a dual Banach algebra with predual \mathcal{A}_* and \mathcal{U} be a Banach algebra such that \mathcal{A} is a Banach \mathcal{U} -bimodule with compatible actions, that is

$$\alpha.(ab) = (\alpha.a).b, (\alpha\beta).a = \alpha.(\beta.a) \quad (a, b \in \mathcal{A}, \alpha, \beta \in \mathcal{U}).$$

Let E be a dual Banach \mathcal{A} -bimodule. E is called *normal* if for each $x \in E$, the maps

$$\mathcal{A} \rightarrow E, \quad a \rightarrow \begin{cases} a.x \\ x.a \end{cases}$$

are w_{k^*} -continuous. If moreover E is a \mathcal{U} -bimodule such that for $a \in \mathcal{A}$, $\alpha \in \mathcal{U}$ and $x \in E$

$$\alpha.(a.x) = (\alpha.a).x, (a.\alpha).x = a.(\alpha.x), (\alpha.x).a = \alpha.(x.a),$$

then E is called a *normal Banach left $\mathcal{A} - \mathcal{U}$ -module*. Similarly for the right and two sided actions. Also, E is called *commutative*, if

$$\alpha.x = x.\alpha \quad (\alpha \in \mathcal{U}, x \in E).$$

A module homomorphism from \mathcal{A} to \mathcal{A} is a map $\varphi : \mathcal{A} \rightarrow \mathcal{A}$ with

$$\begin{aligned} \varphi(a + b) &= \varphi(a) + \varphi(b), \quad \varphi(ab) = \varphi(a)\varphi(b) \\ \varphi(\alpha.a) &= \alpha.\varphi(a), \quad \varphi(a.\alpha) = \varphi(a).\alpha \quad (a, b \in \mathcal{A}, \alpha \in \mathcal{U}). \end{aligned}$$

Since \mathcal{A} is a dual Banach algebra, then multiplication in \mathcal{A} is w_{k^*} -continuous. Consider \mathcal{A} as dual \mathcal{A} -module with predual \mathcal{A}_* . So we shall suppose that \mathcal{A} takes w_{k^*} -topology. $\mathcal{HOM}_{w_{k^*}}(\mathcal{A})$ will denote the space of all bounded module homomorphism that is w_{k^*} -continuous.

A bounded map $D : \mathcal{A} \rightarrow E$ is called a module φ -derivation if

$$\begin{aligned} D(a \pm b) &= D(a) \pm D(b), \quad D(ab) = D(a).\varphi(b) + \varphi(a).D(b) \\ D(\alpha.a) &= \alpha.D(a), \quad D(a.\alpha) = D(a).\alpha \quad (a, b \in \mathcal{A}, \alpha \in \mathcal{U}). \end{aligned}$$

When E is commutative, each $x \in E$ defines a module φ -derivation

$$D_x(a) = \varphi(a).x - x.\varphi(a) \quad (a \in \mathcal{A}).$$

Derivations of this form are called *inner module φ -derivation*.

Definition 2.1. Let \mathcal{A} be a dual Banach algebra, \mathcal{U} be a Banach algebra such that \mathcal{A} is a Banach \mathcal{U} -module and $\varphi \in \mathcal{HOM}_{w_{k^*}}(\mathcal{A})$. \mathcal{A} is called φ -Connes module amenable if for any commutative normal Banach $\mathcal{A}\mathcal{U}$ -module E , each w_{k^*} -continuous module φ -derivation $D : \mathcal{A} \rightarrow E$ is inner.

Recall that if φ is identity map on \mathcal{A} , then id-Connes module amenability is called Connes module amenability. Also, by the proof of [1, Proposition 2.1], Connes amenability of \mathcal{A} implies its Connes module amenability in the case where \mathcal{U} has a bounded approximate identity for \mathcal{A} . Example 9 shows that the converse is false. Hence Connes module amenability is weaker than Connes amenability. Throughout this paper, \mathcal{A} is a Banach algebra that is a Banach \mathcal{U} -module.

Theorem 2.2. *Let \mathcal{A} be a dual Banach algebra and $\varphi \in \mathcal{HOM}_{w_{k^*}}(\mathcal{A})$. If φ is an epimorphism and \mathcal{A} is φ -Connes module amenable, then \mathcal{A} is Connes-module amenable.*

Proof. Let E be a commutative normal Banach $\mathcal{A} - \mathcal{U}$ -module and $D : \mathcal{A} \rightarrow E$ be a w_{k^*} -continuous module derivation. Set $d = D \circ \varphi$. The mapping $d : \mathcal{A} \rightarrow E$ is a module φ -derivation. Since $\varphi \in \mathcal{HOM}_{w_{k^*}}(\mathcal{A})$, then d is w_{k^*} -continuous. Thus there exists $f \in E$ such that $d(a) = f.\varphi(a) - \varphi(a).f$ for all $a \in \mathcal{A}$. Let $b \in \mathcal{A}$, there exists $a \in \mathcal{A}$ such that $\varphi(a) = b$. Hence

$$D(b) = D(\varphi(a)) = d(a) = f.\varphi(a) - \varphi(a).f = f.b - b.f.$$

This shows that \mathcal{A} is Connes-module amenable. □

Theorem 2.3. *Let \mathcal{A} be a dual Arens regular Banach algebra and $\varphi \in \mathcal{HOM}_{w_{k^*}}(\mathcal{A})$. Then the following are equivalent:*

- (i) \mathcal{A} is φ -Connes module amenable.
- (ii) \mathcal{A}^{**} is φ^{**} -Connes module amenable.

Proof. (i) \Rightarrow (ii) Let E be a commutative normal Banach \mathcal{A}^{**} - \mathcal{U} -module and $D : \mathcal{A}^{**} \rightarrow E$ be a w_{k^*} -continuous module φ^{**} -derivation. Let $\theta : \mathcal{A} \rightarrow \mathcal{A}^{**}$ be the canonical map. It is known that θ is w_{k^*} -continuous. Define a module action of \mathcal{A} on E by letting $x \bullet a = x \cdot \theta(a)$, $a \bullet x = \theta(a) \cdot x$ ($a \in \mathcal{A}, x \in E$). It can be shown that this module action is well defined and turns E into a normal Banach \mathcal{A} - \mathcal{U} -module. We define a derivation $\tilde{D} : \mathcal{A} \rightarrow E$ by letting $\tilde{D} = D \circ \theta$. Then we have

$$\begin{aligned} \tilde{D}(ab) = D \circ \theta(ab) &= D \circ \theta(a) \cdot \varphi^{**}(\theta(b)) + \varphi^{**}(\theta(a)) \cdot D \circ \theta(b) \\ &= D \circ \theta(a) \cdot \theta(\varphi(b)) + \theta(\varphi(a)) \cdot D \circ \theta(b) \\ &= D \circ \theta(a) \bullet \varphi(b) + \varphi(a) \bullet D \circ \theta(b) \\ &= \tilde{D}(a) \bullet \varphi(b) + \varphi(a) \bullet \tilde{D}(b). \end{aligned}$$

Thus \tilde{D} is a module φ -derivation that is w_{k^*} -continuous. Since \mathcal{A} is φ -Connes module amenable, then there exists $x \in E$ such that

$$\begin{aligned} \tilde{D}(a) &= D \circ \theta(a) = x \bullet \varphi(a) - \varphi(a) \bullet x \\ &= x \cdot \theta(\varphi(a)) - \theta(\varphi(a)) \cdot x. \end{aligned}$$

Let $G \in \mathcal{A}^{**}$. As $\theta(\mathcal{A})$ is w_{k^*} -dense in \mathcal{A}^{**} , there exists a net $\{g_\alpha\}$ in \mathcal{A} such that $\theta(g_\alpha) \rightarrow G$ in the w_{k^*} -topology. Also it is known that φ^{**} is w_{k^*} -continuous, then $\varphi^{**}(\theta(g_\alpha)) \rightarrow \varphi^{**}(G)$. Hence

$$\begin{aligned} D(G) = \lim_{\alpha} D \circ \theta(g_\alpha) &= \lim_{\alpha} x \cdot \theta \circ \varphi(g_\alpha) - \theta \circ \varphi(g_\alpha) \cdot x \\ &= \lim_{\alpha} x \cdot \varphi \circ \theta(g_\alpha) - \varphi \circ \theta(g_\alpha) \cdot x \\ &= x \cdot \varphi^{**}(G) - \varphi^{**}(G) \cdot x \end{aligned}$$

(ii) \Rightarrow (i) Let E be a commutative normal Banach \mathcal{A} - \mathcal{U} -module and $D : \mathcal{A} \rightarrow E$ be a w_{k^*} -continuous module φ -derivation. Let $\pi : (\mathcal{A}_*)^{***} \rightarrow (\mathcal{A}_*)^*$ by $\pi(F) = F |_{\theta(\mathcal{A}_*)}$ be the Dixmier projection. It is well known that the Dixmier projection from \mathcal{A}^{**} onto \mathcal{A} is a module homomorphism [14]. Then E is a Banach \mathcal{A}^{**} - \mathcal{U} -module with the bimodule multiplications

$$F \bullet x = \pi(F) \cdot x, \quad x \bullet F = x \cdot \pi(F) \quad (x \in E, F \in \mathcal{A}^{**}).$$

It is routinely checked that E is a commutative normal Banach \mathcal{A}^{**} - \mathcal{U} -module. Now set $D \circ \pi : \mathcal{A}^{**} \rightarrow E$. We have

$$\begin{aligned} D \circ \pi(FG) &= D(\pi(F)\pi(G)) = D \circ \pi(F) \cdot \varphi \circ \pi(G) + \varphi \circ \pi(F) \cdot D \circ \pi(G) \\ &= D \circ \pi(F) \cdot \varphi^{**} \circ \pi(G) + \varphi^{**} \circ \pi(F) \cdot D \circ \pi(G) \\ &= D \circ \pi(F) \cdot \pi(\varphi^{**}(G)) + \pi(\varphi^{**}(F)) \cdot D \circ \pi(G) \\ &= D \circ \pi(F) \bullet \varphi^{**}(G) + \varphi^{**}(F) \bullet D \circ \pi(G). \end{aligned}$$

Since \mathcal{A}^{**} is φ^{**} -Connes module amenable, then there exists $x \in E$ such that

$$\begin{aligned} D \circ \pi(F) = \varphi^{**}(F) \bullet x - x \bullet \varphi^{**}(F) &= \pi(\varphi^{**}(F)) \cdot x - x \cdot \pi(\varphi^{**}(F)) \\ &= \varphi^{**}(\pi(F)) \cdot x - x \cdot \varphi^{**}(\pi(F)). \end{aligned}$$

Therefore $D(a) = \varphi(a) \cdot x - x \cdot \varphi(a)$ for all $a \in \mathcal{A}$, and hence D is inner. \square

Theorem 2.4. *Let \mathcal{A} be a commutative dual Banach algebra and $\varphi \in \mathcal{HOM}_{w_{k^*}}(\mathcal{A})$. If \mathcal{A} is φ -Connes module amenable, then \mathcal{A} has a bounded approximate identity for $\varphi(\mathcal{A})$.*

Proof. Let \mathcal{A} be a commutative Banach \mathcal{A} - \mathcal{U} -module whose underlying space is \mathcal{A} , but on which \mathcal{A} acts via

$$a \cdot x := ax, \quad x \cdot a := 0 \quad (a \in \mathcal{A}, x \in \mathcal{A}).$$

Let $I : \mathcal{A} \rightarrow \mathcal{A}$ be the identity map. It is easy to see that $I \circ \varphi$ is a module φ -derivation. Since \mathcal{A} is φ -Connes module amenable, there exists $e \in \mathcal{A}$ such that

$$\begin{aligned} I \circ \varphi(a) &= \varphi(a) \cdot e - e \cdot \varphi(a) \\ \varphi(a) &= \varphi(a) \cdot e. \end{aligned}$$

The element e has the desired properties. \square

Theorem 2.5. *Let \mathcal{A} be a dual Banach algebra and $\varphi \in \mathcal{HOM}_{w_{k^*}}(\mathcal{A})$. If \mathcal{A} is φ -Connes module amenable, then \mathcal{A} is $\lambda \circ \varphi$ -Connes module amenable for any $\lambda \in \mathcal{HOM}_{w_{k^*}}(\mathcal{A})$.*

Proof. Let E be a commutative normal Banach $\mathcal{A} - \mathcal{U}$ -module and $D : \mathcal{A} \rightarrow E$ be a module $\lambda \circ \varphi$ -derivation that is w_{k^*} -continuous. If E is equipped with the module operation by

$$a \bullet x = \lambda(a) \cdot x, \quad x \bullet a = x \cdot \lambda(a), \quad (a \in \mathcal{A}, x \in E)$$

then E becomes a commutative normal Banach $\mathcal{A} - \mathcal{U}$ -module. We have

$$\begin{aligned} D(ab) &= D(a) \cdot \lambda \circ \varphi(b) + \lambda \circ \varphi(a) \cdot D(b) \\ &= D(a) \bullet \varphi(b) + \varphi(a) \bullet D(b). \end{aligned}$$

Thus, there exists $f \in E$ such that

$$D(a) = f \bullet \varphi(a) - \varphi(a) \bullet f = f. \lambda \circ \varphi(a) - \lambda \circ \varphi(a). f \quad (a \in \mathcal{A}).$$

This shows that D is inner. □

Theorem 2.6. *Let \mathcal{A} be a unital dual Banach algebra and also $\varphi \in \mathcal{HOM}_{w_k^*}(\mathcal{A})$. Then \mathcal{A} is φ -Connes module amenable if and only if for any unital commutative Banach $\mathcal{A}\mathcal{U}$ -module E , each module φ -derivation $D : \mathcal{A} \rightarrow E$ is inner.*

Proof. Let E be a commutative normal Banach $\mathcal{A}\mathcal{U}$ -bimodule with predual E_* , and consider $l : E \rightarrow E$ and $r : E \rightarrow E$ by $l(x) = e_{\mathcal{A}}x$ and $r(x) = xe_{\mathcal{A}}$. put $E_1 = (id - l) \circ r(E)$, $E_2 = (id - r) \circ l(E)$, $E_3 = (id - l) \circ (id - r)(E)$ and $E_4 = l \circ r(E)$. The verification that $E = E_1 \oplus E_2 \oplus E_3 \oplus E_4$ is routine. It is the direct sum of E_i for $i = 1, 2, 3, 4$. Then E_1 is equipped with the module operation by

$$(x - e_{\mathcal{A}}x). a = x. a - e_{\mathcal{A}}x. a, a(x - e_{\mathcal{A}}x) = a. x - a. e_{\mathcal{A}}.x = 0$$

It is easy to see that E_1 is a commutative normal Banach $\mathcal{A}\mathcal{U}$ -bimodule by predual $(1 - e_{\mathcal{A}}).E_*.e_{\mathcal{A}}$. Let $\pi_1 : E \rightarrow E_1$ be the projection map. Then $\pi_1 \circ D$ is a module φ -derivation from \mathcal{A} to E_1 that is w_k^* -continuous. Since \mathcal{A} has a left zero action on E_1 , then we have

$$\begin{aligned} \pi_1 \circ D(a) &= \pi_1 \circ D(e_{\mathcal{A}}.a) = \pi_1 \circ D(e_{\mathcal{A}}).\varphi(a) + \varphi(e_{\mathcal{A}}).\pi_1 \circ D(a) \\ &= \pi_1 \circ D(e_{\mathcal{A}}).\varphi(a) = \pi_1 \circ D(e_{\mathcal{A}}).\varphi(a) - \varphi(a).\pi_1 \circ D(e_{\mathcal{A}}) \end{aligned}$$

Also, a routine verification shows that $\pi_2 \circ D = ad_{\pi_2 \circ D(e_{\mathcal{A}})}$ and $\pi_3 \circ D = 0$.

Now, let $\pi_4 \circ D : \mathcal{A} \rightarrow E_4$. It is obvious that $\pi_4 \circ D$ is a module φ -derivation. We can show that E_4 is a commutative normal Banach $\mathcal{A}\mathcal{U}$ -bimodule with predual $e_{\mathcal{A}}.E_*.e_{\mathcal{A}}$. By our assumption, $\pi_4 \circ D$ is inner. □

Let \mathcal{A} and \mathcal{U} are dual Banach algebras. Let \mathcal{A} be a dual Banach \mathcal{U} -module and $\mathcal{A} \hat{\otimes} \mathcal{A}$ denote the projective tensor product of \mathcal{A} and \mathcal{A} . Let $\mathcal{A}_* \otimes_w \mathcal{A}_*$ be the injective tensor product of \mathcal{A}_* with itself. Then we have a canonical map from $\mathcal{A} \hat{\otimes} \mathcal{A}$ into $(\mathcal{A}_* \otimes_w \mathcal{A}_*)^*$ which has closed range if \mathcal{A} has the bounded approximation property. For more details, see [18]. Let I be the closed ideal of $\mathcal{A} \hat{\otimes} \mathcal{A}$ generated by elements of the form $\alpha.(a \otimes b) - (a \otimes b).\alpha$, for $a, b \in \mathcal{A}$ and $\alpha \in \mathcal{U}$. $\mathcal{A} \hat{\otimes}_{\mathcal{U}} \mathcal{A}$ is defined to be the quotient Banach space $\frac{\mathcal{A} \hat{\otimes} \mathcal{A}}{I}$ [15]. Let J be the closed ideal of \mathcal{A} generated by elements of the form $(\alpha.a).b - a.(b.\alpha)$. Since J is weak*-closed, then the quotient algebra $\frac{\mathcal{A}}{J}$ is again dual with predual

${}^\perp J = \{\phi \in \mathcal{A}_* : \langle \phi, a \rangle = 0 \text{ for all } a \in J\}$. Moreover, $\mathcal{A} \hat{\otimes}_{\mathcal{U}} \mathcal{A} \cong \frac{\mathcal{A} \hat{\otimes} \mathcal{A}}{I}$ and $\frac{\mathcal{A}}{J}$ could be regarded as a Banach $\mathcal{A}\mathcal{U}$ -module. Let $\mathcal{L}_{w^*}^2(\frac{\mathcal{A}}{J}, \mathbb{C})$ denote the separately w_k^* -continuous 2-linear maps from $\frac{\mathcal{A}}{J} \times \frac{\mathcal{A}}{J}$ to \mathbb{C} . Note that the dual Banach $\mathcal{A}\mathcal{U}$ -module $\mathcal{L}_{w^*}^2(\frac{\mathcal{A}}{J}, \mathbb{C})$ need not be normal. Let $\tilde{w} : \mathcal{A} \hat{\otimes}_{\mathcal{U}} \mathcal{A} \rightarrow \frac{\mathcal{A}}{J}$ be the multiplication operator, $\tilde{w}(a \otimes b + I) = ab + J$. Since the quotient map is continuous and open, then it is immediate that \tilde{w}^* maps ${}^\perp J$ into $\mathcal{L}_{w^*}^2(\frac{\mathcal{A}}{J}, \mathbb{C})$. It follows that \tilde{w}^{**} drops to an $\mathcal{A}\mathcal{U}$ -module homomorphism $\tilde{w}^{**} : \mathcal{L}_{w^*}^2(\frac{\mathcal{A}}{J}, \mathbb{C})^* \rightarrow \frac{\mathcal{A}}{J}$. Recall a few definitions from [10](with a different notation, however). Given $F \in \mathcal{L}_{w^*}^2(\frac{\mathcal{A}}{J}, \mathbb{C})$ and $M \in \mathcal{L}_{w^*}^2(\frac{\mathcal{A}}{J}, \mathbb{C})^*$, we put

$$\langle M, F \rangle = \int F dM =: \int_{\mathcal{A} \hat{\otimes}_{\mathcal{U}} \mathcal{A}} F(a + J, b + J) dM(a + J, b + J).$$

More generally, let E be a dual Banach space and let $F : \frac{\mathcal{A}}{J} \times \frac{\mathcal{A}}{J} \rightarrow E$ be a bilinear map such that $a + J \rightarrow F(a + J, b + J)$ and $b + J \rightarrow F(a + J, b + J)$ are w_k^* -continuous. We define $\int F dM \in E$ by

$$\langle \int F dM, x \rangle = \int \langle F(a + J, b + J), x \rangle dM(a + J, b + J),$$

where $a, b \in \mathcal{A}, x \in E_*$. Let $\varphi \in \mathcal{HOM}_{w_k^*}(\mathcal{A})$ such that $\varphi(J) \subseteq J$. Then the map $\tilde{\varphi} : \frac{\mathcal{A}}{J} \rightarrow \frac{\mathcal{A}}{J}$ by $\tilde{\varphi}(a + J) = \varphi(a) + J$ could be considered as an element of $\mathcal{HOM}_{w_k^*}(\frac{\mathcal{A}}{J})$.

Definition 2.7. Let \mathcal{A} be a dual Banach algebra. An element $M \in \mathcal{L}_{w^*}^2(\frac{\mathcal{A}}{J}, \mathbb{C})^*$ is called a φ -module normal virtual diagonal for \mathcal{A} if $\tilde{w}^{**}(M)$ is an identity for $\frac{\varphi(\mathcal{A})}{J}$ and

$$M. \tilde{\varphi}(c + J) = \tilde{\varphi}(c + J). M \quad (c \in \mathcal{A}).$$

Note that with the above notation $M. (c + J) = (c + J). M$ is equivalent to

$$\int F(ca + J, b + J) dM(a + J, b + J) = \int F(a + J, bc + J) dM(a + J, b + J).$$

Theorem 2.8. *Let \mathcal{A} and \mathcal{U} be dual Banach algebras, let \mathcal{A} be a unital dual Banach \mathcal{U} -module and let \mathcal{A} has an id-module normal virtual diagonal. Then \mathcal{A} is id-Connes module amenable.*

Proof. Let E be a commutative normal Banach \mathcal{A} - \mathcal{U} -module. We first note that \mathcal{A} has an identity. From Theorem 5, it is therefore sufficient for \mathcal{A} to be id-Connes module amenable that we suppose that E is unital. Let $D : \mathcal{A} \rightarrow E$ be a module derivation that is w_{k^*} -continuous. It is straightforward to see that E is a normal Banach $\frac{\mathcal{A}}{J}$ - \mathcal{U} -module. Let $E = (E_*)^*$. Since E is commutative, then $D = 0$ on J . Thus we have $\tilde{D} : \frac{\mathcal{A}}{J} \rightarrow E$, $\tilde{D}(a + J) := D(a)$ ($a \in \mathcal{A}$). To each $x \in E_*$, there corresponds $V_x : \frac{\mathcal{A}}{J} \times \frac{\mathcal{A}}{J} \rightarrow \mathbb{C}$ via $V_x(a + J, b + J) = \langle x, (a + J)\tilde{D}(b + J) \rangle$ ($a, b \in \mathcal{A}$). It is routinely checked that $V_x \in \mathcal{L}_{w^*}^2(\frac{\mathcal{A}}{J}, \mathbb{C})$. For each $a, b \in \mathcal{A}$ and $a_* \in \mathcal{A}_*$ we have

$$\begin{aligned} \langle \int ab + JdM, a_* + J^\perp \rangle &= \langle \int \tilde{w}(a \otimes b + I)dM, a_* + J^\perp \rangle \\ &= \int \langle \tilde{w}(a \otimes b + I), a_* + J^\perp \rangle dM \\ &= \int \langle a \otimes b + I, \tilde{w}^*(a_* + J^\perp) \rangle dM \\ &= \langle \int a \otimes b + IdM, \tilde{w}^*(a_* + J^\perp) \rangle \\ &= \langle M, \tilde{w}^*(a_* + J^\perp) \rangle = \langle \tilde{w}^{**}(M), a_* + J^\perp \rangle, \end{aligned}$$

Now, put $f(x) = \langle M, v_x \rangle$ ($x \in E_*$). Let $c \in \mathcal{A}$. We have

$$\begin{aligned} &\langle (c + J). f - f. (c + J), x \rangle \\ &= \langle f, x. (c + J) - (c + J). x \rangle \\ &= \langle M, V_{x. (c+J)-(c+J). x} \rangle \\ &= \int V_{x. (c+J)-(c+J). x}(a + J, b + J)dM \\ &= \int \langle x. (c + J) - (c + J). x, (a + J)\tilde{D}(b + J) \rangle dM \\ &= \int \langle x, (c + J)(a + J)\tilde{D}(b + J) - (a + J)\tilde{D}(b + J)(c + J) \rangle dM \\ &= \int \langle x, (ca + J)\tilde{D}(b + J) - (a + J)\tilde{D}(b + J)(c + J) \rangle dM, \end{aligned}$$

and so

$$\begin{aligned}
 & \langle (c + J). f - f. (c + J), x \rangle \\
 &= \int \langle x, (a + J)\tilde{D}(bc + J) - (a + J)\tilde{D}(b + J)(c + J) \rangle dM \\
 &= \int \langle x, (a + J)\tilde{D}(b + J)(c + J) + (a + J)(b + J)\tilde{D}(c + J) \\
 &\quad - (a + J)\tilde{D}(b + J)(c + J) \rangle dM \\
 &= \int \langle (a + J)(b + J)\tilde{D}(c + J), x \rangle dM \\
 &= \int \langle (ab + J)\tilde{D}(c + J), x \rangle dM \\
 &= \int \langle (ab + J), x \rangle dM. \tilde{D}(c + J) \\
 &= \langle \tilde{w}^{**}(M). \tilde{D}(c + J), x \rangle.
 \end{aligned}$$

All in all, $D(c) = c. f - f. c$ holds. \square

Let \mathcal{A} be a commutative Banach \mathcal{U} -bimodule. Consider $\mathcal{A} \hat{\otimes}_{\mathcal{U}} \mathcal{A}$ with the product specified by $(a \otimes b)(c \otimes d) = ac \otimes bd$. Let $\varphi \otimes \varphi$ denote the element of $\mathcal{HOM}_{w_{k^*}}(\mathcal{A} \hat{\otimes}_{\mathcal{U}} \mathcal{A})$ satisfying $\varphi \otimes \varphi(a \otimes b) = \varphi(a) \otimes \varphi(b)$ for all $a, b \in \mathcal{A}$. $\varphi \otimes \varphi$ induces a map $\varphi \otimes_{\mathcal{U}} \varphi \in \mathcal{HOM}_{w_{k^*}}(\mathcal{A} \hat{\otimes}_{\mathcal{U}} \mathcal{A})$ with $\varphi \otimes_{\mathcal{U}} \varphi(a \otimes b) = \varphi(a) \otimes \varphi(b) + I$ [7].

Theorem 2.9. *Let \mathcal{A} and \mathcal{U} be dual Banach algebras, let \mathcal{A} be a unital dual Banach \mathcal{U} -module and let $\mathcal{A} \hat{\otimes}_{\mathcal{U}} \mathcal{A}$ be a dual Banach algebra and $\varphi \in \mathcal{HOM}_{w_{k^*}}(\mathcal{A})$. If \mathcal{A} is φ -Connes module amenable, then $\mathcal{A} \hat{\otimes}_{\mathcal{U}} \mathcal{A}$ is $\varphi \otimes_{\mathcal{U}} \varphi$ -Connes module amenable.*

Proof. Let E be a commutative normal Banach $\mathcal{A} \hat{\otimes}_{\mathcal{U}} \mathcal{A}$ - \mathcal{U} -module and $\hat{D} : \mathcal{A} \hat{\otimes}_{\mathcal{U}} \mathcal{A} \rightarrow E$ be a module $\varphi \otimes_{\mathcal{U}} \varphi$ -derivation that is w_{k^*} -continuous. Consider the quotient map $\pi : \mathcal{A} \hat{\otimes}_{\mathcal{U}} \mathcal{A} \rightarrow \mathcal{A} \hat{\otimes}_{\mathcal{U}} \mathcal{A}$. Define

$$(a \otimes b). x = \pi(a \otimes b) \ominus x, \quad x. (a \otimes b) = x \ominus \pi(a \otimes b) \quad (a, b \in \mathcal{A}, x \in E)$$

Since π is w_{k^*} -continuous, then E is a normal Banach $\mathcal{A} \hat{\otimes}_{\mathcal{U}} \mathcal{A}$ - \mathcal{U} -module. Put $\hat{D} \circ \pi : \mathcal{A} \hat{\otimes}_{\mathcal{U}} \mathcal{A} \rightarrow E$. It is easy to see that $\hat{D} \circ \pi$ is a module $\varphi \otimes \varphi$ -derivation that is w_{k^*} -continuous. If $\hat{D} \circ \pi$ is inner, then \hat{D} is inner. Therefore in the following we prove that $D = \hat{D} \circ \pi$ is inner. For with $e_{\mathcal{A}}$ an identity for \mathcal{A} we define

$$a \Delta x = (a \otimes e_{\mathcal{A}}). x, \quad x \Delta a = x. (a \otimes e_{\mathcal{A}}) \quad (a \in \mathcal{A}, x \in E).$$

For $a \in \mathcal{A}$, $x \in E$ and $\alpha \in \mathcal{U}$, we get

$$\begin{aligned} a \Delta (\alpha \cdot x) - (a \cdot \alpha) \Delta x &= (a \otimes e_{\mathcal{A}}) \cdot (\alpha \cdot x) - (a \cdot \alpha \otimes e_{\mathcal{A}}) \cdot x \\ &= (a \otimes e_{\mathcal{A}}) \cdot (\alpha \cdot x) - (\alpha \cdot a \otimes e_{\mathcal{A}}) \cdot x \\ &= (a \otimes e_{\mathcal{A}}) \cdot (\alpha \cdot x) - (\alpha \cdot (a \otimes e_{\mathcal{A}})) \cdot x \\ &= (a \otimes e_{\mathcal{A}}) \cdot (\alpha \cdot x) - ((a \otimes e_{\mathcal{A}}) \cdot \alpha) \cdot x \\ &= (a \otimes e_{\mathcal{A}}) \cdot (\alpha \cdot x) - (a \otimes e_{\mathcal{A}}) \cdot (\alpha \cdot x) = 0 \end{aligned}$$

and the same for the right or two-sided actions. So E is a commutative normal Banach $\mathcal{A}\mathcal{U}$ -bimodule. Put $D_{\mathcal{A}} : \mathcal{A} \rightarrow E$, $D_{\mathcal{A}}(a) = D(a \otimes e_{\mathcal{A}})$, then

$$\begin{aligned} D_{\mathcal{A}}(ab) &= D(ab \otimes e_{\mathcal{A}}) \\ &= D(a \otimes e_{\mathcal{A}}) \cdot \varphi \otimes \varphi(b \otimes e_{\mathcal{A}}) + \varphi \otimes \varphi(a \otimes e_{\mathcal{A}}) \cdot D(b \otimes e_{\mathcal{A}}) \\ &= D_{\mathcal{A}}(a) \Delta \varphi(b) + \varphi(a) \Delta D_{\mathcal{A}}(b). \end{aligned}$$

Since \mathcal{A} is φ -Connes module amenable, there is $u \in E$ such that $D_{\mathcal{A}} = ad_u$. Therefore, $\tilde{D} = D - ad_u$ vanishes on $\mathcal{A} \otimes e_{\mathcal{A}}$. Setting

$$a \nabla x = (e_{\mathcal{A}} \otimes a) \cdot x, \quad x \nabla a = x \cdot (e_{\mathcal{A}} \otimes a) \quad (a \in \mathcal{A}, x \in E)$$

makes E into an $\mathcal{A}\mathcal{U}$ -bimodule. Let us now, $D'_{\mathcal{A}}(a) = \tilde{D}(e_{\mathcal{A}} \otimes a)$ ($a \in \mathcal{A}$). Set $K = \{e \in E_* : \langle \tilde{D}(e_{\mathcal{A}} \otimes a), e \rangle = 0\}$. Since \tilde{D} is w_{k^*} -continuous, by a similar argument of [17, Theorem 4.9] we have $(\frac{E_*}{K})^* = \overline{\tilde{D}(e_{\mathcal{A}} \otimes a)}^{w_k^*}$. Further, $\overline{\tilde{D}(e_{\mathcal{A}} \otimes a)}^{w_k^*}$ is a w_{k^*} -closed submodule of E .

All in all $\overline{\tilde{D}(e_{\mathcal{A}} \otimes a)}^{w_k^*}$ is a commutative normal Banach $\mathcal{A}\mathcal{U}$ -module.

Then there is $v \in \overline{\tilde{D}(e_{\mathcal{A}} \otimes a)}^{w_k^*}$ such that

$$\tilde{D}(e_{\mathcal{A}} \otimes a) = D'_{\mathcal{A}}(a) = \varphi(a) \nabla v - v \nabla \varphi(a) = \varphi \otimes \varphi(e_{\mathcal{A}} \otimes a) \cdot v - v \cdot \varphi \otimes \varphi(e_{\mathcal{A}} \otimes a)$$

and $\tilde{D} - ad_v|_{(e_{\mathcal{A}} \otimes \mathcal{A})} = \{0\}$. Consequently $\tilde{D} - ad_v = D - ad_u - ad_v$ vanishes on $\mathcal{A} \hat{\otimes} \mathcal{A}$. This complete the proof. \square

3. χ -CONNES MODULE AMENABILITY OF SEMIGROUP ALGEBRAS

A discrete semigroup S is called an inverse semigroup if for each $x \in S$ there is a unique element $x^* \in S$ such that $xx^*x = x$ and $x^*xx^* = x^*$. An element $e \in S$ is called an idempotent if $e = e^* = e^2$. The set of idempotent elements of S is denoted by E . For $s \in S$, we define $L_s, R_s : S \rightarrow S$ by $L_s(t) = st, R_s(t) = ts, (t \in S)$. If for each $s \in S$, L_s and R_s are finite-to-one maps, then we say that S is weakly cancellative.

Before turning our result, we note that if S is a weakly cancellative semigroup, then $l^1(S)$ is a dual Banach algebra with predual $c_0(S)$ [8].

In Theorem 2.8 it is shown that if a unital Banach algebra \mathcal{A} has an id -module normal virtual diagonal, then \mathcal{A} is id -Connes module amenable. It would be interesting to know that the converse holds for inverse semigroup algebra $l^1(S)$.

For an inverse semigroup S , we consider an equivalence relation on S where $s \sim t$ if and only if there is $e \in E$ such that $se = te$. The quotient semigroup $S_G = \frac{S}{\sim}$ is a group [13]. It is easy to see that E is a commutative subsemigroup of S . Therefore, $l^1(S)$ is a Banach $l^1(E)$ -module with compatible canonical actions. Let $l^1(E)$ acts on $l^1(S)$ by the multiplication from right and trivially from left, that is

$$\delta_e \cdot \delta_s = \delta_s, \delta_s \cdot \delta_e = \delta_{se} = \delta_s * \delta_e \quad (s \in S, e \in E).$$

With above notation, $l^1(S_G)$ is a quotient of $l^1(S)$ and so the above action of $l^1(E)$ on $l^1(S)$ lifts to an action of $l^1(E)$ on $l^1(S_G)$, making it a Banach $l^1(E)$ -module [1].

Theorem 3.1. *Let S be a weakly cancellative semigroup. Let S be an inverse semigroup with idempotents E , let $l^1(S)$ be a Banach $l^1(E)$ -module and let $\chi \in \mathcal{HOM}_{w_k^*}(l^1(S))$. If $l^1(S)$ is χ -Connes module amenable, then $l^1(S)$ has a χ -module normal virtual diagonal.*

Proof. Let $\pi : S \rightarrow S_G$ be the quotient map. By [1, Lemma 3.2], we define a bimodule action of $l^1(S)$ on $l^\infty(S_G)$ by

$$\delta_s \cdot x = \delta_{\pi(s)} * x, \quad x \cdot \delta_s = x * \delta_{\pi(s)} \quad (s \in S, x \in l^\infty(S_G)).$$

Since $c_0(S_G)$ is an introverted subspace of $l^\infty(S_G)$ [9], then $l^\infty(S_G)^*$ is a normal Banach $l^1(S)$ - $l^1(E)$ -module. Choose $n \in l^\infty(S_G)^*$ with $\langle n, 1 \rangle = 1$, and define $D : l^1(S) \rightarrow l^\infty(S_G)^*$ by $D(\delta_s) = \chi(\delta_s) \cdot n - n \cdot \chi(\delta_s)$. Moreover, D attains its values in the weak*-closed submodule $(\frac{l^\infty(S_G)}{\mathbb{C}})^*$. Since $l^1(S)$ is χ -Connes module amenable, then D is inner.

Consequently, there exists $\tilde{n} \in (\frac{l^\infty(S_G)}{\mathbb{C}})^*$ such that $D(\delta_s) = ad_{\tilde{n}}$, so

$$\tilde{\chi}(\delta_{\pi(s)}) \cdot n - n \cdot \tilde{\chi}(\delta_{\pi(s)}) = \tilde{\chi}(\delta_{\pi(s)}) \cdot \tilde{n} - \tilde{n} \cdot \tilde{\chi}(\delta_{\pi(s)}).$$

For each $f \in l^\infty(S_G)$,

$$\langle \tilde{\chi}(\delta_{\pi(s)}) \cdot (n - \tilde{n}) - (n - \tilde{n}) \cdot \tilde{\chi}(\delta_{\pi(s)}), f \rangle = 0.$$

Now put $m := n - \tilde{n} \in l^\infty(S_G)^*$, we have

$$\langle \tilde{\chi}(\delta_{\pi(s)}) \cdot m - m \cdot \tilde{\chi}(\delta_{\pi(s)}), f \rangle = 0.$$

By a similar argument as in [18, Lemma 7.1.1], there exists a net $\{f_\alpha\}$ of $l^1(S_G)$ such that $\int f_\alpha = 1$ and $\| \tilde{\chi}(\delta_{\pi(s)}) * f_\alpha - f_\alpha * \tilde{\chi}(\delta_{\pi(s)}) \| \rightarrow 0$.

Now let $f \in c_0(S_G \times S_G)$. Take $\epsilon > 0$ and consider a compact set K such that $\|f(x)\|_{S_G \setminus K} < \sqrt{\epsilon}$ and

$$\sup_{s \in K} \|\tilde{\chi}(\delta_{\pi(s)}) * f_\alpha - f_\alpha * \tilde{\chi}(\delta_{\pi(s)})\| < \frac{\sqrt{\epsilon}}{\|f\|}.$$

Since the quotient map is continuous and open, then by [20, Proposition 3.1] we have $\mathcal{L}_{w_k^*}^2(l^1(S_G), \mathbb{C}) = c_0(S_G \times S_G)$. Then we may define

$$\langle M, f \rangle = \lim_{\alpha} \int f(\tilde{\chi}(\delta_{\pi(x^*)}), \tilde{\chi}(\delta_{\pi(x)})) f_\alpha(x) dx.$$

By the above argument, for each $s \in S$ there exists α_0 such that for each $\alpha > \alpha_0$, $\|\tilde{\chi}(\delta_{\pi(s)}) * f_\alpha - f_\alpha * \tilde{\chi}(\delta_{\pi(s)})\| < \frac{\sqrt{\epsilon}}{2}$. Hence

$$\begin{aligned} & \langle \tilde{\chi}(\delta_{\pi(s)}) \cdot M - M \cdot \tilde{\chi}(\delta_{\pi(s)}), f \rangle = \langle M, f \cdot \tilde{\chi}(\delta_{\pi(s)}) - \tilde{\chi}(\delta_{\pi(s)}) \cdot f \rangle \\ & = \lim_{\alpha} \int \left(f(\tilde{\chi}(\delta_{\pi(s)\pi(x^*)}), \tilde{\chi}(\delta_{\pi(x)})) - f(\tilde{\chi}(\delta_{\pi(x^*)}), \tilde{\chi}(\delta_{\pi(xs)})) \right) f_\alpha(x) dx \\ & \leq \|f\|_{S_G \setminus K} \|\tilde{\chi}(\delta_{\pi(s)}) * f_\alpha - f_\alpha * \tilde{\chi}(\delta_{\pi(s)})\| \\ & + \|f\|_K \|\tilde{\chi}(\delta_{\pi(s)}) * f_\alpha - f_\alpha * \tilde{\chi}(\delta_{\pi(s)})\| < \epsilon. \end{aligned}$$

Also for each s

$$\begin{aligned} \tilde{w}^{**}(M) \cdot \tilde{\chi}(\delta_{\pi(s)}) & = \langle M, \tilde{w}^*(\tilde{\chi}(\delta_{\pi(s)})) \rangle \\ & = \lim_{\alpha} \int (\tilde{w}^*(\tilde{\chi}(\delta_{\pi(s)})))(\tilde{\chi}(\delta_{\pi(x^*)}), \tilde{\chi}(\delta_{\pi(x)})) f_\alpha(x) dx \\ & = \lim_{\alpha} \int \tilde{\chi}(\delta_{\pi(s)}) \tilde{\chi}(\delta_{\pi(x^*)}) \tilde{\chi}(\delta_{\pi(x)}) f_\alpha(x) dx \\ & = \lim_{\alpha} \int \tilde{\chi}(\delta_{\pi(s)}) \delta_{\pi(x^*)} \delta_{\pi(x)} f_\alpha(x) dx \\ & = \lim_{\alpha} \tilde{\chi}(\delta_{\pi(s)}) \int f_\alpha(x) dx = \tilde{\chi}(\delta_{\pi(s)}). \end{aligned}$$

Consequently, M is a χ -normal module virtual diagonal for $l^1(S)$. \square

Corollary 3.2. *Let S be a weakly cancellative semigroup, let S be an inverse semigroup with idempotents E and let $l^1(S)$ be a Banach $l^1(E)$ -module. Then the following are equivalent:*

- (i) $l^1(S)$ is Connes module amenable.
- (ii) $l^1(S)$ has a module normal virtual diagonal.

Proof. This follows immediately from Theorem 2.8 and Theorem 3.1. \square

Example 10. Let (\mathbb{N}, \vee) be the semigroup of positive integers with maximum operation. Since \mathbb{N} is weakly cancellative, then $l^1(\mathbb{N})$ is a dual Banach algebra with predual $c_0(\mathbb{N})$. By [8, Theorem 5.13], $l^1(\mathbb{N})$

is not Connes amenable. Moreover $l^1(\mathbb{N})$ is module amenable on $l^1(E_{\mathbb{N}})$, so it is Connes module amenable (see [2]).

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Ali Ghaffari

Department of Mathematics, University of Semnan, P.O. Box 35195-363, Semnan, Iran.

Email: aghaffari@semnan.ac.ir

Samaneh Javadi

Faculty of Engineering- East Guilan, University of Guilan, P.O. Box 44891-63157, Rudsar, Iran.

Email: s.javadi62@gmail.com

Ebrahim Tamimi

Department of Mathematics, University of Semnan, P.O. Box 35195-363, Semnan, Iran.

Email: tamimi-ebrahim@yahoo.com

φ -CONNES MODULE AMENABILITY OF DUAL BANACH ALGEBRAS

A. GHAFFARI, S. JAVADI AND E. TAMIMI

φ -کنز میانگین‌پذیری مدولی از جبرهای باناخ دوگان

علی غفاری^۱، سمانه جوادی^۲ و ابراهیم تمیمی^۱

^۱گروه ریاضی محض، دانشکده علوم ریاضی، دانشگاه سمنان، سمنان، ایران

^۲دانشکده مهندسی گیلان-شرق، دانشگاه گیلان، رودسر، ایران

در این مقاله φ -کنز میانگین‌پذیری مدولی از جبرهای باناخ دوگان A را تعریف می‌کنیم که در آن φ یک هم‌ریختی مدولی کراندار از A به A بوده که ضعیف ستاره پیوسته است. بررسی φ مدول نرمال و قطرهای واقعی از اهداف این مقاله است. فرض کنید S یک نیم‌گروه معکوس و حذفی ضعیف بوده و E زیر نیم‌گروه از خودتوان‌های S باشد. اگر χ یک هم‌ریختی مدولی کراندار و ضعیف ستاره پیوسته از $l^1(S)l^1(S)l^1(E)$ χ کنز میانگین‌پذیر مدولی باشد، آنگاه دارای یک χ -مدول نرمال و قطر واقعی است. در حالی که $\chi = id$ عکس این موضوع درست است.

کلمات کلیدی: جبرهای باناخ، میانگین‌پذیری مدولی، اشتقاق و جبر نیم‌گروهی.