

ANNIHILATOR OF LOCAL COHOMOLOGY MODULES
UNDER THE RING EXTENSION $R \subset R[X]$

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ABSTRACT. Let R be a commutative Noetherian ring, I an ideal of R and M a non-zero R -module. In this paper, we calculate the extension of annihilator of local cohomology modules $H_I^t(M)$, $t \geq 0$, under the ring extension $R \subset R[X]$ (resp., $R \subset R[[X]]$). By using this extension we will present some of the faithfulness conditions of local cohomology modules, and show that if the Lynch's conjecture [11] holds in $R[[X]]$, then it will hold in R .

1. INTRODUCTION

Throughout this paper, R denotes a commutative Noetherian ring (with identity) and I is an ideal of R . The local cohomology modules $H_I^i(M)$, $i = 0, 1, 2, \dots$, of an R -module M with respect to I were introduced by Grothendieck [9]. They arise as the derived functors of the left exact functor $\Gamma_I(-)$, where for an R -module M , $\Gamma_I(M)$ is the submodule of M consisting of all elements annihilated by some power of I , i.e., $\bigcup_{n=1}^{\infty} (0 :_M I^n)$. There is a natural isomorphism

$$H_I^i(M) \cong \varinjlim_{n \geq 1} \text{Ext}_R^i(R/I^n, M).$$

We refer the reader to [9] or [5] for more details about local cohomology.

MSC(2010): Primary: 13D45; Secondary: 14B15, 13E05.

Keywords: Annihilator, cohomological dimension, faithfully flat, local cohomology, zero-divisor.

Received: 7 April 2019, Accepted: 18 December 2019.

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Recall that, for an R -module M , the *cohomological dimension* of M with respect to I , denoted by $\text{cd}(I, M)$, is defined as

$$\text{cd}(I, M) := \sup\{i \in \mathbb{Z} : H_i^t(M) \neq 0\}.$$

One of the important problems in commutative algebra is determining the annihilator of local cohomology modules. This problem has been studied by several authors; see, for example, [1, 2, 3, 10, 11, 13]. In this paper, we will calculate the extension of annihilator of local cohomology modules, under the ring extension $R \subset R[X]$ (resp., $R \subset R[[X]]$), as the first main result.

Lynch in [11] conjectured the following:

For every Noetherian local ring (R, \mathfrak{m}) and any ideal I of R , if $\text{cd}(I, R) = t > 0$ then $\dim R / \text{Ann}_R H_I^t(R) = \dim R / \Gamma_I(R)$.

An another aim of this paper is to find a relation between the Lynch's conjecture in R and $R[[X]]$.

2. MAIN RESULTS

The following lemma will be quite useful in this section.

Lemma 2.1. *Let R be a Noetherian ring and M be a non-zero R -module. Let X be an indeterminate over R . Then for every monic polynomial $f \in R[X]$ of positive degree, the following statements hold:*

- (i) $\Gamma_{fR[X]}(M \otimes_R R[X]) = 0$.
- (ii) For every positive integer n ,

$$(0 :_{H_{fR[X]}^1(M \otimes_R R[X])} f^n) \cong M[X] / f^n M[X].$$

In particular, $H_{fR[X]}^1(M \otimes_R R[X]) \neq 0$.

- (iii) $\text{cd}(IR[X] + fR[X], R[X]) = \text{cd}(I, R) + 1$.

Proof. See [4, Lemma 2.9 and Theorem 2.10]. □

The next theorem is the first main result of this paper.

Theorem 2.2. *Let R be a Noetherian ring, I an ideal of R and M a non-zero R -module. Let $H_I^t(M) \neq 0$, for integer $t \geq 0$. If $J := \text{Ann}_R H_I^t(M)$, then*

$$\text{Ann}_{R[X]} H_{IR[X]}^t(M[X]) = \text{Ann}_{R[X]} H_{IR[X]+fR[X]}^{t+1}(M[X]) = JR[X],$$

where X is an indeterminate over R and $f \in R[X]$ is a monic polynomial of positive degree.

Proof. Since $R[X]$ is a faithfully flat R -algebra, it follows from [5, Theorem 4.3.2] that $H_I^t(M) \longrightarrow H_{IR[X]}^t(M[X])$ is injective. Therefore,

$$J = \text{Ann}_R H_I^t(M) = \text{Ann}_R H_{IR[X]}^t(M[X]) = \text{Ann}_{R[X]} H_{IR[X]}^t(M[X]) \cap R.$$

So $JR[X] \subseteq \text{Ann}_{R[X]} H_{IR[X]}^t(M[X])$.

On the other hand, using Lemma 2.1 and [14, Corollary 1.4] yield the following isomorphism

$$H_{IR[X]+fR[X]}^{t+1}(M[X]) \cong H_{fR[X]}^1(H_{IR[X]}^t(M[X])) \neq 0.$$

So

$$\text{Ann}_{R[X]} H_{IR[X]}^t(M[X]) \subseteq \text{Ann}_{R[X]} H_{fR[X]}^1(H_{IR[X]}^t(M[X])),$$

and hence

$$JR[X] \subseteq \text{Ann}_{R[X]} H_{IR[X]+fR[X]}^{t+1}(M[X]).$$

Now, we claim that $\text{Ann}_{R[X]} H_{fR[X]}^1(H_{IR[X]}^t(M[X])) \subseteq JR[X]$. Let $g = a_0 + a_1X + \dots + a_nX^n$ be a non-zero polynomial with $a_j \in R$, for all $0 \leq j \leq n$, and let $g \notin JR[X]$ but $g \in \text{Ann}_{R[X]} H_{fR[X]}^1(H_{IR[X]}^t(M[X]))$. Since $g \notin JR[X]$, it follows that there exists $a_{j'} \in R \setminus J$, for $0 \leq j' \leq n$, such that $a_{j'}H_I^t(M) \neq 0$ and hence $a_{j'}b \neq 0$ for some $b \in H_I^t(M)$. It is clear that

$$g \notin \text{Ann}_{R[X]}((H_I^t(M))[X]) = \text{Ann}_{R[X]} H_{IR[X]}^t(M[X]).$$

On the other hand, using Lemma 2.1 yields the following exact sequence

$$0 \longrightarrow (H_I^t(M))[X] \longrightarrow ((H_I^t(M))[X])_f \longrightarrow H_{fR[X]}^1(H_{IR[X]}^t(M[X])) \longrightarrow 0,$$

which implies that $((H_I^t(M))[X])_f / (H_I^t(M))[X] \cong H_{fR[X]}^1(H_{IR[X]}^t(M[X]))$. Thus

$$g((H_I^t(M))[X])_f \subseteq (H_I^t(M))[X].$$

In fact, $g(((H_I^t(M))[X])[1/f]) \subseteq (H_I^t(M))[X]$. Let $m > n$ be an integer, and set $h := b/f^m \in ((H_I^t(M))[X])[1/f]$. Since f is a monic polynomial of positive degree, it follows that f^m is a monic polynomial of degree at least m . Since $gh \in (H_I^t(M))[X]$, it follows that $gb \in f^m(H_I^t(M))[X]$. But, $0 \neq gb = a_0b + a_1bX + \dots + a_nbX^n$ and $n < m$, which is a contradiction. Therefore,

$$\begin{aligned} JR[X] &\subseteq \text{Ann}_{R[X]}((H_I^t(M))[X]) \\ &= \text{Ann}_{R[X]} H_{IR[X]}^t(M[X]) \\ &\subseteq \text{Ann}_{R[X]} H_{fR[X]}^1(H_{IR[X]}^t(M[X])) \\ &\subseteq JR[X]. \end{aligned}$$

Thus, we obtain that

$$\text{Ann}_{R[X]} H_{IR[X]}^t(M[X]) = \text{Ann}_{R[X]} H_{IR[X]+fR[X]}^{t+1}(M[X]) = JR[X].$$

□

With a similar argument, we have the following corollary.

Corollary 2.3. *Let R be a Noetherian ring, I an ideal of R and M a non-zero R -module. Let $H_I^t(M) \neq 0$, for integer $t \geq 0$. If $J := \text{Ann}_R H_I^t(M)$, then*

$$\text{Ann}_{R[[X]]} H_{IR[[X]]}^t(M[[X]]) = \text{Ann}_{R[[X]]} H_{IR[[X]]+XR[[X]]}^{t+1}(M[[X]]) = JR[[X]].$$

□

Corollary 2.4. *Let R be a Noetherian ring, I an ideal of R and M a non-zero R -module. Let X be an indeterminate over R and f be a monic polynomial of positive degree. For an integer $t \geq 0$, $\text{Ann}_R H_I^t(M) = 0$ if and only if $\text{Ann}_{R[X]} H_{IR[X]+fR[X]}^{t+1}(M[X]) = 0$ (resp., $\text{Ann}_{R[[X]]} H_{IR[[X]]+XR[[X]]}^{t+1}(M[[X]]) = 0$).*

Proof. The assertion follows from Theorem 2.2 (resp., Corollary 2.3). □

The following theorem will be useful in the proof of Corollary 2.6.

Theorem 2.5. *Let R be a (not necessarily local) Noetherian ring, I an ideal of R and M a finitely generated R -module such that $\text{cd}(I, M) = t > 0$. Then $H_I^t(M)$ is not finitely generated.*

Proof. First, it is clear that for every $\mathfrak{p} \in \text{Supp } H_I^t(M)$, $\text{cd}(IR_{\mathfrak{p}}, M_{\mathfrak{p}}) = \text{cd}(I, M) = t$. So, without loss of generality, we may assume that (R, \mathfrak{m}) is a Noetherian local ring and M is a finitely generated R -module. Since $\text{Supp } M = \text{Supp } R/\text{Ann } M$, it follows from [7, Theorem 2.2] and [5, Theorem 4.2.1] (the Independence Theorem) that

$$\text{cd}(I, M) = \text{cd}(I, R/\text{Ann } M) = \text{cd}(I(R/\text{Ann } M), R/\text{Ann } M).$$

Since $H_{I(R/\text{Ann } M)}^t(-)$ is a right exact functor and M is an $R/\text{Ann } M$ -module, it follows from [5, Exercise 6.1.8] that

$$\begin{aligned} H_I^t(M)/\mathfrak{m} H_I^t(M) &\cong H_I^t(M) \otimes_R R/\mathfrak{m} \\ &\cong (H_{I(R/\text{Ann } M)}^t(R/\text{Ann } M) \otimes_{R/\text{Ann } M} M) \otimes_R R/\mathfrak{m} \\ &\cong H_{I(R/\text{Ann } M)}^t(R/\text{Ann } M) \otimes_{R/\text{Ann } M} M/\mathfrak{m} M \\ &\cong H_{I(R/\text{Ann } M)}^t(M/\mathfrak{m} M) \cong H_I^t(M/\mathfrak{m} M) = 0. \end{aligned}$$

Therefore, $H_I^t(M) = \mathfrak{m} H_I^t(M)$ and hence by Nakayama's lemma we can deduce that the R -module $H_I^t(M)$ is not finitely generated. □

Corollary 2.6. *Let R be a (not necessarily local) Noetherian ring, I an ideal of R and M a finitely generated R -module. If $\text{cd}(I, M) = 1$, then $\text{Ann}_R H_I^1(M) \subseteq Z_R(M)$.*

Proof. Let $\text{Ann}_R H_I^1(M) \not\subseteq Z_R(M)$. Hence there exists $x \in \text{Ann}_R H_I^1(M)$, such that $x \notin Z_R(M)$. An exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$$

induces the exact sequence

$$\Gamma_I(M/xM) \xrightarrow{f} H_I^1(M) \xrightarrow{x} H_I^1(M),$$

which implies that $H_I^1(M)$ is a finitely generated R -module. But, in view of Theorem 2.5, this is a contradiction. \square

For the next result we need the following lemma.

Lemma 2.7. *Let R be a commutative Noetherian ring and X be an indeterminate over R . Then every associated prime ideal of $R[X]$ is extended, and hence*

$$\text{Ass}_{R[X]}(R[X]) = \{\mathfrak{p} R[X] : \mathfrak{p} \in \text{Ass}_R(R)\}.$$

Proof. See [8, Theorem]. \square

Theorem 2.8. *Let R be a Noetherian ring, I an ideal of R and M a non-zero R -module. Let X be an indeterminate over R and $f \in R[X]$ be a monic polynomial of positive degree. Then for an integer $t \geq 0$, $\text{Ann}_R H_I^t(M) \subseteq Z_R(R)$ (the set of all zero-divisors of R) if and only if $\text{Ann}_{R[X]} H_{IR[X]+fR[X]}^{t+1}(M[X]) \subseteq Z_{R[X]}(R[X])$.*

Proof. The assertion follows from Theorem 2.2, [12, Theorem 7.5] and Lemma 2.7. \square

Corollary 2.9. *Let R be a Noetherian ring and I an ideal of R with $\text{cd}(I, R) = 1$. Let X be an indeterminate over R and $f \in R[X]$ be a monic polynomial of positive degree. Then*

$$\text{Ann}_{R[X]} H_{IR[X]+fR[X]}^2(R[X]) \subseteq Z_{R[X]}(R[X]).$$

In particular, if R is an integral domain, then

$$\text{Ann}_{R[X]} H_{IR[X]+fR[X]}^2(R[X]) = \text{Ann}_{R[X]} H_{IR[X]}^1(R[X]) = 0.$$

Proof. The first assertion follows from Corollary 2.6 and Theorem 2.8. Moreover, the last assertion is immediate from Corollary 2.4. \square

Remark 2.10. Let R be a Noetherian ring, I an ideal of R and M an R -module. If $H_I^t(M) \neq 0$, for integer t , then it follows from Lemma 2.1 that $H_{IR[X]}^t(M[X])$ is not an injective $R[X]$ -module. So, $\text{injdim}_{R[X]} H_{IR[X]}^t(M[X]) > 0$.

Corollary 2.11. *Let R be a Noetherian ring, I an ideal of R and M an R -module, such that $H_I^t(M) \neq 0$, for integer $t \geq 0$. If*

$$\text{injdim}_{R[X]} H_{IR[X]}^t(M[X]) = 1,$$

then

$$\text{Ann}_{R[X]} H_{IR[X]+fR[X]}^{t+1}(M[X]) = \text{Ann}_{R[X]} H_{IR[X]}^t(M[X]) \subseteq Z_{R[X]}(R[X]),$$

where f is a monic polynomial of positive degree.

In particular, $\text{Ann}_R H_I^t(M) \subseteq Z_R(R)$.

Proof. Since $\text{injdim}_{R[X]} H_{IR[X]}^t(M[X]) = 1$, there is an exact sequence

$$0 \longrightarrow H_{IR[X]}^t(M[X]) \longrightarrow \mathbb{E}_0 \longrightarrow \mathbb{E}_1 \longrightarrow 0,$$

as an injective resolution of $H_{IR[X]}^t(M[X])$, which induces the following exact sequence

$$\Gamma_{fR[X]}(\mathbb{E}_1) \longrightarrow H_{fR[X]}^1(H_{IR[X]}^t(M[X])) \longrightarrow 0.$$

If

$$\begin{aligned} \text{Ann}_{R[X]} H_{IR[X]+fR[X]}^{t+1}(M[X]) &= \text{Ann}_{R[X]} H_{fR[X]}^1(H_{IR[X]}^t(M[X])) \\ &\not\subseteq Z_{R[X]}(R[X]), \end{aligned}$$

then there exists $g \in \text{Ann}_{R[X]} H_{fR[X]}^1(H_{IR[X]}^t(M[X])) \setminus Z_{R[X]}(R[X])$. So, we have the following exact sequence

$$\Gamma_{fR[X]}(\mathbb{E}_1)/g\Gamma_{fR[X]}(\mathbb{E}_1) \longrightarrow H_{fR[X]}^1(H_{IR[X]}^t(M[X])) \longrightarrow 0.$$

Since $H_{fR[X]}^1(H_{IR[X]}^t(M[X])) \neq 0$, it follows that $\Gamma_{fR[X]}(\mathbb{E}_1) \neq g\Gamma_{fR[X]}(\mathbb{E}_1)$. On the other hand, $\Gamma_{fR[X]}(\mathbb{E}_1)$ is an injective $R[X]$ -module. Thus the exact sequence

$$0 \longrightarrow R[X] \xrightarrow{g} R[X] \longrightarrow R[X]/gR[X] \longrightarrow 0,$$

induces the exact sequence $\Gamma_{fR[X]}(\mathbb{E}_1) \xrightarrow{g} \Gamma_{fR[X]}(\mathbb{E}_1) \longrightarrow 0$. Therefore, $\Gamma_{fR[X]}(\mathbb{E}_1) = g\Gamma_{fR[X]}(\mathbb{E}_1)$, which is a contradiction. \square

We need the following notation in the proof of Corollary 2.12.

Notation. [6, Theorem A.11] Let $(R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ be a homomorphism of Noetherian local rings. If M is a finitely generated R -module and N is an R -flat finitely generated S -module, then

$$\dim_S(M \otimes_R N) = \dim_R M + \dim_S N/\mathfrak{m}N.$$

Corollary 2.12. *Let (R, \mathfrak{m}) be a commutative Noetherian local ring, I an ideal of R and i be a non-negative integer. Let X be an indeterminate over R , and set $S := R[[X]]$. Then, the following statements are equivalent:*

- (i) $\dim_R R/\text{Ann}_R H_I^i(R) = \dim_R R/\Gamma_I(R)$.
- (ii) $\dim_S S/\text{Ann}_S H_{IS}^i(S) = \dim_S S/\Gamma_{IS}(S)$.

Proof. Since $R \rightarrow S$ is a faithfully flat ring homomorphism, it follows from Corollary 2.3 that

$$R/\text{Ann}_R H_I^i(R) \otimes_R S \cong S/\text{Ann}_S H_{IS}^i(S).$$

Also using [5, Theorem 4.3.2] yields the following isomorphism

$$R/\Gamma_I(R) \otimes_R S \cong S/\Gamma_{IS}(S).$$

Now, the assertion follows from the Notation. □

Now, we are ready to state and prove the second main result of this paper.

Corollary 2.13. *Let (R, \mathfrak{m}) be a commutative Noetherian local ring and let I be an ideal of R with $\text{cd}(I, R) = t > 0$. If the Lynch's conjecture holds in $R[[X]]$, then it holds in R .*

Proof. By the assumption we have

$$\dim_{R[[X]]} R[[X]]/\text{Ann}_{R[[X]]} H_{\mathbb{J}}^{\text{cd}(\mathbb{J}, R[[X]])}(R[[X]]) = \dim_{R[[X]]} R[[X]]/\Gamma_{\mathbb{J}}(R[[X]]),$$

for any ideal \mathbb{J} of $R[[X]]$ with $\text{cd}(\mathbb{J}, R[[X]]) > 0$. Since $R[[X]]$ is a faithfully flat R -algebra, it follows easily that

$$\text{cd}(IR[[X]], R[[X]]) = \text{cd}(I, R) = t > 0.$$

So, we have

$$\dim_{R[[X]]} R[[X]]/\text{Ann}_{R[[X]]} H_{IR[[X]]}^t(R[[X]]) = \dim_{R[[X]]} R[[X]]/\Gamma_{IR[[X]]}(R[[X]]).$$

Now using Corollary 2.12 yields the assertion. □

Acknowledgments

The authors are deeply grateful to the referee for a very careful reading of the manuscript and many valuable suggestions.

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ANNIHILATOR OF LOCAL COHOMOLOGY MODULES UNDER THE
RING EXTENSION $R \subset R[X]$

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پوچساز مدول‌های کوهمولوژی موضعی تحت توسیع حلقه‌ای $R \subset R[X]$

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فرض کنید R حلقه‌ای جابجایی و نوتری، I ایده‌آلی از R و M یک R -مدول غیرصفر باشد. ما در این مقاله توسیع پوچساز مدول‌های کوهمولوژی موضعی $H_I^t(M)$ ، $t \geq 0$ ، را تحت توسیع حلقه‌ای $R \subset R[X]$ (به ترتیب $R \subset R[[X]]$) محاسبه می‌کنیم. با استفاده از این توسیع، برخی از شرایط وفاداری مدول‌های کوهمولوژی موضعی را ارائه داده، و نشان می‌دهیم که اگر حدس لینچ [۱۱]، در $R[[X]]$ برقرار باشد، آنگاه در R نیز برقرار خواهد بود.

کلمات کلیدی: پوچساز، بعد کوهمولوژیکی، یکدست وفادار، کوهمولوژی موضعی، مقسوم‌علیه صفر.