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SOME RESULTS ON THE COMPLEMENT OF THE INTERSECTION GRAPH OF SUBGROUPS OF A FINITE GROUP

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Abstract. In this article we consider groups G such that G admits at least one nontrivial subgroup (recall that a subgroup H of G is said to be nontrivial if $H \notin \{G, \{e\}\}$). Let G be a group. Recall that the intersection graph of subgroups of G, denoted by $\Gamma(G)$, is an undirected graph whose vertex set is the set of all nontrivial subgroups of G and distinct vertices H, K are joined by an edge in this graph if and only if $H \cap K \neq \{e\}$. Let G be a finite group. The aim of this article is to investigate the interplay between the group-theoretic properties of a finite group G and the graph-theoretic properties of the complement of $\Gamma(G)$.

1. Introduction

Let G be a group which admits at least one nontrivial subgroup. Recall that the *intersection graph of* G, denoted by $\Gamma(G)$ is an undirected simple graph whose vertex set is the set of all nontrivial subgroups of G and distinct vertices H, K are joined by an edge in this graph if and only if $H \cap K \neq \{e\}$. The intersection graphs of groups have been investigated by several algebraists (for example, refer the articles [1, 4, 7, 8, 9, 11, 12]). Let $G = (V, E)$ be a simple graph. Recall from [2, Definition 1.1.13] that the *complement of* G, denoted by G^c is a graph whose vertex set is V and distinct vertices u, v are joined by an

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edge in G^c if and only if there is no edge joining u and v in G. Thus for a group G which admits at least one nontrivial subgroup, $(\Gamma(G))^c$ is a graph whose vertex set is the set of all nontrivial subgroups of G and distinct vertices H, K are joined by an edge in $(\Gamma(G))^c$ if and only if $H \cap K = \{e\}$. The groups considered in this article are finite which admit at least one nontrivial subgroup. Let G be a finite group. The purpose of this article is to investigate the effect of certain graph parameters of $(\Gamma(G))^c$ on the group structure of G.

It is useful to recall the following definitions and results from graph theory before we give an account of results that are proved on $(\Gamma(G))^c$, where G is a finite group which admits at least one nontrivial subgroup. The graphs considered in this article are undirected and simple. Let $G = (V, E)$ be a graph. Let $a, b \in V, a \neq b$. Recall from [2] that the *distance* between a and b, denoted by $d(a, b)$ is defined as the length of a shortest path in G between a and b if such a path exists in G. Otherwise, we define $d(a, b) = \infty$. We define $d(a, a) = 0$. A graph $G = (V, E)$ is said to be *connected* if for any distinct $a, b \in V$, there exists a path in G between a and b. Let $G = (V, E)$ be a connected graph. Recall from [2, Definition 4.2.1] that the *diameter* of G, denoted by $diam(G)$ is defined as $diam(G) = sup\{d(a, b) : a, b \in V\}$. Let $a \in V$. The *eccentricity* of a, denoted by $e(a)$ is defined as $e(a) = sup\{d(a, b) : b \in V\}$. The *radius* of G, denoted by $r(G)$ is defined as $r(G) = min\{e(a) : a \in V\}$.

Let $G = (V, E)$ be a graph. Suppose that G contains a cycle. Recall from [2, p. 159] that the *girth* of G, denoted by $girth(G)$ is the length of a shortest cycle in G. If G does not contain any cycle, then we set $girth(G) = \infty$. A complete graph on n vertices is denoted by K_n. Recall from [2, Definition 1.2.2] that a *clique* of G is a complete subgraph of G. Let $G = (V, E)$ be a simple graph. Suppose that there exists $k \in \mathbb{N}$ such that any clique of G is a clique on at most k vertices. Then the *clique number* of G, denoted by $\omega(G)$ is defined as the largest positive integer n such that G contains a clique on n vertices. If G contains a clique on n vertices for all $n \geq 1$, then we set $\omega(G) = \infty$.

Let $G = (V, E)$ be a graph. Recall from [2, p.129] that a *vertex coloring* of G is a mapping $f : V \to S$, where S is a set of distinct colors. A vertex coloring $f : V \to S$ is said to be *proper* if adjacent vertices of G receive distinct colors of S; that is, if u and v are adjacent in G, then $f(u) \neq f(v)$. The *chromatic number* of G, denoted by $\chi(G)$ is the minimum number of colors needed for a proper vertex coloring of G. It is clear that for any graph G, $\omega(G) \leq \chi(G)$.

Let G be a group. Recall that a nontrivial subgroup H of G is said to be a *minimal subgroup* of G if there is no nontrivial subgroup K of

G such that K is properly contained in H. A nontrivial subgroup H of G is said to be a *maximal subgroup* of G if there is no nontrivial subgroup K of G such that H is properly contained in K. If G is a finite group with at least one nontrivial subgroup, then it is clear that G admits at least one minimal (respectively, one maximal) subgroup of G. Let G be a finite group with at least one nontrivial subgroup. Let $\mathcal{C} = \{H : H \text{ is a minimal subgroup of } G\}$. As in [9], we denote the subgroup of G generated by $\cup_{H \in \mathcal{C}} H$ by N_G. In Section 2 of this article, we discuss some results regarding the connectedness of $(\Gamma(G))^c$. Let G be a finite group with at least two nontrivial subgroups. It is shown in Proposition 2.1 that $(\Gamma(G))^c$ is connected if and only if $N_G = G$. And in the case $(\Gamma(G))^c$ is connected, it is verified in Proposition 2.1 that $diam((\Gamma(G))^c) \leq 3$. In Lemma 2.5 and Proposition 2.6, we characterize finite groups G which admit at least two nontrivial subgroups such that $(\Gamma(G))^c$ is complete. Let G be a finite abelian group which admits at least two nonrivial subgroups. With the help of fundamental theorem of finite abelian groups [6, Theorem 2.14.1, p.109] and Proposition 2.1, we are able to determine the structure of finite abelian groups G such that $(\Gamma(G))^c$ is connected (see Propositions 2.8 and 2.9). Moreover, in the case when $(\Gamma(G))^c$ is connected, we characterize finite abelian groups G such that $diam((\Gamma(G))^c) = 1, 2$ or 3 (see Propositions 2.8 and 2.11). Furthermore, in the case when $(\Gamma(G))^c$ is connected, we determine $r((\Gamma(G))^c)$ (see Proposition 2.8 and Remark 2.13).

Let $n \geq 3$ and let S_n denote the *symmetric group of degree* n. With the help of Proposition 2.1, it is verified in Proposition 2.14 that $(\Gamma(S_n))^c$ is connected. Moreover, it is shown that $diam((\Gamma(S_3))^c) = 1$ and for $n \geq 4$, it is proved that $diam((\Gamma(S_n))^c) = r((\Gamma(S_n))^c) = 2$ (see Proposition 2.14 and Remark 2.15). Let $n \geq 4$ and let A_n denote the *alternating group of degree* n. It is shown in Proposition 2.17 that $(\Gamma(A_n))^c$ is connected and $diam((\Gamma(A_n))^c) = 2$. It is observed in Proposition 2.18(i) that $r((\Gamma(A_4))^c) = 1$ and for any $n \geq 5$, it is shown in Proposition 2.18(ii) that H is any minimal subgroup of A_n with either $o(H) \in \{2, 3\}$ or $o(H) \equiv 1(mod 4)$, then $e(H) \geq 2$ in $(\Gamma(A_n))^c$. Let $n \geq 3$ and let D_n denote the *dihedral group of degree* n. It is shown that $(\Gamma(D_n))^c$ is connected and moreover, the values of n are classified according as $diam((\Gamma(D_n))^c)$ is either 1, 2 or 3 (see Remark 2.19 and Proposition 2.20). Let $n \geq 4$ be such that n is not a prime number. It is proved in Remark 2.21 that $r((\Gamma(D_n))^c) = 2$.

In Section 3 of this article, we discuss some results regarding the girth of $(\Gamma(G))^c$, where G is a finite group which admits at least one nontrivial subgroup. It is proved in Proposition 3.1 that $\omega((\Gamma(G))^c) = \chi((\Gamma(G))^c) = k$, where k is the number of minimal subgroups of G.

It is noted in Proposition 3.2 that $girth((\Gamma(G))^c) = 3$ if and only if G has at least three minimal subgroups. It is observed in Remark 3.4 that if $o(G)$ is divisible by at least three distinct prime numbers, then $girth((\Gamma(G))^c) = 3$. Let G be a finite abelian group such that $o(G)$ is divisible by exactly t distinct prime numbers. Then it is shown in Proposition 3.3 that $\omega((\Gamma(G))^c) = t$ if and only if G is cyclic. Let G be a finite group with $o(G) = p_1^{n_1}p_2^{n_2}$, where p_1, p_2 are distinct prime numbers and $n_i \geq 1$ for each $i \in \{1, 2\}$. If $n_i = 1$ for each $i \in \{1, 2\}$, then it is proved in Proposition 3.5 that $girth((\Gamma(G))^c) \in \{3, \infty\}$. If G is cyclic and if $n_i > 1$ for each $i \in \{1, 2\}$, then it is shown in Proposition 3.7 that $girth((\Gamma(G))^c) = 4$. If G is cyclic and if $n_1 > 1$ and $n_2 = 1$, then it is verified in Proposition 3.8 that the subgraph of $(\Gamma(G))^c$ induced on its nonisolated vertices is a star graph and hence, $girth((\Gamma(G))^c) = \infty$. If G is abelian but not cyclic, then it is proved in Proposition 3.9 that $girth((\Gamma(G))^c) = 3$.

Whenever a set A is a subset of a set B and $A \neq B$, we denote it symbolically by $A \subset B$.

2. MAIN RESULTS

Let G be a finite group admitting at least two nontrivial subgroups. The aim of this section is to characterize G such that $(\Gamma(G))^c$ is connected and also to determine $diam((\Gamma(G))^c)$ in the case when $(\Gamma(G))^c$ is connected.

Proposition 2.1. *Let G be a finite group which admits at least two nontrivial subgroups. Then the following statements are equivalent:*

(i) $(\Gamma(G))^c$ is connected.

(ii) $N_G = G$.

Moreover, if either (i) or (ii) holds, then $diam((\Gamma(G))^c) \leq 3$.

Proof. $(i) \Rightarrow (ii)$ Assume that $(\Gamma(G))^c$ is connected. Let H be a nontrivial subgroup of G. Since G is finite, there exists a minimal subgroup K of G such that $H \supseteq K$. Hence, $H \cap N_G \supseteq K$ and so, $H \cap N_G \neq \{e\}$. If $N_G \neq G$, then we obtain that N_G is an isolated vertex of $(\Gamma(G))^c$. This is impossible since G has at least two nontrivial subgroups and $(\Gamma(G))^c$ is connected. Therefore, $N_G = G$.

$(ii) \Rightarrow (i)$ Assume that $N_G = G$. Let H_1, H_2 be nontrivial subgroups of G with $H_1 \neq H_2$. We now verify that there exists a path of length at most three between H_1 and H_2 in $(\Gamma(G))^c$. We can assume that H_1 and H_2 are not adjacent in $(\Gamma(G))^c$. If H is any nontrivial subgroup of G, then as $N_G = G$, it follows that there exists a minimal subgroup K of G such that $K \not\subseteq H$.

Case(1): There exists a minimal subgroup K of G such that $K \nsubseteq H_1$ and $K \nsubseteq H_2$.

Observe that $H_1 \cap K = H_2 \cap K = \{e\}$. Hence, $H_1 - K - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(G))^c$.

Case(2): There exists a minimal subgroup W_1 of G such that $W_1 \nsubseteq H_1$ but $W_1 \subseteq H_2$ and there exists a minimal subgroup W_2 of G such that $W_2 \nsubseteq H_2$ but $W_2 \subseteq H_1$.

It is clear that $H_1 \cap W_1 = H_2 \cap W_2 = W_1 \cap W_2 = \{e\}$ and so, $H_1 - W_1 - W_2 - H_2$ is a path of length three between H_1 and H_2 in $(\Gamma(G))^c$.

This proves that $(\Gamma(G))^c$ is connected and $diam((\Gamma(G))^c) \leq 3$.

The proof of the moreover part is contained in the proof of $(ii) \Rightarrow (i)$ of this Proposition. □

Let G be a finite group which admits at least two nontrivial subgroups. We next try to characterize G such that $(\Gamma(G))^c$ is complete.

Remark 2.2. Let G be a group. It is not hard to verify that G has a unique nontrivial subgroup if and only if G is a finite cyclic group with $o(G) = p^2$, where p is a prime number.

Lemma 2.3. *Let G be a finite group which admits at least one nontrivial subgroup. Then $(\Gamma(G))^c$ is complete if and only if every nontrivial subgroup of G is minimal.*

Proof. Assume that $(\Gamma(G))^c$ is complete. Let H be a nontrivial subgroup of G. Let K be a nontrivial subgroup of G such that $K \subseteq H$. If $K \neq H$, then as H, K are adjacent in $(\Gamma(G))^c$, we obtain that $H \cap K = \{e\}$. This implies that $K = H \cap K = \{e\}$. This is a contradiction and so, H is a minimal subgroup of G.

Conversely, assume that any nontrivial subgroup of G is minimal. Let H_1, H_2 be nontrivial subgroups of G such that $H_1 \neq H_2$. Then $H_1 \cap H_2 = \{e\}$ and so, H_1 and H_2 are adjacent in $(\Gamma(G))^c$. This shows that $(\Gamma(G))^c$ is complete. □

Remark 2.4. Let G be a finite group which admits at least one nontrivial subgroup. If K is any minimal subgroup of G, then $o(K)$ is a prime number.

Proof. Suppose that $o(K)$ is composite. Let p be a prime number such that p divides $o(K)$. We know from Cauchy's theorem [6, Theorem 2.11.3, p.87] that there exists a subgroup H of K such that $o(H) = p$. It is clear that $\{e\} \subset H \subset K$. This implies that K is not a minimal subgroup of G. This is a contradiction. Therefore, $o(K)$ is a prime number. □

Lemma 2.5. *Let G be a finite group with at least two nontrivial subgroups. Suppose that $o(G) = p^n$, where p is a prime number and $n \geq 2$. Then the following statements are equivalent:*
(i) $(\Gamma(G))^c$ *is complete.*
(ii) $n = 2$ *and G is not cyclic.*

Proof. $(i) \Rightarrow (ii)$ Assume that $(\Gamma(G))^c$ is complete. Then we know from Lemma 2.3 that any nontrivial subgroup of G is minimal. Suppose that $n \geq 3$. Note that p^2 is a divisor of $o(G)$. Hence, we obtain from [6, Theorem 2.12.1, p.92] that there exists a subgroup H of G such that $o(H) = p^2$. We know from Remark 2.4 that H is not a minimal subgroup of G. This is a contradiction. Therefore, $n \leq 2$. Since G has at least two nontrivial subgroups, we obtain that $n \geq 2$ and so, $n = 2$. This shows that $o(G) = p^2$. As a cyclic group of order p^2 has a unique nontrivial subgroup, it follows that G is not cyclic.
$(ii) \Rightarrow (i)$ Assume that $o(G) = p^2$, where p is a prime number and G is not cyclic. We know from [6, Corollary, p.86] that G is abelian. Let $g \in G, g \neq e$. It follows as a consequence of Lagrange's theorem [6, Corollary 1, p.41] that $o(g)$ is a divisor of $o(G) = p^2$. Since G is not cyclic, we obtain that $o(g) = p$. Hence, it follows from [10, Example 2.5, p.146] that there exist cyclic subgroups A_1, A_2 of G such that $o(A_i) = p$ for each $i \in \{1, 2\}$ and G is the internal direct product of A_1 and A_2. It is clear from Lagrange's theorem [6, Theorem 2.4.1, p.41] that any nontrivial subgroup of G is of order p and it is well-known that there are exactly $p + 1$ subgroups of G each of order p. Therefore, $(\Gamma(G))^c$ is K_{p+1}. □

Let G be a finite group such that $o(G)$ is divisible by at least two distinct prime numbers. In Proposition 2.6, we characterize G such that $(\Gamma(G))^c$ is complete.

Proposition 2.6. *Let G be a finite group such that $o(G)$ is divisible by at least two distinct prime numbers. Then the following statements are equivalent:*
(i) $(\Gamma(G))^c$ *is complete.*
(ii) $o(G) = p_1p_2$*, where p_1 and p_2 are distinct prime numbers.*

Proof. $(i) \Rightarrow (ii)$ Assume that $(\Gamma(G))^c$ is complete. We know from Lemma 2.3 that each nontrivial subgroup of G is minimal. Let $o(G) = \prod_{i=1}^{k} p_i^{n_i}$ be the factorization of $o(G)$ into product of prime numbers (here $p_1, p_2, \ldots, p_k$ are distinct prime numbers and $n_i \in \mathbb{N}$ for each $i \in \{1, 2, \ldots, k\}$). We claim that $n_1 = n_2 = \cdots = n_k = 1$. Suppose that $n_i > 1$ for some $i \in \{1, 2, \ldots, k\}$. We know from [6, Theorem 2.12.1, p.92] that there exists a subgroup H of G such that $o(H) = p_i^{n_i}$.

We know from Remark 2.4 that H is not a minimal subgroup of G. This is a contradiction. Therefore, $n_i = 1$ for each $i \in \{1, 2, \ldots, k\}$.

We next verify that $k = 2$. By hypothesis, $k \geq 2$. We know from Cauchy's theorem [6, Theorem 2.11.3, p.87] that for each $i \in \{1, 2, \ldots, k\}$, there exists a subgroup P_i of G such that $o(P_i) = p_i$. We claim that P_i is normal in G for at least one $i \in \{1, 2, \ldots, k\}$. Suppose that P_i is not normal in G for each $i \in \{1, 2, \ldots, k\}$. Let $i \in \{1, \ldots, k\}$. Observe that $N(P_i) \supseteq P_i$, where $N(P_i)$ is the normalizer of P_i in G. Since P_i is not normal in G, it follows that $N(P_i) \neq G$. Hence, $N(P_i)$ is a nontrivial subgroup of G. As any nontrivial subgroup of G is minimal, we obtain that $N(P_i) = P_i$. Note that P_i is a p_i-Sylow subgroup of G. We know from [6, Lemma 2.12.6, p.99] that the number of p_i-Sylow subgroups in G equals $\frac{o(G)}{o(N(P_i))} = \frac{o(G)}{o(P_i)} = \frac{o(G)}{p_i}$. Let $\{P_i = P_{i1}, P_{i2}, \ldots, P_{i\frac{o(G)}{p_i}}\}$ be the set of all p_i-Sylow subgroups of G. As any element g of a p_i-Sylow subgroup with $g \neq e$ is of order p_i, it follows that G has exactly $\frac{o(G)}{p_i}(p_i - 1)$ elements of order p_i. As any nontrivial subgroup of G is minimal, it follows that if $x \in G$ with $x \neq e$, then $o(x) = p_i$ for some $i \in \{1, 2, \ldots, k\}$. It is now clear from the above discussion that $o(G) = \frac{o(G)}{p_1}(p_1 - 1) + \frac{o(G)}{p_2}(p_2 - 1) + \cdots + \frac{o(G)}{p_k}(p_k - 1) + 1$. This implies that $1 = k - (\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k}) + \frac{1}{o(G)}$. We can assume that $2 \leq p_1 < p_2 < \cdots < p_k$. Hence, we obtain that $k - 1 + \frac{1}{o(G)} < \frac{k}{2}$. This is a contradiction. Therefore, P_i is normal in G for at least one $i \in \{1, 2, \ldots, k\}$. Fix $i \in \{1, 2, \ldots, k\}$ such that P_i is normal in G. Suppose that $k \geq 3$. Let $j \in \{1, 2, \ldots, k\} \setminus \{i\}$. Observe that P_iP_j is a subgroup of G and as $P_i \cap P_j = \{e\}$, it follows from [6, Theorem 2.5.1, p.45] that $o(P_iP_j) = p_ip_j$. Note that P_iP_j is a nontrivial subgroup of G and is not minimal. This is in contradiction to the assumption that $(\Gamma(G))^c$ is complete. Therefore, $k = 2$. Hence, $o(G) = p_1p_2$, where p_1, p_2 are distinct prime numbers.

$(ii) \Rightarrow (i)$ Assume that $o(G) = p_1p_2$, where p_1 and p_2 are distinct prime numbers. It follows from Lagrange's theorem that any nontrivial subgroup of G is of order either p_1 or p_2. Hence, any nontrivial subgroup of G is minimal and so, we obtain from Lemma 2.3 that $(\Gamma(G))^c$ is complete. $\square$

Remark 2.7. Let G be a finite group with $o(G) = p_1p_2$, where p_1, p_2 are distinct primes. In this remark, we mention some well-known facts about the structure of G. If G is abelian, then G is necessarily cyclic and in such a case, $(\Gamma(G))^c$ is K_2. Suppose that G is not abelian. We can assume that $p_1 < p_2$. We know from [6, Theorem 2.12.3 and Lemma 2.12.6, p.100, p.99] that G has a unique subgroup H with $o(H) = p_2$

and has exactly p_2 subgroups of G each of order p_1. Hence, $(\Gamma(G))^c$ is K_{p_2+1}.

Let G be a finite abelian group which admits at least two nontrivial subgroups. We next proceed to discuss regarding the characterization of G such that $(\Gamma(G))^c$ is connected and determine its diameter when it is connected. First, we consider finite abelian groups with $o(G) = p^n$, where p is prime number and $n \geq 2$.

Proposition 2.8. *Let G be a finite abelian group with $o(G) = p^n$, where p is a prime number and $n \geq 2$. Then the following statements are equivalent:*

(i) $(\Gamma(G))^c$ is connected.

(ii) G is the internal direct product of cyclic subgroups $A_1, A_2, \ldots, A_n$ with $o(A_i) = p$ for each $i \in \{1, 2, \ldots, n\}$.

Moreover, in the case when $(\Gamma(G))^c$ is connected, $diam((\Gamma(G))^c) = 1$ if $n = 2$ and $diam((\Gamma(G))^c) = r((\Gamma(G))^c) = 2$ if $n \geq 3$.

Proof. $(i) \Rightarrow (ii)$ Assume that $(\Gamma(G))^c$ is connected. We know from Proposition 2.1 that $N_G = G$. Since $o(G) = p^n$ where p is a prime number, we obtain that any minimal subgroup of G is of order p. As G is the subgroup of G generated by all its minimal subgroups, it follows that each element $g \in G$ with $g \neq e$ is of order p. We know from [10, Example 2.5, p.146] that there exist cyclic subgroups $A_1, A_2, \ldots, A_n$ of G satisfying the following properties: $o(A_i) = p$ for each $i \in \{1, 2, \ldots, n\}$ and G is the internal direct product of $A_1, A_2, \ldots, A_n$. This shows that G is the internal direct product of cyclic subgroups $A_1, A_2, \ldots, A_n$ with $o(A_i) = p$ for each $i \in \{1, 2, \ldots, n\}$.

$(ii) \Rightarrow (i)$ Assume that there exist cyclic subgroups $A_1, A_2 \ldots, A_n$ with $o(A_i) = p$ for each $i \in \{1, 2, \ldots, n\}$ and G is the internal direct product of $A_1, A_2, \ldots, A_n$.

Suppose that $n = 2$. Then we know from the proof of $(ii) \Rightarrow (i)$ of Lemma 2.5 that $(\Gamma(G))^c$ is K_{p+1}. Therefore, $diam((\Gamma(G))^c) = 1$.

Let us next suppose that $n \geq 3$. Let H_1, H_2 be two distinct nontrivial subgroups of G with $H_1 \neq H_2$. We show that there exists a path of length at most two between H_1 and H_2 in $(\Gamma(G))^c$. We can assume that H_1 and H_2 are not adjacent in $(\Gamma(G))^c$. If H_1, H_2 are not comparable under the inclusion relation, then it is clear that $H_1 \cup H_2$ is not a subgroup of G and therefore, $H_1 \cup H_2 \neq G$. Let $g \in G$ be such that $g \notin H_! \cup H_2$. Let $K =< g >$. Note that $o(K) = p$ and $H_i \cap K = \{e\}$ for each $i \in \{1, 2\}$. Hence, $H_1 - K - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(G))^c$. Suppose that H_1 and H_2 are comparable under the inclusion relation. We can assume without loss of generality that $H_1 \subset H_2$. Since $H_2 \neq G$, it follows that $A_i \not\subseteq H_2$

for some $i \in \{1, 2, \ldots, n\}$. As $o(A_i) = p$, it follows that $H_2 \cap A_i = \{e\}$ and so, $H_1 \cap A_i = \{e\}$. Hence, $H_1 - A_i - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(G))^c$. This shows that $(\Gamma(G))^c$ is connected and $diam((\Gamma(G))^c) \leq 2$. We next verify that $e(S) \geq 2$ in $(\Gamma(G))^c$ for any nontrivial subgroup S of G. Note that $o(S) = p^i$ for some i with $1 \leq i < n$. Observe that there exists a subgroup W of S with $o(W) = p$. If $i > 1$, then $W \neq S$ and S and W are not adjacent in $(\Gamma(G))^c$ and so, $d(S, W) \geq 2$ in $(\Gamma(G))^c$. Suppose that $i = 1$. Now, $A_k \not\subseteq S$ for some $k \in \{1, 2, \ldots, n\}$. Hence, $A_k \cap S = \{e\}$. Observe that SA_k is a subgroup of G and it follows from [6, Theorem 2.5.1, p.45] that $o(SA_k) = p^2$. As $o(G) = p^n$ with $n \geq 3$, it is clear that SA_k is a nontrivial subgroup of G. Since $S \cap SA_k \neq \{e\}$, we get that S and SA_k are not adjacent in $(\Gamma(G))^c$. Therefore, $d(S, SA_k) \geq 2$ in $(\Gamma(G))^c$. This proves that $e(S) \geq 2$ in $(\Gamma(G))^c$ for each nontrivial subgroup S of G. This proves that $diam((\Gamma(G))^c) = r((\Gamma(G))^c) = 2$.

The proof of the moreover part is contained in the proof of $(ii) \Rightarrow (i)$ of this Proposition. □

Let G be a finite abelian group with $o(G) = \prod_{i=1}^{k} p_i^{n_i}$, where $k \geq 2$ and $p_1, p_2, \ldots, p_k$ are distinct prime numbers and $n_i \geq 1$ for each $i \in \{1, 2, \ldots, k\}$. We next proceed to characterize G such that $(\Gamma(G))^c$ is connected and determine its diameter when it is connected.

Proposition 2.9. *Let G be a finite abelian group such that $o(G) = \prod_{i=1}^{k} p_i^{n_i}$, where $k \geq 2$ and $p_1, p_2, \ldots, p_k$ are distinct prime numbers and $n_i \geq 1$ for each $i \in \{1, 2, \ldots, k\}$. For each $i \in \{1, 2, \ldots, k\}$, let P_i denote the unique p_i-Sylow subgroup of G. Then the following statements are equivalent:*

(i) $(\Gamma(G))^c$ is connected.

(ii) Given $i \in \{1, 2, \ldots, k\}$, either $o(P_i) = p_i$ or $(\Gamma(P_i))^c$ is connected.

Proof. $(i) \Rightarrow (ii)$ Assume that $(\Gamma(G))^c$ is connected. Since $k \geq 2$, G has at least two nontrivial subgroups. Indeed, P_i is a nontrivial subgroup of G for each $i \in \{1, 2, \ldots, k\}$ and $o(P_i) = p_i^{n_i}$ for each $i \in \{1, 2, \ldots, k\}$. It is well-known that G is the internal direct product of $P_1, P_2, \ldots, P_k$. As $(\Gamma(G))^c$ is connected, we obtain from $(i) \Rightarrow (ii)$ of Proposition 2.1 that $N_G = G$. Let $g \in G$, $g \neq e$. It follows from $N_G = G$ that $o(g) = \prod_{j \in A} p_j$ for some nonempty subset A of $\{1, 2, \ldots, k\}$. Let $i \in \{1, 2, \ldots, k\}$. Suppose that $o(P_i) \neq p_i$. Hence, $n_i \geq 2$. As any element x of P_i with $x \neq e$ is of order p_i, it follows from [10, Example 2.5, p.146] that there exist cyclic subgroups $A_{i1}, A_{i2}, \ldots, A_{in_i}$ of P_i such that $o(A_{ij}) = p_i$ for each $j \in \{1, 2, \ldots, n_i\}$ and P_i is the

internal direct product of $A_{i1}, A_{i2}, \ldots, A_{in_i}$. Now, it follows from $(ii) \Rightarrow (i)$ of Proposition 2.8 that $(\Gamma(P_i))^c$ is connected.
$(ii) \Rightarrow (i)$ It is well-known that G is the internal direct product of $P_1, P_2, \ldots, P_k$. Let $g \in G$, $g \neq e$. Now, there exist unique elements $x_1, x_2, \ldots x_k$ with $x_i \in P_i$ for each $i \in \{1, 2, \ldots, k\}$ such that $g = \prod_{i=1}^{k} x_i$. As $g \neq e$, it follows that $x_i \neq e$ for at least one $i \in \{1, 2, \ldots, k\}$. Let $i \in \{1, 2, \ldots .k\}$ be such that $x_i \neq e$. By assumption, either $o(P_i) = p_i$ or $(\Gamma(P_i))^c$ is connected. If $o(P_i) = p_i$, then $o(x_i) = p_i$. Suppose that $(\Gamma(P_i))^c$ is connected. Then it follows from $(i) \Rightarrow (ii)$ of Proposition 2.8 that $o(x_i) = p_i$. Hence, in any case $o(x_i) = p_i$. Now, it follows from $g = \prod_{i=1}^{k} x_i$ that $g \in N_G$ and so, $N_G = G$. Therefore, we obtain from $(ii) \Rightarrow (i)$ of Proposition 2.1 that $(\Gamma(G))^c$ is connected. □

Let G be a finite abelian group and let $o(G)$ be as in the statement of Proposition 2.9. Suppose that $(\Gamma(G))^c$ is connected. In Proposition 2.11, we determine $diam((\Gamma(G))^c)$. We use Lemma 2.10 in the proof of Proposition 2.11.

Lemma 2.10. *Let G be a finite abelian group such that G has at least two nontrivial subgroups. Suppose that $(\Gamma(G))^c$ is connected. Then the following hold:*

(i) $diam((\Gamma(G))^c) = 2$ if and only if G admits a nontrivial subgroup which is not a minimal subgroup of G and if H_1, H_2 are distinct maximal subgroups of G with $H_1 \cap H_2 \neq \{e\}$, then H_1 and H_2 are isomorphic.

(ii) $diam((\Gamma(G))^c) = 3$ if and only if there exist nonisomorphic maximal subgroups H_1, H_2 of G such that $H_1 \cap H_2 \neq \{e\}$.

Proof. Since $(\Gamma(G))^c$ is connected, we know from the proof of $(ii) \Rightarrow (i)$ of Proposition 2.1 that $diam((\Gamma(G))^c) \leq 3$.
(i) Assume that $diam((\Gamma(G))^c) = 2$. We know from Lemma 2.3 that G admits at least one nontrivial subgroup which is not a minimal subgroup of G. Let H_1, H_2 be distinct maximal subgroups of G such that $H_1 \cap H_2 \neq \{e\}$. Note that H_1 and H_2 are not adjacent in $(\Gamma(G))^c$. As $diam((\Gamma(G))^c) = 2$, there exists a nontrivial subgroup K of G such that $H_1 - K - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(G))^c$. Hence, $H_i \cap K = \{e\}$ for each $i \in \{1, 2\}$. Let $i \in \{1, 2\}$. As H_i is a maximal subgroup of G, we obtain that $H_iK = G$. Therefore, we obtain from the second isomorphism theorem of groups [3, Theorem 2.3, p.98] that $\frac{G}{K} = \frac{H_iK}{K} \cong \frac{H_i}{H_i \cap K} = H_i$ for each $i \in \{1, 2\}$.

Conversely, assume that G admits at least one nontrivial subgroup which is not a minimal subgroup and any two distinct maximal subgroups of G which are not adjacent in $(\Gamma(G))^c$ are isomorphic. As there exists at least one nontrivial subgroup of G which is not a minimal subgroup of G, it follows from Lemma 2.3 that $diam((\Gamma(G))^c) \geq 2$. Let W_1, W_2 be nontrivial subgroups of G. We prove that there exists a path of length at most two between W_1 and W_2 in $(\Gamma(G))^c$. We can assume that W_1 and W_2 are not adjacent in $(\Gamma(G))^c$. That is, $W_1 \cap W_2 \neq \{e\}$. Let H_i be a maximal subgroup of G such that $W_i \subseteq H_i$ for each $i \in \{1, 2\}$. Observe that $H_1 \cap H_2 \neq \{e\}$. It can happen that $H_1 = H_2$. And in the case, $H_1 \neq H_2$, we know from the assumption that H_1 and H_2 are isomorphic. Thus in any case, $o(H_1) = o(H_2)$. Hence, we obtain that $o(\frac{G}{H_1}) = o(\frac{G}{H_2})$. Since $\frac{G}{H_i}$ is an abelian simple group, we get that $\frac{G}{H_i}$ is a cyclic group for each $i \in \{1, 2\}$ with $o(\frac{G}{H_1}) = o(\frac{G}{H_2}) = p$, where p is a prime number. As $(\Gamma(G))^c$ is connected, we know from $(i) \Rightarrow (ii)$ of Proposition 2.1 that $N_G = G$. Let $i \in \{1, 2\}$. As $H_i \neq G$, there exists a minimal subgroup M_i of G such that $M_i \not\subseteq H_i$. We know from Remark 2.4 that $o(M_i)$ is a prime number. Since H_i is a maximal subgroup of G, we obtain that $H_iM_i = G$. It follows from $H_i \cap M_i = \{e\}$ and [6, Theorem 2.5.1, p.45] that $o(G) = o(H_i)o(M_i)$. Therefore, we obtain that $o(M_i) = \frac{o(G)}{o(H_i)} = o(\frac{G}{H_i}) = p$. If $M_2 \not\subseteq H_1$, then it follows from $H_1 \cap M_2 = H_2 \cap M_2 = \{e\}$ that $W_i \cap M_2 = \{e\}$ for each $i \in \{1, 2\}$ and so, $W_1 - M_2 - W_2$ is a path of length two between W_1 and W_2 in $(\Gamma(G))^c$. Similarly, if $H_2 \cap M_1 = \{e\}$, then it follows that $W_1 - M_1 - W_2$ is a path of length two between W_1 and W_2 in $(\Gamma(G))^c$. Suppose that $M_2 \subseteq H_1$ and $M_1 \subseteq H_2$. Note that M_i is a cyclic group with $o(M_i) = p$ for each $i \in \{1, 2\}$. Let $g_1 \in M_1 \backslash M_2$ and let $g_2 \in M_2 \backslash M_1$. Observe that $o(g_1g_2) = p$ and let us denote $< g_1g_2 >$ by M. It is clear that M is a minimal subgroup of G and $M \not\subseteq H_i$ for each $i \in \{1, 2\}$. Therefore, $W_i \cap M \subseteq H_i \cap M = \{e\}$ for each $i \in \{1, 2\}$. Hence, we obtain that $W_1 - M - W_2$ is a path of length two between W_1 and W_2 in $(\Gamma(G))^c$. Therefore, we get that $diam((\Gamma(G))^c) = 2$.

(ii) Assume that $diam((\Gamma(G))^c) = 3$. Let W_1, W_2 be distinct nontrivial subgroups of G such that $d(W_1, W_2) = 3$ in $(\Gamma(G))^c$. Let $i \in \{1, 2\}$. Let H_i be a maximal subgroup of G such that $W_i \subseteq H_i$. From $W_1 \cap W_2 \neq \{e\}$, it follows that $H_1 \cap H_2 \neq \{e\}$. If $H_1 \cong H_2$ as groups, then it follows from the proof of the if part of (i) that $d(W_1, W_2) = 2$ in $(\Gamma(G))^c$. This is in contradiction to the assumption that $d(W_1, W_2) = 3$ in $(\Gamma(G))^c$. Therefore, H_1 and H_2 are nonisomorphic. This proves that there exist nonisomorphic maximal subgroups H_1, H_2 of G such that $H_1 \cap H_2 \neq \{e\}$.

Conversely, assume that there exist nonisomorphic maximal subgroups H_1, H_2 of G such that $H_1 \cap H_2 \neq \{e\}$. It follows from the proof of the only if part of (i) that $d(H_1, H_2) \geq 3$ in $(\Gamma(G))^c$ and so, $diam((\Gamma(G))^c) \geq 3$. Therefore, we obtain that $diam((\Gamma(G))^c) = 3$. □

Proposition 2.11. *Let G be a finite abelian group. Let $o(G) = \prod_{i=1}^{k} p_i^{n_i}$, where $k \geq 2$ and $p_1, p_2, \ldots, p_k$ are distinct prime numbers, and $n_i \geq 1$ for each $i \in \{1, 2, \ldots, k\}$. Suppose that $(\Gamma(G))^c$ is connected. Then the following hold.*

If $k = 2$, then $diam((\Gamma(G))^c) = 1$ if and only if $n_1 = n_2 = 1$. If $n_i \geq 2$ for some $i \in \{1, 2\}$, then $diam((\Gamma(G))^c) = 3$.

If $k \geq 3$, then $diam((\Gamma(G))^c) = 3$.

Proof. Suppose that $(\Gamma(G))^c$ is connected. For each $i \in \{1, 2, \ldots, k\}$, let P_i denote the unique p_i-Sylow subgroup of G. We know that G is the internal direct product of $P_1, P_2, \ldots, P_k$.

Suppose that $k = 2$. If $n_1 = n_2 = 1$, then P_1, P_2 are the only nontrivial subgroups of G and $(\Gamma(G))^c$ is K_2 and so, $diam((\Gamma(G))^c) = 1$. Suppose that $n_i \geq 2$ for some $i \in \{1, 2\}$. Without loss of generality, we can assume that $n_1 \geq 2$. Let W_i be a subgroup of P_i with $o(W_i) = p_i^{n_i-1}$ for each $i \in \{1, 2\}$. Let H_1 be the internal direct product of W_1 and P_2 and H_2 be the internal direct product of P_1 and W_2. Observe that $o(H_1) = p_1^{n_1-1}p_2^{n_2}$ and $o(H_2) = p_1^{n_1}p_2^{n_2-1}$. It is clear that H_1 and H_2 are nonisomorphic maximal subgroups of G with $H_1 \cap H_2 \neq \{e\}$. Hence, it follows from Lemma 2.10(ii) that $diam((\Gamma(G))^c) = 3$.

Suppose that $k \geq 3$. Let W_1 be the internal direct product of $P_1, P_2, \ldots, P_{k-1}$. Let W_2 be the internal direct product of $P_2, \ldots, P_k$. Let U be a subgroup of P_1 with $o(U) = p_1^{n_1-1}$ and let W be a subgroup of P_k with $o(W) = p_k^{n_k-1}$. Let H_1 be the internal direct product of W_1 and W and let H_2 be the internal direct product of W_2 and U. It is clear that $o(H_1) = (\prod_{i=1}^{k-1} p_i^{n_i})p_k^{n_k-1}$, $o(H_2) = p_1^{n_1-1}(\prod_{j=2}^{k} p_j^{n_j})$, H_1 and H_2 are nonisomorphic maximal subgroups of G with $H_1 \cap H_2 \neq \{e\}$. Therefore, we obtain from Lemma 2.10(ii) that $diam((\Gamma(G))^c) = 3$. □

Remark 2.12. Let G be a finite group which admits at least two nontrivial subgroups. If $(\Gamma(G))^c$ is connected, then $e(H) \leq 2$ in $(\Gamma(G))^c$ for any minimal subgroup H of G.

Proof. Let H be a minimal subgroup of G. Let W be any nontrivial subgroup of G with $W \neq H$. We claim that $d(H, W) \leq 2$ in $(\Gamma(G))^c$. We can assume that H and W are not adjacent in $(\Gamma(G))^c$. Hence, $H \cap W \neq \{e\}$. As H is a minimal subgroup of G, it follows that $H \subset W$. Since $(\Gamma(G))^c$ is connected, we know from $(i) \Rightarrow (ii)$ of Proposition 2.1 that $N_G = G$. It follows from $W \neq G$ that there exists a minimal subgroup S of G such that $S \not\subseteq W$. Observe that $H \cap S = W \cap S = \{e\}$. Therefore, $H - S - W$ is a path of length two between H and W in $(\Gamma(G))^c$. This proves that $d(H,W) \leq 2$ in $(\Gamma(G))^c$ for any nontrivial subgroup W of G and so, $e(H) \leq 2$ in $(\Gamma(G))^c$ for any minimal subgroup H of G. □

Remark 2.13. Let G be a finite abelian group and let $o(G) = \prod_{i=1}^{k} p_i^{n_i}$, where $k \geq 2$ and $p_1, p_2, \ldots, p_k$ are distinct prime numbers and $n_i \geq 1$for each $i \in \{1, 2, \ldots, k\}$. Suppose that $(\Gamma(G))^c$ is connected and in the case $k = 2$, either $n_1 > 1$ or $n_2 > 1$. Then $r((\Gamma(G))^c) = 2$.

Proof. Let H be any minimal subgroup of G. We know from Remark 2.12 that $e(H) \leq 2$ in $(\Gamma(G))^c$.

In the case $k \geq 3$, it is clear that if H is a minimal subgroup of G, then there exists at least one nontrivial subgroup W of G such that $H \subset W$ and so, H and W are not adjacent in $(\Gamma(G))^c$. In the case $k = 2$, we are assuming that either $n_1 > 1$ or $n_2 > 1$. Hence, in this case also, given a minimal subgroup H of G, there exists a nontrivial subgroup W of G such that $H \subset W$ and so, H and W are not adjacent in $(\Gamma(G))^c$. Therefore, $d(H,W) \geq 2$ in $(\Gamma(G))^c$. It is already shown that $e(H) \leq 2$ in $(\Gamma(G))^c$ for any minimal subgroup H of G. This proves that $e(H) = 2$ in $(\Gamma(G))^c$ for any minimal subgroup H of G. As for a given nontrivial subgroup W of G, there exists a minimal subgroup H of G such that $H \subseteq W$, it follows that $e(W) \geq 2$ in $(\Gamma(G))^c$. Therefore, we obtain that $r((\Gamma(G))^c) = 2$. □

Let $n \geq 3$. Let S_n denote the symmetric group of degree n. We know from [6, Lemma 2.10.2, p.78] that any $\sigma \in S_n$ is a product of transpositions. If $\tau = (i,j)$ is any transposition, then $o(\tau) = 2$ in S_n. Therefore, $N_{S_n} = S_n$ and so, we obtain from $(ii) \Rightarrow (i)$ of Proposition 2.1 that $(\Gamma(S_n))^c$ is connected. In Proposition 2.14, we determine $diam((\Gamma(S_n))^c)$.

Proposition 2.14. *Let $n \geq 3$. Then $(\Gamma(S_n))^c$ is connected and $diam((\Gamma(S_3))^c) = 1$, whereas $diam((\Gamma(S_n))^c) = 2$ for all $n \geq 4$.*

Proof. It is already noted above that $(\Gamma(S_n))^c$ is connected. Observe that $o(S_3) = 6 = 2 \times 3$ and S_3 is not abelian. We know from Remark 2.7 that $(\Gamma(S_3))^c$ is a complete graph on four vertices. Therefore, we obtain that $diam((\Gamma(S_3))^c) = 1$. Let $n \geq 4$. Let $\sigma = (1,2,3,4)$. Let $H =< \sigma >$ and let $K =< \sigma^2 >$. Observe that $o(H) = 4$ and $o(K) = 2$ and $H \cap K = K$ is nontrivial. Hence, H and K are not adjacent in $(\Gamma(S_n))^c$. Therefore, $diam((\Gamma(S_n))^c) \geq 2$. We next verify that $diam((\Gamma(S_n))^c) \leq 2$. Let H_1, H_2 be any nontrivial subgroups of S_n with $H_1 \neq H_2$. We claim that there exists a path of length at most two between H_1 and H_2 in $(\Gamma(S_n))^c$. We can assume that H_1 and H_2

are not adjacent in $(\Gamma(S_n))^c$. It is well-known that S_n is generated by the set of 2-cycles $\{(1,i) : i \in \{2,3,\ldots,n\}\}$. Since $H_1 \neq S_n$, it follows that $(1,i) \notin H_1$ for some $i \in \{2,3,\ldots,n\}$. If $(1,i) \notin H_2$, then with $H =< (1,i) >$, we get that $H_i \cap H = \{e\}$ for each $i \in \{1,2\}$. Hence, $H_1 - H - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(S_n))^c$. Suppose that $(1,i) \in H_2$. As $H_2 \neq S_n$, we obtain that there exists $j \in \{2,3,\ldots,n\}$ such that $(1,j) \notin H_2$. It is clear that $i \neq j$. If $(1,j) \notin H_1$, then $H_1 - < (1,j) > - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(S_n))^c$. Suppose that $(1,j) \in H_1$. Thus $(1,j) \in H_1 \backslash H_2$ and $(1,i) \in H_2 \backslash H_1$. Let $\rho = (1,i)(1,j)$. Note that $\rho = (1,j,i)$ is a cycle of length 3 and let $H_3 =< (1,j,i) >$. It is clear that $H_3 = \{e, \rho, \rho^2\}$ and $\rho \notin H_1 \cup H_2$. Hence, we get that $H_i \cap H_3 = \{e\}$ for each $i \in \{1,2\}$. Therefore, $H_1 - H_3 - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(S_n))^c$. From the above discussion, it is clear that $diam((\Gamma(S_n))^c) \leq 2$ and so, $diam((\Gamma(S_n))^c) = 2$. □

Remark 2.15. Let $n \geq 4$. Then $r((\Gamma(S_n))^c) = 2$.

Proof. Let $n \geq 4$. We know from Proposition 2.14 that $(\Gamma(S_n))^c$ is connected and $diam((\Gamma(S_n))^c) = 2$. Therefore, $e(H) \leq 2$ in $(\Gamma(S_n))^c$ for each nontrivial subgroup H of S_n. Hence, to prove this remark, it is enough to show that $e(H) \geq 2$ in $(\Gamma(S_n))^c$ for any nontrivial subgroup H of S_n. Let H be any nontrivial subgroup of S_n. If H is not a minimal subgroup of S_n, then it is clear that $e(H) \geq 2$ in $(\Gamma(S_n))^c$. Hence, we can assume that H is a minimal subgroup of S_n. Note that either $H \subseteq A_n$ or $H \nsubseteq A_n$, where A_n is the alternating group of degree n. It is known that $o(A_n) = \frac{o(S_n)}{2}$ [6, Lemma 2.10.3, p.80] Thus, A_n is a maximal subgroup of S_n and is a normal subgroup of S_n. If $H \subseteq A_n$, then as A_n is not a minimal subgroup of S_n, it follows that $H \neq A_n$. Therefore, it follows from $H \cap A_n = H \neq \{e\}$ that H and A_n are not adjacent in $(\Gamma(S_n))^c$. Hence, $e(H) \geq 2$ in $(\Gamma(S_n))^c$. Suppose that $H \nsubseteq A_n$. Then $A_n H = S_n$ and $H \cap A_n = \{e\}$. We know from [6, Theorem 2.5.1, p.45] that $o(S_n) = o(A_n)o(H)$ and so, $o(H) = 2$. Let $\sigma \in S_n$ be such that $H = \{e, \sigma\}$. Note that $o(\sigma) = 2$. It follows from [6, Lemma 2.10.1, p.78] that there exist disjoint transpositions $\tau_1, \ldots, \tau_k$ such that $\sigma = \prod_{i=1}^{k} \tau_i$. As σ is an odd permutation, k must be odd. Suppose that $k = 1$. Let $\sigma = \tau_1 = (i_1, i_2)$. As $n \geq 4$, there exist distinct symbols i_3, i_4 such that $i_3, i_4 \in \{1,2,\ldots,n\} \backslash \{i_1, i_2\}$. Let K be the subgroup of S_n generated by $\{\sigma, (i_1, i_3)\}$. It is clear that $(i_1, i_4) \notin K$ and so, $K \neq S_n$. Since $(i_1, i_3) \notin H$, it follows that $H \subset K$. Hence, H and K are not adjacent in $(\Gamma(S_n))^c$ and so, $e(H) \geq 2$ in $(\Gamma(S_n))^c$. Suppose that k is odd and $k \geq 3$. Let $\tau_1 = (i_1, i_2), \tau_2 = (i_3, i_4), \ldots, \tau_k = (i_{2k-1}, i_{2k})$. Let K be the subgroup of S_n generated by $\{\sigma, (i_1, i_2)\}$. It is clear that

$(i_1, i_3) \notin K$ and so, $K \neq S_n$. Since $(i_1, i_2) \in K \backslash H$, it follows that $H \subset K$. Hence, H and K are not adjacent in $(\Gamma(S_n))^c$. Therefore, we get that $e(H) \geq 2$ in $(\Gamma(S_n))^c$. This proves that $r((\Gamma(S_n))^c) = 2$. $\square$

Remark 2.16. Let $n \geq 4$. It is well-known that A_n is generated by the set of 3-cycles $\{(1, 2, i) : i \in \{3, 4, \ldots, n\}\}$ [10, Proposition 4.5.1, p.55]. If $\sigma \in S_n$ is any 3-cycle, then $o(\sigma) = 3$ and so, $< \sigma >= \{e, \sigma, \sigma^2\}$. It follows from the above given arguments that $N_{A_n} = A_n$ and so, we obtain from $(ii) \Rightarrow (i)$ of Proposition 2.1 that $(\Gamma(A_n))^c$ is connected. We prove in Proposition 2.17 that $diam((\Gamma(A_n))^c) = 2$.

Proposition 2.17. *Let $n \geq 4$. Then $(\Gamma(A_n))^c$ is connected and $diam((\Gamma(A_n))^c) = 2$.*

Proof. It is noted in Remark 2.16 that $(\Gamma(A_n))^c$ is connected. Let $H = \{e, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}$ and $K = \{e, (1, 2)(3, 4)\}$. It is clear that H, K are subgroups of A_n and $H \cap K = K$ is nontrivial. Hence, H and K are not adjacent in $(\Gamma(A_n))^c$. Therefore, $d(H, K) \geq 2$ in $(\Gamma(A_n))^c$ and so, $diam((\Gamma(A_n))^c) \geq 2$. We next verify that $diam((\Gamma(A_n))^c)) \leq 2$. Let H_1, H_2 be nontrivial subgroups of A_n with $H_1 \neq H_2$. We show that there exists a path of length at most two between H_1 and H_2 in $(\Gamma(A_n))^c$. We can assume that H_1 and H_2 are not adjacent in $(\Gamma(A_n))^c$. Since $H_1 \neq A_n$, $(1, 2, i) \notin H_1$ for some $i \in \{3, 4, \ldots, n\}$. If $(1, 2, i) \notin H_2$, then $H_1- < (1, 2, i) > -H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(A_n))^c$. Suppose that $(1, 2, i) \in H_2$. As $H_2 \neq A_n$, there exists $j \in \{3, 4, \ldots, n\}$ such that $(1, 2, j) \notin H_2$. If $(1, 2, j) \notin H_1$, then $H_1- < (1, 2, j) > -H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(A_n))^c$. Suppose that $(1, 2, j) \in H_1$. Now, $(1, 2, j) \in H_1 \backslash H_2$ and $(1, 2, i) \in H_2 \backslash H_1$. Let $\rho = (1, 2, i)(1, 2, j)$. Observe that $\rho \notin H_1 \cup H_2$ and $\rho = (1, i)(2, j)$. Let $H_3 =< \rho >$. As $H_3 = \{e, \rho\}$, we obtain that $H_i \cap H_3 = \{e\}$ for each $i \in \{1, 2\}$ and so, $H_1 - H_3 - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(A_n))^c$. This proves that $diam((\Gamma(A_n))^c) \leq 2$ and so, we obtain that $diam((\Gamma(A_n))^c) = 2$. $\square$

Proposition 2.18. *Let $n \geq 4$. Then the following hold.*

(i) $r((\Gamma(A_4))^c) = 1$.

(ii) Let $n \geq 5$. Let H be a minimal subgroup of A_n. If $o(H) \in \{2, 3\}$ or $o(H) \equiv 1(mod4)$, then there exists a nontrivial subgroup W of A_n such that $H \subset W$.

Proof. Let $n \geq 4$. It is proved in Proposition 2.17 that $(\Gamma(A_n))^c$ is connected and $diam((\Gamma(A_n))^c) = 2$.

(i) We verify that $r((\Gamma(A_4))^c) = 1$. Let $\sigma = (1, 2, 3)$ and let $H =< \sigma >$. Observe that $o(H) = 3$. We claim that $e(H) = 1$ in $(\Gamma(A_4))^c$. Let

K be any nontrivial subgroup of A_4 with $K \neq H$. We assert that $H \cap K = \{e\}$. Suppose that $H \cap K \neq \{e\}$. Then as H is a minimal subgroup of A_4, it follows that $H \subset K$. It follows from Lagrange's theorem [6, Theorem 2.4.1, p.41] that $o(K) = 3t$ for some $t \in \mathbb{N}$ with $t \geq 2$. As $o(K)$ is a divisor of $o(A_4) = 12$ and $K \neq A_4$, it follows that $o(K) = 6$. This is impossible since it is well-known that A_4 has no subgroup of order 6 [10, Example 3.3.6, p.75] Therefore, $H \cap K = \{e\}$ and so, H and K are adjacent in $(\Gamma(A_4))^c$. This proves that $e(H) = 1$ in $(\Gamma(A_4))^c$ and therefore, $r((\Gamma(A_4))^c) = 1$.
(ii) Let $n \geq 5$. Let H be a minimal subgroup of A_n. Note that $o(H) = p$, where p is a prime number and $H =< \sigma >$ for any $\sigma \in H\backslash\{e\}$. We discuss two cases.
Case(1): $p = 2$.

Let $\sigma \in H\backslash\{e\}$. As $\sigma \in A_n$ and $o(\sigma) = 2$, it follows from [6, Lemma 2.10.1, p.78] that there exist disjoint transpositions $(i_1, i_2), (i_3, i_4), \ldots, (i_{2k-1}, i_{2k})$ with $k \geq 2$ is even and is such that $\sigma = \prod_{s=1}^{k}(i_{2s-1}, i_{2s})$. If $k = 2$, then $\sigma = (i_1, i_2)(i_3, i_4)$. Observe that $W = \{e, \sigma, (i_1, i_3)(i_2, i_4), (i_1, i_4)(i_2, i_3)\}$ is a nontrivial subgroup of A_n such that $o(W) = 4$ and $H \subset W$. As H and W are not adjacent in $(\Gamma(A_n))^c$, it follows that $e(H) \geq 2$ in $(\Gamma(A_n))^c$. Suppose that $k \geq 4$. Let $\sigma_1 = (i_1, i_2)(i_3, i_4)$ and let $\sigma_2 = \prod_{s=3}^{k}(i_{2s-1}, i_{2s})$. Note that $\sigma_1, \sigma_2 \in A_n$, $o(\sigma_i) = 2$ for each $i \in \{1, 2\}$ and $\sigma_1\sigma_2 = \sigma = \sigma_2\sigma_1$ and $W_1 = \{e, \sigma_1, \sigma_2, \sigma\}$ is a nontrivial subgroup of A_n with $o(W_1) = 4$ and $H \subset W_1$. Hence, H and W_1 are not adjacent in $(\Gamma(A_n))^c$ and so, $e(H) \geq 2$ in $(\Gamma(A_n))^c$.
Case(2): p is odd.

Let $\sigma \in H\backslash\{e\}$. Note that $\sigma \in A_n$ and $o(\sigma) = p$. Hence, it follows from [6, Lemma 2.10.1, p.78] that there exists $t \in \mathbb{N}$ and disjoint cycles $C_1, \ldots, C_t$ such that C_i is of length p for each $i \in \{1, \ldots, t\}$ and $\sigma = \prod_{i=1}^{t} C_i$. Suppose that $t = 1$. Then $\sigma = C_1 = (i_1, i_2, i_3, \ldots, i_p)$. Observe that either $p = 3$ or $p \geq 5$. Assume that $p = 3$. Let $i_4 \in \{1, 2, 3, \ldots, n\}\backslash\{i_1, i_2, i_3\}$. Let $W = \{e, (i_1, i_2, i_3), (i_1, i_3, i_2), (i_1, i_2, i_4), (i_1, i_4, i_2), (i_1, i_3, i_4), (i_1, i_4, i_3), (i_2, i_3, i_4), (i_2, i_4, i_3), (i_1, i_2)(i_3, i_4), (i_1, i_3)(i_2, i_4), (i_1, i_4)(i_2, i_3)\}$. Note that W is a nontrivial subgroup of A_n with $o(W) = 12$, $W \cong A_4$ as groups, and $H \cap W = H$. Therefore, H and W are not adjacent in $(\Gamma(A_n))^c$ and so, $e(H) \geq 2$ in $(\Gamma(A_n))^c$. Suppose that $p \geq 5$. Note that $\sigma = (i_1, i_2, i_3 \ldots, i_p)$. Suppose that $p \equiv 1(mod 4)$. Let $\tau \in S_n$ be given by $\tau(i_1) = i_1, \tau(i_j) = i_{p-j+2}$ for each $j \in \{2, 3, \ldots, p\}$. Observe that $\tau = \prod_{j=2}^{\frac{p+1}{2}}(i_j, i_{p-j+2})$ is the product of $\frac{p-1}{2}$ disjoint transpositions. As $\frac{p-1}{2}$ is even, we obtain that $\tau \in A_n$. Observe that $\sigma^p = e, \tau^2 = e, \sigma^{p-1}\tau = \tau\sigma$. Let W be the subgroup of A_n generated by $\{\sigma, \tau\}$. Note that $W = \{e, \sigma, \sigma^2, \ldots, \sigma^{p-1}, \tau, \sigma\tau, \sigma^2\tau, \ldots, \sigma^{p-1}\tau = \tau\sigma\}$ and $W \cong D_p$ as groups, where D_p is the dihedral group of degree p. It is clear that W is a nontrivial subgroup of A_n and $H \subset W$. Hence, $H \cap W = H \neq \{e\}$ and so, H and W are not adjacent in $(\Gamma(A_n))^c$. Therefore, $e(H) \geq 2$ in $(\Gamma(A_n))^c$.

Suppose that p is an odd prime number and σ is the product of t ($t \geq 2$) disjoint cycles $C_1, C_2, \ldots, C_t$ such that C_i is of length p for each $i \in \{1, 2, \ldots, t\}$. Let $\sigma_1 = C_1$ and let $\sigma_2 = \prod_{j=2}^{t} C_j$. Note that $\sigma_i \in A_n$ for each $i \in \{1, 2\}$, $o(\sigma_1) = o(\sigma_2) = p$, and $\sigma = \sigma_1\sigma_2 = \sigma_2\sigma_1$. Let W be the subgroup of A_n generated by $\{\sigma_1, \sigma_2\}$. Let $H_1 =< \sigma_1 >$ and let $H_2 =< \sigma_2 >$. It is clear that $o(H_1) = o(H_2) = p$, $H_1 \cap H_2 = \{e\}$, and $W = H_1H_2$. It follows from [6, Theorem 2.5.1, p.45] that $o(W) = o(H_1)o(H_2) = p^2$. Observe that $H =< \sigma >\subset W$ and so, $H \cap W = H \neq \{e\}$. Hence, H and W are not adjacent in $(\Gamma(A_n))^c$. Therefore, $e(H) \geq 2$ in $(\Gamma(A_n))^c$.

Thus for any $n \geq 5$, it is shown that if H is any minimal subgroup of A_n with $o(H) \in \{2, 3\}$ or $o(H) \equiv 1(mod 4)$, then there exists a nontrivial subgroup W of A_n such that $H \subset W$ and so, $e(H) \geq 2$ in $(\Gamma(A_n))^c$. □

Remark 2.19. Let $n \geq 3$. Recall from [3, Theorem 5.2, p.87 and p.88] that the *dihedral group of degree* n denoted by D_n is the subgroup of S_n generated by σ and τ, where σ is the cycle given by $\sigma = (1, 2, 3, \ldots, n)$ and τ is given by $\tau(1) = 1, \tau(i) = n - i + 2$ for each $i \in \{2, 3, \ldots, n\}$. Note that $o(\sigma) = n$, $o(\tau) = 2$, $\sigma^{n-1}\tau = \tau\sigma$, $o(D_n) = 2n$ and indeed, $D_n = \{e, \sigma, \sigma^2, \ldots, \sigma^{n-1}, \tau, \sigma\tau, \sigma^2\tau, \ldots, \sigma^{n-1}\tau = \tau\sigma\}$. Observe that $D_3 = S_3$ and it is already shown in Proposition 2.14 that $(\Gamma(S_3))^c$ is connected and $diam((\Gamma(S_3))^c) = 1$. Hence, in discussing the connectedness of $(\Gamma(D_n))^c$, we can assume that $n \geq 4$. Suppose that n is a prime number. Then n is odd and $o(D_n) = 2n$ is the product of two distinct prime numbers. As D_n is not abelian, it follows from Remark 2.7 that $(\Gamma(D_n))^c$ is a complete graph on $n + 1$ vertices. Therefore, in our discussion regarding the connectedness of $(\Gamma(D_n))^c$, we can assume that $n \geq 4$ and n is not a prime number. We prove in Proposition 2.20 that $(\Gamma(D_n))^c$ is connected and moreover, we determine $diam((\Gamma(D_n))^c)$.

Proposition 2.20. *Let $n \geq 4$ and suppose that n is not a prime number. Then $(\Gamma(D_n))^c$ is connected. Moreover, the following hold.*

(i) $diam((\Gamma(D_n))^c) = 2$ if either n is odd or $n = 2m$, where $m \geq 3$ is odd.

(ii) $diam((\Gamma(D_n))^c) = 3$ if $n = 2^k t$, where $k \geq 2$ and $t \geq 1$ is odd.

Proof. We know that D_n is the subgroup of S_n generated by σ and τ, where σ and τ are mentioned as above in Remark 2.19. Note that $o(\sigma) = n, o(\tau) = o(\sigma^i\tau) = 2$ for each $i \in \{1, 2, \ldots, n-1\}$. It is clear that D_n has at least two nontrivial subgroups and as D_n is generated by $\sigma\tau$ and τ, it follows that $N_{D_n} = D_n$. Therefore, we obtain from $(ii) \Rightarrow (i)$ of Proposition 2.1 that $(\Gamma(D_n))^c$ is connected. Moreover, we know from the proof of $(ii) \Rightarrow (i)$ of Proposition 2.1 that $diam((\Gamma(D_n))^c) \leq 3$.

Let $n \geq 4$ and suppose that n is not a prime number. Then $< \sigma >$ is not a minimal subgroup of D_n. Let H be a nontrivial subgroup of $< \sigma >$ such that $H \subset < \sigma >$. Observe that H and $< \sigma >$ are not adjacent in $(\Gamma(D_n))^c$. Hence, $d(H, < \sigma >) \geq 2$ in $(\Gamma(D_n))^c$ and so, we obtain that $diam((\Gamma(D_n))^c) \geq 2$. Let H_1, H_2 be nontrivial subgroups of D_n with $H_1 \neq H_2$. Suppose that $\tau \notin H_1 \cup H_2$. Note that $K = < \tau >$ is a subgroup of D_n with $o(K) = 2$ and $H_i \cap K = \{e\}$ for each $i \in \{1, 2\}$. Hence, $H_1 - K - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(D_n))^c$. As $o(\sigma\tau) = 2$, it follows that if $\sigma\tau \notin H_1 \cup H_2$, then $H_1 - < \sigma\tau > - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(D_n))^c$. Hence, in finding $d(H_1, H_2)$ in $(\Gamma(D_n))^c$, we can assume that $\tau, \sigma\tau \in H_1 \cup H_2$. Since D_n is generated by $\sigma\tau$ and τ, both τ and $\sigma\tau$ cannot be in H_i for each $i \in \{1, 2\}$. Without loss of generality, we can assume that $\tau \in H_1 \backslash H_2$ and $\sigma\tau \in H_2 \backslash H_1$. Since D_n is generated by τ and $\tau\sigma$ and as $\tau \in H_1$, it follows that $\tau\sigma \notin H_1$.

(i) Suppose that $n \geq 4$ and n is odd. We claim that $\tau\sigma \notin H_2$. For, if $\tau\sigma \in H_2$, then $(\sigma\tau)(\tau\sigma) = \sigma^2 \in H_2$. As n is odd, $o(\sigma) = o(\sigma^2) = n$. This implies that $\sigma \in H_2$ and so, $H_2 = D_n$. This is a contradiction. Therefore, $\tau\sigma \notin H_2$. It is now clear that $H_1 - < \tau\sigma > - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(D_n))^c$. This proves that for any nontrivial subgroups H_1, H_2 of D_n with $H_1 \neq H_2$, $d(H_1, H_2) \leq 2$ in $(\Gamma(D_n))^c$. Therefore, we get that $diam((\Gamma(D_n))^c) \leq 2$ and so, $diam((\Gamma(D_n))^c) = 2$.

Suppose that $n = 2m$, where $m \geq 3$ and m is odd. If $\tau\sigma \notin H_2$, then $H_1 - < \tau\sigma > - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(D_n))^c$. Suppose that $\tau\sigma \in H_2$. Then it follows that $(\sigma\tau)(\tau\sigma) = \sigma^2 \in H_2$. Thus $\sigma^2, \sigma\tau \in H_2$ and so, $< \sigma^2, \sigma\tau > \subseteq H_2$. As $\sigma \notin H_2$, $\sigma^2 \in H_2$, and m is odd we obtain that $\sigma^m \notin H_2$. Suppose that $\sigma^m \notin H_1$. Let $K = < \sigma^m >$. Note that $o(K) = 2$ and $H_i \cap K = \{e\}$ for each $i \in \{1, 2\}$. Hence, $H_1 - K - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(D_n))^c$. Suppose that $\sigma^m \in H_1$. As $\sigma \notin H_1$, it follows that $\sigma^2 \notin H_1$. Since $\tau \in H_1$, we obtain that $\sigma^2\tau \notin H_1$. As $\sigma^2 \in H_2$ and $\tau \notin H_2$, we obtain that $\sigma^2\tau \notin H_2$. Thus $\sigma^2\tau \notin H_1 \cup H_2$. As $o(< \sigma^2\tau >) = 2$, we obtain that $H_i \cap < \sigma^2\tau > = \{e\}$ for each $i \in \{1, 2\}$. Therefore, $H_1 - < \sigma^2\tau > - H_2$ is a path of length two between H_1

and H_2 in $(\Gamma(D_n))^c$. It follows from the above given arguments that $diam((\Gamma(D_n))^c) \leq 2$ and so, $diam((\Gamma(D_n))^c) = 2$.
(ii) Suppose that $n = 2^k t$, where $k \geq 2$ and $t \geq 1$ is odd. Let H_1 be the subgroup of D_n generated by σ^2 and τ and let H_2 be the subgroup of D_n generated by σ^2 and $\sigma\tau$. Observe that $< \sigma^2 >$ is a characteristic subgroup of $< \sigma >$. Since $[D_n :< \sigma >] = 2$, it follows that $< \sigma >$ is a normal subgroup of D_n. Therefore, we obtain from [6, Problem 9, p.70] that $< \sigma^2 >$ is a normal subgroup of D_n. Therefore, $H_1 =< \sigma^2 >< \tau >$ and $H_2 =< \sigma^2 >< \sigma\tau >$. Note that $o(< \sigma^2 >) = 2^{k-1}t$ and $o(< \tau >) = o(\sigma\tau >) = 2$ and $< \sigma^2 > \cap < \tau >=< \sigma^2 > \cap < \sigma\tau >= \{e\}$. Therefore, we obtain from [6, Theorem 2.5.1, p.45] that $o(H_1) = o(H_2) = (2^{k-1}t)(2) = 2^k t$. Hence, H_1 and H_2 are maximal subgroups of D_n and they are also normal subgroups of D_n. Since $\sigma^2 \in H_1 \cap H_2$, it follows that H_1 and H_2 are not adjacent in $(\Gamma(D_n))^c$. We claim that there exists no path of length two between H_1 and H_2 in $(\Gamma(D_n))^c$. Suppose that there exists a path of length two between H_1 and H_2 in $(\Gamma(D_n))^c$. Let H_3 be a nontrivial subgroup of D_n such that $H_1 - H_3 - H_2$ is a path of length two in $(\Gamma(D_n))^c$. Then $H_i \cap H_3 = \{e\}$ for each $i \in \{1, 2\}$. Note that $H_3 \not\subseteq H_1$ and H_1 is a maximal and a normal subgroup of D_n. Therefore, we obtain that $H_1 H_3 = D_n$. Hence, $o(H_1)o(H_3) = o(D_n)$ and so, $o(H_3) = 2$. Observe that $S = \{\sigma^{2^{k-1}t}, \tau, \sigma\tau, \sigma^2\tau, \ldots, \sigma^{n-1}\tau\}$ is the set of all elements of order 2 in D_n. Hence, $H_3 =< s >$ for some $s \in S$. Note that $\{\sigma^{2^{k-1}t}, \tau, \sigma^2\tau, \ldots, \sigma^{n-2}\tau\} \subseteq H_1$ and $\{\sigma\tau, \sigma^3\tau, \ldots, \sigma^{n-1}\tau\} \subseteq H_2$. This implies that $S \subseteq H_1 \cup H_2$ and so, either $H_3 \subseteq H_1$ or $H_3 \subseteq H_2$. This is a contradiction. Therefore, there exists no path of length two between H_1 and H_2 in $(\Gamma(D_n))^c$. Hence, we obtain that $diam((\Gamma(D_n))^c) \geq 3$ and as $diam((\Gamma(D_n))^c) \leq 3$, it follows that $diam((\Gamma(D_n))^c) = 3$. □

Remark 2.21. Let $n \geq 4$ be such that n is not a prime number. Then $r((\Gamma(D_n))^c) = 2$.

Proof. It is already noted in Remark 2.19 that $(\Gamma(D_n))^c$ is connected and $diam((\Gamma(D_n))^c)$ is determined in Proposition 2.20. Let σ, τ be as mentioned in Remark 2.19. Let H be any minimal subgroup of D_n. We know from Remark 2.12 that $e(H) \leq 2$ in $(\Gamma(D_n))^c$. We next verify that $e(H) \geq 2$ in $(\Gamma(D_n))^c$. We consider the following cases.
Case(1): $H \subseteq< \sigma >$.

Note that $H =< \sigma^{\frac{n}{p}} >$ for some prime number p such that p is a divisor of n. Observe that $o(H) = p$. Since H is a characteristic subgroup of $< \sigma >$ and $< \sigma >$ is a normal subgroup of D_n, we obtain from [6, Problem 9, p.70] that H is a normal subgroup of D_n. Let K be the subgroup of D_n generated by $\sigma^{\frac{n}{p}}$ and τ. Observe that $K = H <$

$\tau >$. As $o(\tau) = 2$ and $H\cap < \tau >= \{e\}$, it follows from [6, Theorem 2.5.1, p.45] that $o(K) = 2p < 2n = o(D_n)$. Hence, K is a nontrivial subgroup of D_n. Since $H \cap K = H \neq \{e\}$, we get that H and K are not adjacent in $(\Gamma(D_n))^c$. Therefore, $d(H, K) \geq 2$ in $(\Gamma(D_n))^c$ and so, $e(H) \geq 2$ in $(\Gamma(D_n))^c$.

Case(2): $H \not\subseteq < \sigma >$.

In this case $H =< \sigma^i\tau >$ for some $i \in \{0, 1, \ldots, n-1\}$. Note that $o(H) = 2$. Let p be a prime number such that p is a divisor of n. Let K be the subgroup of D_n generated by $\sigma^{\frac{n}{p}}$ and $\sigma^i\tau$. Using the same arguments as in Case(1), we obtain that $o(K) = 2p < 2n = o(D_n)$. Hence, K is a nontrivial subgroup of D_n. It is clear that H and K are not adjacent in $(\Gamma(D_n))^c$. Therefore, $d(H, K) \geq 2$ in $(\Gamma(D_n))^c$. This proves that $e(H) \geq 2$ in $(\Gamma(D_n))^c$.

Therefore, $e(H) = 2$ in $(\Gamma(D_n))^c$ for any minimal subgroup H of D_n. It is clear that if K is any nontrivial subgroup of D_n which is not minimal, then $e(K) \geq 2$ in $(\Gamma(D_n))^c$. Therefore, we obtain that $r((\Gamma(D_n))^c) = 2$. □

Proposition 2.22. *Let $G, \overline{G}$ be finite groups such that both of them admit at least two nontrivial subgroups. Let $\phi : G \to \overline{G}$ be a surjective homomorphism of groups. If $(\Gamma(G))^c$ is connected, then $(\Gamma(\overline{G}))^c$ is also connected. Moreover, if $diam((\Gamma(G))^c) \leq 2$, then $diam((\Gamma(\overline{G}))^c) \leq 2$.*

Proof. Let e denote the identity element of G and let us denote the identity element of $\overline{G}$ by $\overline{e}$. Let us denote $Ker\phi$ by N. It is clear that $N \neq G$. If $N = \{e\}$, then $G \cong \overline{G}$ as groups. Hence, the graphs $(\Gamma(G))^c$ and $(\Gamma(\overline{G}))^c$ are isomorphic. Therefore, there is nothing to prove in this case. So, we can assume that $N \neq \{e\}$. Let $y \in \overline{G}$, $y \neq \overline{e}$. Since ϕ is a surjective homomorphism from G onto $\overline{G}$, there exists $x \in G\backslash\{e\}$ such that $y = \phi(x)$. We are assuming that $(\Gamma(G))^c$ is connected. Therefore, we obtain from $(i) \Rightarrow (ii)$ of Proposition 2.1 that $N_G = G$. Note that there exist $k \geq 1$ and elements $g_1, \ldots, g_k \in G$ such that $o(g_i)$ is a prime number for each $i \in \{1, \ldots, k\}$ and $x = \prod_{i=1}^k g_i$. Hence, $y = \phi(x) = \prod_{i=1}^k \phi(g_i)$. Since $y \neq \overline{e}$, it follows that $\phi(g_i) \neq \overline{e}$ for at least one $i \in \{1, \ldots, k\}$ and for such an i, $o(\phi(g_i)) = o(g_i)$ is a prime number. The above discussion implies that $N_{\overline{G}} = \overline{G}$. Therefore, we obtain from $(ii) \Rightarrow (i)$ of Proposition 2.1 that $(\Gamma(\overline{G}))^c$ is connected.

We next prove the moreover part. Suppose that $diam((\Gamma(G))^c) \leq 2$. We show that $diam((\Gamma(\overline{G}))^c) \leq 2$. Let W_1, W_2 be nontrivial subgroups of $\overline{G}$ with $W_1 \neq W_2$. We now show that there exists a path of length at most two between W_1 and W_2 in $(\Gamma(\overline{G}))^c$. We can assume that W_1 and W_2 are not adjacent in $(\Gamma(\overline{G}))^c$. We know from [6, Lemma 2.7.5, p.63] that there exist nontrivial subgroups H_1, H_2 of G with $N \subset H_i$ for each $i \in \{1,2\}$ and $W_i = \phi(H_i)$ for each $i \in \{1,2\}$. It is clear that $H_1 \neq H_2$ and as $H_1 \cap H_2 \neq \{e\}$, we obtain that H_1 and H_2 are not adjacent in $(\Gamma(G))^c$. We are assuming that $diam((\Gamma(G))^c) \leq 2$. Hence, there exists a nontrivial subgroup K of G such that $H_1 - K - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(G))^c$. We assert that $W_i \cap \phi(K) = \{\overline{e}\}$. for each $i \in \{1,2\}$. Let $i \in \{1,2\}$. Let $z \in W_i \cap \phi(K)$. Then $z = \phi(h_i) = \phi(k)$ for some $h_i \in H_i$ and $k \in K$. Hence, $kh_i^{-1} \in N \subset H_i$ and so, $k \in H_i \cap K = \{e\}$. Therefore, $z = \phi(k) = \phi(e) = \overline{e}$. This shows that $W_i \cap \phi(K) = \{\overline{e}\}$ for each $i \in \{1,2\}$. From $H_1 \cap K = \{e\}$ and $N \subset H_1$, it follows that $\phi(K) \neq \{\overline{e}\}$. Hence, $W_1 - \phi(K) - W_2$ is a path of length two between W_1 and W_2 in $(\Gamma(\overline{G}))^c$. This proves that $diam((\Gamma(\overline{G}))^c) \leq 2$. □

Remark 2.23. Let $G, \overline{G}$ be finite groups such that both G and $\overline{G}$ admit at least two nontrivial subgroups. Let $\phi : G \to \overline{G}$ be a surjective homomorphism of groups. Suppose that $(\Gamma(G))^c$ is connected. Then $(\Gamma(\overline{G}))^c$ is connected. If $diam((\Gamma(\overline{G}))^c) = 3$, then $diam((\Gamma(G))^c) = 3$.

Proof. We know from Proposition 2.22 that $(\Gamma(\overline{G}))^c$ is connected. If $diam((\Gamma(\overline{G}))^c) = 3$, then it follows from Proposition 2.22 that $diam((\Gamma(G))^c) \geq 3$. We know from the proof of $(ii) \Rightarrow (i)$ of Proposition 2.1 that $diam((\Gamma(G))^c) \leq 3$ and so, we get that $diam((\Gamma(G))^c) = 3$. □

3. SOME MORE RESULTS

Let G be a finite group which admits at least one nontrivial subgroup. The aim of this section is to determine $\omega((\Gamma(G))^c)$ and $girth((\Gamma(G))^c)$.

Proposition 3.1. *Let G be a finite group. Then $\omega((\Gamma(G))^c) = \chi((\Gamma(G))^c) = k$, where k is the number of minimal subgroups of G.*

Proof. Since G is a finite group with at least one nontrivial subgroup, G has at least one minimal subgroup and G has only a finite number of minimal subgroups. Let k be the number of minimal subgroups of G. Let $\{W_1, \ldots, W_k\}$ be the set of all minimal subgroups of G. Since $W_i \cap W_j = \{e\}$ for all distinct $i, j \in \{1, 2, \ldots, k\}$, it follows that the subgraph of $(\Gamma(G))^c$ induced on $\{W_1, \ldots, W_k\}$ is a clique on k vertices. Therefore, we get that $\omega((\Gamma(G))^c) \geq k$. We next verify that the vertices of $(\Gamma(G))^c$ can be properly colored using a set of k distinct colors. Let $\{c_1, \ldots, c_k\}$ be a set of k distinct colors. Now, color W_i with c_i for each $i \in \{1, \ldots, k\}$. Let H be any nontrivial subgroup of G. It is clear that H contains a minimal subgroup of G. Let $i \in \{1, \ldots, k\}$ be least with the property that $H \supseteq W_i$. Then color H using c_i. We claim that the above assignment of colors is a proper vertex coloring of $(\Gamma(G))^c$. Let H_1, H_2 be nontrivial subgroups of G such that H_1 and H_2 are adjacent in $(\Gamma(G))^c$. Hence, $H_1 \cap H_2 = \{e\}$. Let $i \in \{1, \ldots, k\}$ be least with the property that $H_1 \supseteq W_i$ and let $j \in \{1, \ldots, k\}$ be least with the property that $H_2 \supseteq W_j$. Note that H_1 receives color c_i and H_2 receives color c_j. As $H_1 \cap H_2 = \{e\}$, it is clear that $i \neq j$ and so, $c_i \neq c_j$. This shows that $(\Gamma(G))^c$ can be properly colored using a set of k distinct colors. Therefore, we obtain that $\chi((\Gamma(G))^c) \leq k \leq \omega((\Gamma(G))^c) \leq \chi((\Gamma(G))^c)$. This proves that $\omega((\Gamma(G))^c) = \chi((\Gamma(G))^c) = k$. □

Proposition 3.2. *Let G be a finite group. Then $girth((\Gamma(G))^c) = 3$ if and only if G has at least three minimal subgroups.*

Proof. Assume that $girth((\Gamma(G))^c) = 3$. Then there exist nontrivial subgroups H_1, H_2, H_3 such that $H_1 - H_2 - H_3 - H_1$ is a cycle of length three in $(\Gamma(G))^c$. Note that $H_1 \cap H_2 = H_2 \cap H_3 = H_3 \cap H_1 = \{e\}$. Let $i \in \{1, 2, 3\}$. Let W_i be a minimal subgroup of G such that $W_i \subseteq H_i$ for each $i \in \{1, 2, 3\}$. Observe that $W_1 \cap W_2 = W_2 \cap W_3 = W_3 \cap W_1 = \{e\}$. Hence, $W_i \neq W_j$ for all distinct $i, j \in \{1, 2, 3\}$. Therefore, G has at least three minimal subgroups.

Conversely, assume that G has at least three minimal subgroups. We know from Proposition 3.1 that $\omega((\Gamma(G))^c) = k$, where k is the number of minimal subgroups of G. As $k \geq 3$, it follows that $girth((\Gamma(G))^c) = 3$. □

Proposition 3.3. *Let G be a finite group. Let $o(G) = \prod_{i=1}^{t} p_i^{n_i}$ be the factorization of $o(G)$ into product of prime numbers (here, $p_1, \ldots, p_t$ are distinct prime numbers and $n_i \geq 1$ for each $i \in \{1, \ldots, t\}$ and in the case $t = 1$, $n_1 > 1$). Then $\omega((\Gamma(G))^c) = t$ if and only if for each $i \in \{1, \ldots, t\}$, G has only one subgroup W_i with $o(W_i) = p_i$. Moreover, if G is abelian, then $\omega((\Gamma(G))^c) = t$ if and only if G is cyclic.*

Proof. We know from Proposition 3.1 that $\omega((\Gamma(G))^c) = k$, where k is the number of minimal subgroups of G. Therefore, $\omega((\Gamma(G))^c) = t$ if and only if G has exactly t minimal subgroups. Let $i \in \{1, \ldots, t\}$. Since p_i is a divisor of $o(G)$, we know from Cauchy's theorem [6, Theorem 2.11.3, p.87] that there exists a subgroup W_i of G with $o(W_i) = p_i$. It is clear that W_i is a minimal subgroup of G for each $i \in \{1, \ldots, t\}$. Observe that if W is any minimal subgroup of G, then $o(W) = p_i$ for some $i \in \{1, \ldots, t\}$. Hence, $\omega((\Gamma(G))^c) = t$ if and only if $\{W_1, \ldots, W_t\}$ is the set of all minimal subgroups of G. Therefore, we obtain that $\omega((\Gamma(G))^c) = t$ if and only if for each $i \in \{1, \ldots, t\}$, there exists only one subgroup W_i of G with $o(W_i) = p_i$.

We next verify the moreover part of this Proposition. If G is cyclic, then for each divisor d of $o(G)$, there exists a unique subgroup H of G with $o(H) = d$. Hence, for each $i \in \{1, \ldots, t\}$, W_i is the only subgroup of G with $o(W_i) = p_i$. Therefore, $\omega((\Gamma(G))^c) = t$. Conversely, assume that G is abelian and $\omega((\Gamma(G))^c) = t$. For each $i \in \{1, \ldots, t\}$, let P_i be the unique p_i-Sylow subgroup of G. Note that $o(P_i) = p_i^{n_i}$ for each $i \in \{1, \ldots, t\}$ and G is the internal direct product of $P_1, \ldots, P_t$. It is clear that W_i is the only subgroup of P_i with $o(W_i) = p_i$. We assert P_i is cyclic for each $i \in \{1, \ldots, t\}$. Suppose that P_i is not cyclic for some $i \in \{1, \ldots, t\}$. Then $n_i > 1$ and we know from the proof of the fundamental theorem of finite abelian groups [6, Theorem 2.14.1, p.109] that there exist $s \geq 2$ and cyclic subgroups $A_1, A_2, \ldots, A_s$ of P_i such that $o(A_1) = p_i^{n_{i1}}, o(A_2) = p_i^{n_{i2}}, \ldots, o(A_s) = p_i^{n_{is}}$ with $n_{i1} \geq n_{i2} \cdots \geq n_{is} \geq 1$ and P_i is the internal direct product of $A_1, A_2, \ldots, A_s$. We know from [3, Problem 6, p.154] that the number of minimal subgroups of P_i equals $\frac{p_i^s - 1}{p_i - 1} = 1 + p_i + \cdots + p_i^{s-1} \geq 2$, since $s \geq 2$. This is impossible as W_i is the only minimal subgroup of P_i. This proves that P_i is cyclic for each $i \in \{1, \ldots, t\}$. As $(o(P_i), o(P_j)) = 1$ for all distinct $i, j \in \{1, \ldots, t\}$, it follows from [6, Problem 6, p.108] that G is cyclic. □

Remark 3.4. Let G be a finite group such that $o(G)$ is divisible by at least three distinct prime numbers p_1, p_2, and p_3. We know from Cauchy's theorem [6, Theorem 2.11.3, p.87] that for each $i \in \{1, 2, 3\}$, there exists a subgroup W_i of G such that $o(W_i) = p_i$. It is clear that W_i is a minimal subgroup of G for each $i \in \{1, 2, 3\}$ and hence, we obtain from Proposition 3.2 that $girth((\Gamma(G))^c) = 3$.

Proposition 3.5. *Let G be a finite group such that $o(G) = p_1p_2$, where p_1 and p_2 are distinct prime numbers. Then $girth((\Gamma(G))^c) \in \{3, \infty\}$.*

Proof. We can assume without loss of generality that $p_1 < p_2$. It is already noted in Remark 2.7 that $(\Gamma(G))^c$ is either K_2 or K_{p_2+1}. Therefore, we obtain that $girth((\Gamma(G))^c) \in \{3, \infty\}$. □

Lemma 3.6. *Let G be a finite group such that $o(G) = p_1^{n_1}p_2^{n_2}$, where p_1 and p_2 are distinct prime numbers and $n_i > 1$ for each $i \in \{1, 2\}$. Then $girth((\Gamma(G))^c) \leq 4$.*

Proof. Let $i \in \{1, 2\}$. Let $k \in \mathbb{N}$ be such that $k \leq n_i$. We know from [6, Theorem 2.12.1, p.92] that there exists a subgroup H of G such that $o(H) = p_i^k$. Let V_i denote the set of all subgroups H of G such that $o(H) = p_i^k$ for some $k \in \mathbb{N}$ with $k \leq n_i$ for each $i \in \{1, 2\}$. It is clear that each member of V_i is a nontrivial subgroup of G and V_i contains at least n_i elements for each $i \in \{1, 2\}$. As $n_i \geq 2$, it follows that V_i

contains at least two elements for each $i \in \{1, 2\}$. Since $(p_1, p_2) = 1$, it follows from Lagrange's theorem that $H \cap W = \{e\}$ for any $H \in V_1$ and $W \in V_2$. If there exist $H_1, H_2 \in V_1$ such that $H_1 \cap H_2 = \{e\}$, then for any $W \in V_2$, we obtain that $H_1 - W - H_2 - H_1$ is a cycle of length three in $(\Gamma(G))^c$. Similarly, if there exist $W_1, W_2 \in V_2$ such that $W_1 \cap W_2 = \{e\}$, then for any $H \in V_1$, we get that $W_1 - H - W_2 - W_1$ is a cycle of length three in $(\Gamma(G))^c$. Hence, we can assume that no two distinct members of V_i are adjacent in $(\Gamma(G))^c$ for each $i \in \{1, 2\}$. Let $H_1, H_2 \in V_1$ with $H_1 \neq H_2$ and let $W_1, W_2 \in V_2$ with $W_1 \neq W_2$. Note that $H_1 - W_1 - H_2 - W_2 - H_1$ is a cycle of length four in $(\Gamma(G))^c$. This proves that $girth((\Gamma(G))^c) \leq 4$. □

Proposition 3.7. *Let G be a finite cyclic group with $o(G) = p_1^{n_1} p_2^{n_2}$, where p_1, p_2 are distinct prime numbers and $n_i > 1$ for each $i \in \{1, 2\}$. Then $girth((\Gamma(G))^c) = 4$.*

Proof. We know from Lemma 3.6 that $girth((\Gamma(G))^c) \leq 4$. Since G is a cyclic group with $o(G) = p_1^{n_1} p_2^{n_2}$, it follows that G has exactly two minimal subgroups. Hence, we obtain from Proposition 3.2 that $girth((\Gamma(G))^c) \neq 3$ and therefore, $girth((\Gamma(G))^c) = 4$. □

Proposition 3.8. *Let G be a finite cyclic group with $o(G) = p_1^n p_2$, where p_1 and p_2 are distinct prime numbers and $n > 1$. Then $girth((\Gamma(G))^c) = \infty$.*

Proof. Let P_1 be the subgroup of G with $o(P_1) = p_1^n$ and let P_2 be the subgroup of G with $o(P_2) = p_2$. Let V_1 denote the set of all subgroups H of P_1 with $H \neq \{e\}$ and let $V_2 = \{P_2\}$. Since P_1 is cyclic, it is clear that V_1 contains exactly n elements. As is noted in the proof of Lemma 3.6, $H \cap P_2 = \{e\}$ for any $H \in V_1$ and hence, H and P_2 are adjacent in $(\Gamma(G))^c$. Let W_1, W_2 be any two distinct nontrivial subgroups of G such that $W_i \notin V_1 \cup V_2$. Observe that $W_i = H_i P_2$ for some subgroup $H_i \in V_1$ such that $H_i \neq P_1$ for each $i \in \{1, 2\}$. It is clear that $W_i \cap H \neq \{e\}, W_i \cap P_2 \neq \{e\}, W_1 \cap W_2 \neq \{e\}$ for each $i \in \{1, 2\}$ and for any subgroup $H \in V_1$. From the above discussion, we obtain that $V_1 \cup V_2$ is the set of all nonisolated vertices of $(\Gamma(G))^c$ and the subgraph of $(\Gamma(G))^c$ induced on $V_1 \cup V_2$ is a star graph. Indeed, it is $K_{1,n}$. Therefore, we get that $girth((\Gamma(G))^c) = \infty$. □

Proposition 3.9. *Let G be a finite abelian group with $o(G) = p_1^{n_1} p_2^{n_2}$, where p_1 and p_2 are distinct prime numbers. Suppose that G is not cyclic. Then $girth((\Gamma(G))^c) = 3$.*

Proof. We know from Proposition 3.1 that $\omega((\Gamma(G))^c) = k$, where k is the number of minimal subgroups of G. It is clear that $k \geq 2$.

Since G is abelian but not cyclic, we obtain from Proposition 3.3 that $\omega((\Gamma(G))^c) \geq 3$ and therefore, $girth((\Gamma(G))^c) = 3$. □

We mention an example in Example 3.10 to illustrate that the hypothesis that the group G is abelian cannot be omitted in Proposition 3.9. For any $n \geq 2$, we denote the additive group of integers modulo n by $\mathbb{Z}_n$.

Example 3.10. Let Q_8 be the *quaternion group* of order 8 given in [5, Exercise 44, p.187]. Let $G = Q_8 \times \mathbb{Z}_9$ be the external direct product of Q_8 and $\mathbb{Z}_9$. Observe that $o(G) = 2^3 3^2$. Note that $\{1, -1\} \times \{0\}$ and $\{1\} \times \{0, 3, 6\}$ are the only minimal subgroups of G. Hence, we obtain from Proposition 3.2 that $girth((\Gamma(G)))^c) \neq 3$. We know from Lemma 3.6 that $girth((\Gamma(G))^c) \leq 4$ and therefore, $girth((\Gamma(G))^c) = 4$.

Remark 3.11. Let G be a finite group with $o(G) = p^n$, where p is a prime number and $n \geq 2$. If G is cyclic, then G has only one minimal subgroup and so, $girth((\Gamma(G))^c) = \infty$. If G is abelian but not cyclic, then it is already noted in the proof of Proposition 3.3 that G has at least three minimal subgroups and so, we obtain from Proposition 3.2 that $girth((\Gamma(G))^c) = 3$.

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SOME RESULTS ON THE COMPLEMENT OF THE INTERSECTION GRAPH OF SUBGROUPS OF A FINITE GROUP

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نتایجی درباره مکمل گراف اشتراکی زیرگروهای یک گروه متناهی

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در این مقاله گروه‌هایی مانند G را در نظر می‌گیریم که دارای حداقل یک زیرگروه غیربدیهی می‌باشند (یادآوری می‌کنیم که زیرگروه H از گروه G غیربدیهی نامیده می‌شود هرگاه $H \notin \{G, e\}$). فرض کنیم G یک گروه باشد. در این صورت گراف اشتراکی زیرگروه‌های G، که با نماد $\Gamma(G)$ نمایش داده می‌شود، یک گراف بدون جهت می‌باشد که مجموعه‌ی راسی آن مجموعه‌ی تمام زیرگروه‌های غیربدیهی G بوده و رئوس متمایز H و K توسط یک یال در این گراف مجاور می‌باشند اگر و تنها اگر $H \cap K \neq \{e\}$. فرض کنیم G یک گروه متناهی باشد. هدف اصلی این مقاله بررسی ارتباط بین خواص گروهی G و خواص گرافی مکمل گراف $\Gamma(G)$ می‌باشد.

کلمات کلیدی: مکمل گراف اشتراکی زیرگروه‌های یک گروه متناهی، گروه آبلی متناهی، گراف همبند، کمر گراف.