

FILTER REGULAR SEQUENCES AND LOCAL COHOMOLOGY MODULES

J. AZAMI*

ABSTRACT. Let R be a commutative Noetherian ring. In this paper we consider some relations between filter regular sequence, regular sequence and system of parameters over R -modules. Also we obtain some new results about cofiniteness and cominimaxness of local cohomology modules.

1. INTRODUCTION

Throughout this paper, let R denote a commutative Noetherian ring (with identity) and I an ideal of R . For an R -module M , the i^{th} local cohomology module of M with respect to I is defined as

$$H_I^i(M) = \varinjlim_{n \geq 1} \text{Ext}_R^i(R/I^n, M).$$

We refer the reader to [5] or [3] for more details about local cohomology. The concept of filter regular sequence plays an important role in this paper. We say that a sequence x_1, \dots, x_n of elements of I , is an I -filter regular sequence on M , if

$$\text{Supp}_R \left(\frac{(x_1, \dots, x_{i-1})M :_M x_i}{(x_1, \dots, x_{i-1})M} \right) \subseteq V(I),$$

for all $i = 1, \dots, n$. Also, we say that an element $x \in I$ is an I -filter regular sequence on M if $\text{Supp}_R(0 :_M x) \subseteq V(I)$. The concept of an I -filter regular sequence on M is a generalization of the concept of a filter

MSC(2010): Primary: 13D45, 14B15; Secondary: 13E05.

Keywords: Filter regular sequence, regular sequence, system of parameters, local cohomology module.

Received: 26 September 2018, Accepted: 19 May 2019.

*Corresponding author.

regular sequence which has been studied in [18]. Both concepts coincide if I is an \mathfrak{m} -primary ideal of a local ring with maximal ideal \mathfrak{m} . In 1969, A. Grothendieck conjectured that if I is an ideal of R and M is a finitely generated R -module, then the R -modules $\text{Hom}_R(R/I, H_I^i(M))$ are finitely generated for all $i \geq 0$. R. Hartshorne has provided a counterexample to this conjecture in [6]. Also he defined a module T to be I -cofinite if $\text{Supp } T \subseteq V(I)$ and $\text{Ext}_R^i(R/I, T)$ is finitely generated for each $i \geq 0$ and he asked the following question.

For which rings R and ideals I are the modules $H_I^i(M)$ I -cofinite for all i and all finitely generated modules M ?

Hartshorne proved that if I is an ideal of the complete regular local ring R and M a finitely generated R -module, then $H_I^i(M)$ is I -cofinite in two following cases:

- (i) I is principal ideal, (see [6], Corollary 6.3),
- (ii) I is prime ideal with $\dim R/I = 1$, (see [6], Corollary 7.7).

This subject was studied by several authors afterwards, (see [4], [11], [9], [19], [1] and [10]).

Some important results of this paper are as follows:

Theorem 1.1. *Let (R, \mathfrak{m}) be a Noetherian local ring and $M \neq 0$ be a finitely generated R -module of dimension $d \geq 1$. Let $x_1, \dots, x_d \in \mathfrak{m}$ be an \mathfrak{m} -filter regular sequence for M . Then the following statements are holds:*

- (1) x_1, \dots, x_d is a system of parameters for M .
- (2) For each $1 \leq i \leq d$, the R -module $H_{\mathfrak{m}}^i(M)$ is (x_1, \dots, x_i) -cofinite.

Theorem 1.2. *Let (R, \mathfrak{m}) be a Noetherian local ring and I be an ideal of R . Then for every finitely generated R -module $M \neq 0$ of dimension d , the following statements are equivalent:*

- (1) $H_{\mathfrak{m}}^d(M)$ is I -cofinite.
- (2) $H_{\mathfrak{m}}^d(M) \cong H_I^d(M)$.

Theorem 1.3. *Let R be a Noetherian ring, I an ideal of R and $M \neq 0$ be a finitely generated R -module such that $\dim \frac{M}{IM} \leq 1$. If $t \geq 1$ and $x_1, \dots, x_t \in I$ is an I -filter regular sequence for M , then for each $0 \leq i \leq t - 1$, the R -module $H_I^i(M)$ is (x_1, \dots, x_t) -cofinite and $\text{Hom}_R\left(\frac{R}{(x_1, \dots, x_t)}, H_I^t(M)\right)$ is finitely generated.*

For each R -module L , we denote by $\text{Ass}_R L$ the set $\{\mathfrak{p} \in \text{Ass}_R L : \dim R/\mathfrak{p} = \dim L\}$. Also, for any ideal \mathfrak{b} of R , the radical of \mathfrak{b} , denoted by $\text{Rad}(\mathfrak{b})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b} \text{ for some } n \in \mathbb{N}\}$ and we denote $\{\mathfrak{p} \in \text{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{b}\}$ by $V(\mathfrak{b})$. Finally, for each R -module L , we denote by $\text{mAss}_R L$, the minimal elements of $\text{Ass}_R L$. For any unexplained notation and terminology we refer the reader to [3] and [12].

2. MAIN RESULTS

Theorem 2.1. *Let (R, \mathfrak{m}) be a Noetherian local ring and $M \neq 0$ be a finitely generated R -module of dimension $d \geq 1$. Let $x_1, \dots, x_d \in \mathfrak{m}$ be an \mathfrak{m} -filter regular sequence for M . Then*

- (1) x_1, \dots, x_d is a system of parameters for M .
- (2) For each $1 \leq i \leq d$, the R -module $H_{\mathfrak{m}}^i(M)$ is (x_1, \dots, x_i) -cofinite.

Proof. (1). By definition $x_i \notin \cup_{P \in \text{Ass}(\frac{R}{(x_1, \dots, x_{i-1})}) \setminus \{\mathfrak{m}\}} P$ for each $1 \leq i \leq d$, and so $x_i \notin \cup_{P \in \text{Ass}_R(\frac{R}{(x_1, \dots, x_{i-1})})} P$. Therefore x_1, \dots, x_d is a system of parameters for M .

(2). By [8, Proposition 1.2], $H_{(x_1, \dots, x_i)}^j(M) \cong H_{\mathfrak{m}}^j(M)$ for each $0 \leq j \leq i - 1$ and $\dim \text{Supp } H_{(x_1, \dots, x_i)}^j(M) \leq 0$. Hence by [1, Theorem 2.6], the R -module $H_{(x_1, \dots, x_i)}^j(M)$ is (x_1, \dots, x_i) -cofinite. Also for $j > i$, $H_{(x_1, \dots, x_i)}^j(M) = 0$. Thus by [15, Proposition 3.11], the R -module $H_{(x_1, \dots, x_i)}^i(M)$ is also (x_1, \dots, x_i) -cofinite. Since $H_{(x_1, \dots, x_i)}^{i-1}(M)$ is Artinian, it follows from Grothendick vanishing theorem [3, Proposition 6.1], $H_{Rx_{i+1}}^1(H_{(x_1, \dots, x_i)}^{i-1}(M)) = 0$. By [17], there exists an exact sequence as follows $0 \rightarrow H_{Rx_{i+1}}^1(H_{(x_1, \dots, x_i)}^{i-1}(M)) \rightarrow H_{(x_1, \dots, x_{i+1})}^i(M) \rightarrow H_{Rx_{i+1}}^0(H_{(x_1, \dots, x_i)}^i(M)) \rightarrow 0$. Note that this exact sequence shows

$$H_{(x_1, \dots, x_{i+1})}^i(M) \cong H_{Rx_{i+1}}^0(H_{(x_1, \dots, x_i)}^i(M)).$$

Also by [9], we have

$$H_{(x_1, \dots, x_{i+1})}^i(M) \cong H_{\mathfrak{m}}^i(M).$$

Therefore

$$H_{\mathfrak{m}}^i(M) \cong H_{Rx_{i+1}}^0(H_{(x_1, \dots, x_i)}^i(M))$$

and there exists an exact sequence as $0 \rightarrow H_{\mathfrak{m}}^i(M) \rightarrow H_{(x_1, \dots, x_i)}^i(M)$.

Since $\text{Hom}_R\left(\frac{R}{(x_1, \dots, x_i)}, H_{(x_1, \dots, x_i)}^i(M)\right)$ is finitely generated (because $H_{(x_1, \dots, x_i)}^i(M)$ is (x_1, \dots, x_i) -cofinite), it follows that the R -module

$\text{Hom}_R \left(\frac{R}{(x_1, \dots, x_i)}, H_m^i(M) \right)$ is also finitely generated. Now, by [16, Theorem 1.6] and by Artinianess of $H_m^i(M)$, we conclude that $H_m^i(M)$ is (x_1, \dots, x_i) -cofinite. \square

Theorem 2.2. *Let (R, \mathfrak{m}) be a complete Noetherian local ring and $M \neq 0$ be a finitely generated R -module of dimension $d \geq 1$. Let $P \in \text{Ass } M$ be such that $\dim \frac{R}{P} = t \geq 1$. Then for any \mathfrak{m} -filter regular sequence for M such as $x_1, \dots, x_t \in \mathfrak{m}$, $\text{Rad}(P + (x_1, \dots, x_t)) = \mathfrak{m}$. In particular x_1, \dots, x_t is a system of parameters for $\frac{R}{P}$.*

Proof. By Cohen's theorem every complete Noetherian ring is a homomorphic image of a Gorenstein local ring. Then by [2], we have

$$\{q \in \text{Att}_R H_m^t(M) \mid \dim \frac{R}{q} = t\} = \{q \in \text{Ass } M \mid \dim \frac{R}{q} = t\}.$$

Since $P \in \text{Ass } M$ and $\dim \frac{R}{P} = t$, it follows that $P \in \text{Att } H_m^t(M)$. By the previous Theorem, the R -module $H_m^t(M)$ is (x_1, \dots, x_t) -cofinite and so by [16, Theorem 1.6], $\text{Rad}(P + (x_1, \dots, x_t)) = \mathfrak{m}$. \square

Theorem 2.3. *Let (R, \mathfrak{m}) be a Noetherian local ring and I be an ideal of R . Then for every finitely generated R -module $M \neq 0$ of dimension d , the following statements are equivalent.*

- (1) $H_m^d(M)$ is I -cofinite.
- (2) $H_m^d(M) \cong H_I^d(M)$.

Proof. 1 \rightarrow 2 Let $H_m^d(M)$ be I -cofinite module. Then $H_m^d(M) \otimes_R \hat{R}$ is also $I\hat{R}$ -cofinite. Hence by [16, Theorem 1.6], for each $P \in \text{Att}_{\hat{R}}(H_{m\hat{R}}^d(\hat{M})) = \text{Ass}_{\hat{R}}(\hat{M})$, $\text{Rad}(I\hat{R} + P) = \mathfrak{m}\hat{R}$ and so $H_{I\hat{R}}^d(\frac{\hat{R}}{P}) \neq 0$. Therefore $H_{I\hat{R}}^d(\hat{R}) \otimes_{\hat{R}} \frac{\hat{R}}{P} \neq 0$ and $P \in \text{Att}_R H_{I\hat{R}}^d(\hat{R})$. Consequently $\text{Att}_{\hat{R}} H_{m\hat{R}}^d(\hat{R}) \subseteq \text{Att}_{\hat{R}} H_{I\hat{R}}^d(\hat{R}) \subseteq \text{Att } H_{m\hat{R}}^d(\hat{R})$ and so $\text{Att}_{\hat{R}}(H_{m\hat{R}}^d(\hat{R})) = \text{Att}_{\hat{R}}(H_{I\hat{R}}^d(\hat{R}))$. Now by [7], $H_{m\hat{R}}^d(\hat{R}) \cong H_{I\hat{R}}^d(\hat{R})$. Hence we have the following:

$$H_m^d(R) \cong H_{m\hat{R}}^d(\hat{R}) \cong H_{I\hat{R}}^d(\hat{R}) \cong H_I^d(R)$$

(2 \rightarrow 1). By [15], $H_I^d(M)$ is I -cofinite. Since $H_I^d(M) \cong H_m^d(M)$, it follows that $H_m^d(M)$ is I -cofinite. \square

Corollary 2.4. *Let (R, \mathfrak{m}) be a Noetherian local ring of dimension d and I be an ideal of R such that $H_m^d(R)$ is I -cofinite. Then $\text{ara}(I) = d$.*

Proof. The module $H_m^d(R)$ is I -cofinite, hence $H_I^d(R) \cong H_m^d(R) \neq 0$ and so $\text{ara}(I) \geq \text{cd}(I, R) = d$. On the other hand by [14, Corollary 2.8], $\text{ara}(I) \leq d$. \square

Definition 2.5. Let I be an ideal of R . The arithmetic rank of I , denoted by $\text{ara}(I)$, is the least number of elements of R required to generate an ideal which has the same radical as I .

Corollary 2.6. Let (R, \mathfrak{m}) be a Noetherian local ring of dimension $d \geq 0$ and $x_1, \dots, x_{d-1} \in \mathfrak{m}$ be such that $I = (x_1, \dots, x_{d-1})$. Then $\text{Hom}_R\left(\frac{R}{I}, H_m^d(R)\right)$ is not finitely generated.

Proof. By [16, Theorem 1.6], the R -module $\text{Hom}_R\left(\frac{R}{I}, H_m^d(R)\right)$ is finitely generated if and only if $H_m^d(R)$ is I -cofinite. But in this case $\text{ara}(I) = d$. On the other hand $\text{ara}(I) \leq d - 1$ which is a contradiction. \square

Proposition 2.7. Let (R, \mathfrak{m}) be a complete Noetherian local ring and $M \neq 0$ be a finitely generated R -module. Let N be submodule of M such that $\dim N = t \geq 1$. Then any \mathfrak{m} -filter regular sequence for M such as $x_1, \dots, x_t \in \mathfrak{m}$ is a system of parameters for N .

Proof. Let $m \text{Ass}_R N = \{P_1, \dots, P_n\}$, where $m \text{Ass}_R N$ denotes the minimal elements of $\text{Ass}_R N$. Then for each $1 \leq i \leq n$, $\dim \frac{R}{P_i} \leq \dim N = t$ and clearly $\dim \frac{R}{P_i} \geq 1$. Let $j = \dim \frac{R}{P_i}$. Then $j \leq t$ and by Theorem 2.2, $\text{Rad}(P_i + (x_1, \dots, x_j)) = \mathfrak{m}$. Since $(x_1, \dots, x_j) \subseteq (x_1, \dots, x_t)$, it follows that $\text{Rad}(P_i + (x_1, \dots, x_t)) = \mathfrak{m}$. We claim that $\text{Rad}(\cap_{i=1}^n P_i + (x_1, \dots, x_t)) = \mathfrak{m}$. For this, let Q be a minimal prime of $\cap_{i=1}^n P_i + (x_1, \dots, x_t)$. Hence there exists $1 \leq j \leq n$ such that $P_j \subseteq Q$ and so $p_j + (x_1, \dots, x_n) \subseteq Q$. Therefore $\mathfrak{m} = \text{Rad}(P_j + (x_1, \dots, x_t)) \subseteq \text{Rad}(Q) = Q \subseteq \mathfrak{m}$ and consequently $Q = \mathfrak{m}$. But $\cap_{i=1}^n P_i = \text{Rad}(\text{Ann } N)$ shows that

$$\text{Rad}(\text{Ann } N + (x_1, \dots, x_t)) = \mathfrak{m} \text{ and so } \dim_R \frac{N}{(x_1, \dots, x_t)N} = 0.$$

This completes the proof that x_1, \dots, x_t is a system of parameters for N . \square

Corollary 2.8. Let (R, \mathfrak{m}) be a complete Noetherian local ring, M be a finitely generated R -module and N be a submodule of M which is a Cohen-Macaulay with $\dim N = t$. If $x_1, \dots, x_t \in \mathfrak{m}$ is an \mathfrak{m} -filter regular sequence for M , then x_1, \dots, x_t is a N -regular sequence.

Proof. By Proposition 2.7, x_1, \dots, x_t is a system of parameters for N . But N is a Maximal Cohen-Macaulay as an $\frac{R}{\text{Ann } N}$ -module. Also $x_1 + \text{Ann } N, \dots, x_t + \text{Ann } N$ is a system of parameters for $\frac{R}{\text{Ann } N}$. On the other hand every maximal Cohen-Macaulay as an $\frac{R}{\text{Ann } N}$ -module is a balanced big Cohen-Macaulay as an R -module. Set $y_i = x_i + \text{Ann } N$ for each $1 \leq i \leq t$, then y_1, \dots, y_t is an N -regular sequence and this follows that x_1, \dots, x_t is an N -regular sequence. \square

Theorem 2.9. *Let R be a Noetherian ring, I an ideal of R and $M \neq 0$ be a finitely generated R -module such that $\dim \frac{M}{IM} \leq 1$. If $t \geq 1$ and $x_1, \dots, x_t \in I$ is an I -filter regular sequence for M , then for each $0 \leq i \leq t - 1$, the R -module $H_I^i(M)$ is (x_1, \dots, x_t) -cofinite and $\text{Hom}_R \left(\frac{R}{(x_1, \dots, x_t)}, H_I^t(M) \right)$ is finitely generated.*

Proof. For each $0 \leq i \leq t - 1$, we have $H_{(x_1, \dots, x_t)}^i(M) \cong H_I^i(M)$. Then

$$\text{Supp } H_{(x_1, \dots, x_t)}^i(M) = \text{Supp } H_I^i(M) \subseteq \text{Supp } \frac{M}{IM}$$

and for each $0 \leq i \leq t - 1$, $\dim \text{Supp } H_{(x_1, \dots, x_t)}^i(M) \leq 1$. By [1], clearly the R -module $H_{(x_1, \dots, x_{t-1})}^i$ is (x_1, \dots, x_t) -cofinite. Since $H_{(x_1, \dots, x_t)}^i(M) = 0$ for all $i \geq t + 1$, it follows from [15], that $H_{(x_1, \dots, x_t)}^t(M)$ is also (x_1, \dots, x_t) -cofinite. Consequently for each $i \geq 0$, the R -module $H_{(x_1, \dots, x_t)}^i(M)$ is (x_1, \dots, x_t) -Cofinite. Now, let $x_{t+1} \in I$ be such that x_1, \dots, x_{t+1} is I -filter regular sequence. Since $x_{t+1} \in I$ and $H_{(x_1, \dots, x_t)}^{t-1}(M) \cong H_I^{t-1}(M)$ is I -torsion, then $H_{Rx_{t+1}}^1(H_{(x_1, \dots, x_t)}^{t-1}(M)) = 0$. On the other hand by [17], the following exact sequence is hold: $0 \rightarrow H_{Rx_{t+1}}^1(H_{(x_1, \dots, x_t)}^{t-1}(M)) \rightarrow H_{(x_1, \dots, x_{t+1})}^t(M) \rightarrow H_{Rx_{t+1}}^0(H_{(x_1, \dots, x_t)}^t(M)) \rightarrow 0$. But, $H_{(x_1, \dots, x_t)}^t(M) \cong H_I^t(M)$ and so by the above exact sequence, $H_I^t(M) \cong H_{Rx_{t+1}}^0(H_{(x_1, \dots, x_t)}^t(M))$. Since $Rx_{t+1} \subseteq I$, it follows that

$$H_I^0(H_{(x_1, \dots, x_t)}^t(M)) \subseteq H_{Rx_t}^0(H_{(x_1, \dots, x_t)}^t(M)).$$

Also, $H_{Rx_{t+1}}^0(H_{(x_1, \dots, x_t)}^t(M)) \cong H_I^t(M)$ is I -torsion and hence

$$H_{Rx_{t+1}}^0(H_{(x_1, \dots, x_t)}^t(M)) \subseteq H_I^0(H_{(x_1, \dots, x_t)}^t(M)).$$

Then

$$H_I^t(M) \cong \Gamma_{Rx_{t+1}}(H_{(x_1, \dots, x_t)}^t(M)) = \Gamma_I(H_{(x_1, \dots, x_t)}^t(M)).$$

Finally from the exact sequence

$$0 \rightarrow H_I^t(M) \cong H_I^0(H_{(x_1, \dots, x_t)}^t(M)) \rightarrow H_{(x_1, \dots, x_t)}^t(M)$$

and (x_1, \dots, x_t) -cofiniteness of $H_{(x_1, \dots, x_t)}^t(M)$, we conclude that

$\text{Hom}_R\left(\frac{R}{(x_1, \dots, x_t)}, H_I^t(M)\right)$ is finitely generated. \square

Lemma 2.10. *Let M be an R -module and I be an ideal of R such that $\text{Supp } M \subseteq V(I)$. Let $x \in I$ be such that $0 :_M x$ and M/xM are I -cominimax. Then so is M .*

Proof. The proof is similar to the proof of [15, Corollary 3.4]. \square

Theorem 2.11. *With the assumption of Theorem 2.9, the R -module $H_I^t(M)$ is (x_1, \dots, x_t) -cominimax.*

Proof. We prove by induction on t . If $t = 1$, then we set $N = \frac{M}{\Gamma_I(M)}$

and so x_1 is an N -regular element and $H_I^1(N) \cong H_I^1(M)$.

Consider the exact sequence

$$0 \longrightarrow N \xrightarrow{x_1} N \longrightarrow \frac{N}{x_1N} \longrightarrow 0$$

which implies that the following exact sequence

$$\dots \longrightarrow H_I^0\left(\frac{N}{x_1N}\right) \longrightarrow H_I^1(N) \xrightarrow{x_1} H_I^1(N) \longrightarrow H_I^1\left(\frac{N}{x_1N}\right)$$

Clearly the R -module $0 :_{H_I^1(N)} x_1$ is finitely generated, and Rx_1 -cominimax. Set

$$T = \{P \in \text{Supp } H_I^1(N) \mid \dim \frac{R}{P} = 1\}.$$

Then $(H_I^1(N))_P$ for all $P \in T$ is Artinian and Rx_1 -cofinite. Also $T \subseteq \text{Assh } \frac{M}{IM}$ and so is finite. By argument in [1, Theorem 2.6], $\frac{H_I^1(N)}{x_1 H_I^1(N)}$

is minimax. Also $\frac{H_I^1(N)}{x_1 H_I^1(N)}$ and $0 :_{H_I^1(N)} x_1$ are Rx_1 -cominimax and hence $H_I^1(N)$ is also Rx_1 -cominimax.

Now, let $t \geq 2$. Clearly x_1, \dots, x_t is I -filter regular sequence over the R -module $\frac{M}{\Gamma_I(M)}$. Now $H_I^t(M) \cong H_I^t\left(\frac{M}{\Gamma_I(M)}\right)$ and $\frac{M}{\Gamma_I(M)}$ is a finitely generated I -torsion free R -module. We therefore assume in addition that $\Gamma_I(M) = 0$. Since $x_1 \notin \cup_{P \in \text{Ass } M \setminus V(I)} P = \cup_{P \in \text{Ass}(M)} P$, it follows that $(x_1, \dots, x_t) \not\subseteq \cup_{P \in \text{Ass } M} P$.

Set $T := \{P \in \text{Supp } H_I^{t-1}(M) \cup \text{Supp } H_I^t(M) \mid \dim \frac{R}{P} = 1\}$. Hence $T \subseteq \text{Assh}_R \frac{M}{IM}$, and so T is a finite set. Let $T = \{P_1, \dots, P_n\}$. Then for each $i \geq 0$, $\text{Supp } H_{IR_{P_k}}^i(M_{P_k}) \subseteq \{P_k R_{P_k}\}$, where $k = 1, 2, \dots, n$. By [1], for each $t - 1 \leq k \leq t$, $H_{IR_{P_k}}^i(M_{P_k})$ is R_{P_k} -Artinian and $(x_1, \dots, x_t)R_{P_k}$ -cofinite. Also

$$V((x_1, \dots, x_t)R_{P_k}) \cap \text{Att}_{R_{P_k}} H_{IR_{P_k}}^i(M_{P_k}) \subseteq V(P_k R_{P_k}).$$

Set

$$U := \cup_{i=t-1}^t \cup_{k=1}^n \{q \in \text{Spec}(R) \mid qR_{P_k} \in \text{Att}_{R_{P_k}}(H_{IR_{P_k}}^i(M_{P_k}))\}.$$

Therefore $U \cap V(x_1, \dots, x_t) \subseteq T$. Since $(x_1, \dots, x_t) \not\subseteq (\cup_{q \in U \setminus V(I)} q) \cup (\cup_{P \in \text{Ass } M} P)$, it follows that there exists an element $z_1 \in (x_1, \dots, x_t)$ such that $x_1 + z_1 \notin (\cup_{q \in U \setminus V(I)} q) \cup (\cup_{P \in \text{Ass } M} P)$.

Assume that $y_1 = x_1 + z_1$, then $(x_1, \dots, x_t) = (y_1, x_2, \dots, x_t)$ and $y_1 \in I$ is an I -filter regular sequence.

Now if $(x_1, \dots, x_t) = (y_1, x_2, \dots, x_t) \subseteq \cup_{P \in (\text{Ass } \frac{R}{y_1 R}) \setminus V(I)} P$, then there exists $P \in (\text{Ass } \frac{R}{y_1 R}) \setminus V(I)$ such that $(x_1, \dots, x_t) \subseteq P$.

Since $I \not\subseteq P$, it follows that $\frac{x_1}{1}, \dots, \frac{x_t}{1} \in PR_P$ is a R_P -regular sequence and so $\text{grade}((\frac{x_1}{1}, \dots, \frac{x_t}{1}, R_P)) = t$. On the other hand $PR_P \in \text{Ass } \frac{R}{y_1 R}$ and $(y_1, x_2, \dots, x_t)R_P \subseteq PR_P$.

Then $\text{grade}((y_1, x_2, \dots, x_t)R_P, R_P) = 1$ if $t \geq 2$, and so $(y_1, x_2, \dots, x_t) \not\subseteq \cup_{P \in \text{Ass } \frac{R}{y_1 R}} P$. Hence there exists an element $z_2 \in (y_1, x_2, \dots, x_t)$ such that $x_2 + z_2 \notin \cup_{P \in \text{Ass } \frac{R}{y_1 R}} P$. Again, we put $y_2 = x_2 + z_2$, then $(y_1, x_2, \dots, x_t) = (y_1, y_2, x_3, \dots, x_t)$. By the similar argument in the above, we see that there exist elements $y_1, \dots, y_t \in I$ such that $(x_1, \dots, x_t) = (y_1, \dots, y_t)$ and y_1, \dots, y_t is an I -filter regular sequence for M .

The exact sequence

$$0 \longrightarrow M \xrightarrow{y_1} M \longrightarrow \frac{M}{y_1 M} \longrightarrow 0$$

induces a short exact sequence of local cohomology modules

$$0 \longrightarrow \frac{H_I^{t-1}(M)}{y_1 H_I^{t-1}(M)} \longrightarrow H_I^{t-1}(\frac{M}{y_1 M}) \longrightarrow 0 :_{H_I^t(M)} y_1 \longrightarrow 0$$

By a similar proof in [1], we see that $\frac{H_I^{t-1}(M)}{y_1 H_I^{t-1}(M)}$ is a minimax R -module.

Now, by induction hypothesis and since y_2, \dots, y_t is an I -filter regular sequence for $\frac{M}{y_1 M}$, we conclude that the R -module $H_I^{t-1}(\frac{M}{y_1 M})$ is (y_2, \dots, y_t) -cominimax. Also, we note that $(y_2, \dots, y_t) \subseteq (y_1, \dots, y_t)$ and also $\text{Supp } H_I^{t-1}(\frac{M}{y_1 M}) \subseteq V(y_1, \dots, y_t)$. Therefore $H_I^{t-1}(\frac{M}{y_1 M})$ is (y_1, \dots, y_t) -cominimax. Consequently by the above exact sequence $0 :_{H_I^t(M)} y_1$ is also (y_1, \dots, y_t) -cominimax. On the other hand by argument in [1, Theorem 2.6], the R -module $\frac{H_I^t(M)}{y_1 H_I^t(M)}$ is minimax and hence is (y_1, \dots, y_t) -cominimax.

Finally, $y_1 \in (y_1, \dots, y_t) = (x_1, \dots, x_t)$ and the R -modules $0 :_{H_I^t(M)} y_1$ and $\frac{H_I^t(M)}{y_1 H_I^t(M)}$ are both (x_1, \dots, x_t) -cominimax. Thus by lemma 2.9, the R -module $H_I^t(M)$ is also (x_1, \dots, x_t) -cominimax. \square

Acknowledgments

The author is deeply grateful to the referee for a very careful reading of the manuscript and many valuable suggestions.

REFERENCES

1. K. Bahmanpour and R. Naghipour, Cofiniteness of local cohomology modules for ideals of small dimension, *J. Algebra*, **321** (2009), 1997–2011.
2. J. Azami and N. Pourreeza, Attached primes and Artinian modules, *submitted*.
3. M. P. Brodmann and R. Y. Sharp, *Local cohomology; an algebraic introduction with geometric applications*, Cambridge University Press, Cambridge, 1998.
4. D. Delfino and T. Marley, Cofinite modules and local cohomology, *J. Pure Appl. Algebra*, **121** (1997), 45–52.
5. A. Grothendieck, *Local cohomology*, Notes by R. Hartshorne, Lecture Notes in Math., 862, Springer, New York, 1966.
6. R. Hartshorne, Affine duality and cofiniteness, *Invent. Math.*, **9** (1970), 145–164.
7. M. T. Dibaei and S. Yassemi, Top local cohomology modules, *Algebra Colloq.*, **14** (2007), 209–214.
8. K. Khashyarmanesh and S. Salarian, Filter regular sequence and the finiteness of local cohomology modules, *Comm. Algebra*, **26** (1998), 2483–2490.
9. K. Khashyarmanesh, On the finiteness properties of extension and torsion functors of local cohomology modules, *Proc. Amer. Math. Soc.*, **135** (2007), 1319–1327.

10. T. Marley, The associated primes of local cohomology modules over rings of small dimension, *Manuscripta Math.*, **104** (2001), 519–525.
11. T. Marley and J. C. Vassilev, Cofiniteness and associated primes of local cohomology modules, *J. Algebra*, **256** (2002), 180–193.
12. H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, UK, 1986.
13. M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*. Addison-Wesley, 1969.
14. A. Mehrvarz, R. Naghipour and K. Bahmanpour, Arithmetic rank, cohomological dimension and filter regular sequence, *J. Algebra Appl.*, **8**(6) (2009), 855–862.
15. L. Melkersson, Modules cofinite with respect to an ideal, *J. Algebra*, **285** (2005), 649–668.
16. L. Melkersson, Properties of cofinite modules and application to local cohomology, *Math. Proc. Cambridge Philos. Soc.*, **125** (1999), 417–423.
17. P. Schenzel, Proregular sequences, local cohomology, and completion, *Math. Scand.*, **92** (2003), 161–180.
18. P. Schenzel, N. V. Trung, N. T. Cuong, Verallgemeinerte Cohen-Macaulay-Moduln, *Math. Nachr.*, **85** (1978), 57–73.
19. K. I. Yoshida, Cofiniteness of local cohomology modules for ideals of dimension one, *Nagoya Math. J.*, **147** (1997), 179–191.

Jafar A'zami

Department of Mathematics, University of Mohaghegh Ardabili, Ardabil, Iran.

Email: jafar.azamigmail.com, azami@uma.ac.ir

Journal of Algebraic Systems

FILTER REGULAR SEQUENCES AND LOCAL COHOMOLOGY MODULES

J. A'ZAMI

رشته‌های صافی منظم و مدول‌های کوهومولوژی موضعی

جعفر اعظمی

دانشکده علوم، دانشگاه محقق اردبیلی، اردبیل، ایران

فرض کنید R یک حلقه جابجایی و نوتری با عنصر همانی ناصفر باشد. در این مقاله برخی روابط بین رشته‌های صافی منظم، رشته‌های منظم و دستگاه پارامتری را روی R -مدول‌ها بررسی می‌کنیم. همچنین نتایج جدیدی را در ارتباط با هم‌متناهی بودن و هم‌می‌نیماکس بودن مدول‌های کوهومولوژی موضعی به دست می‌آوریم.

کلمات کلیدی: رشته‌های صافی منظم، رشته‌های منظم، دستگاه پارامتری، مدول کوهومولوژی موضعی.