Journal of Algebraic Systems
Vol. 7, No. 2, (2020), pp 301-314

SERRE SUBCATEGORY, LOCAL HOMOLOGY AND LOCAL COHOMOLOGY

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ABSTRACT. This paper deals with local homology modules and local cohomology modules contained in a Serre subcategory of the category of R-modules. For an ideal $\mathfrak{a}$ of R, we define the concept of the condition $C^{\mathfrak{a}}$ on a Serre category which is dual to the condition $C_{\mathfrak{a}}$ of Melkersson. As a main result we show that the local homology module $H_i^{\mathfrak{a}}(M)$ of a minimax R-module M of any Serre category $\mathcal{S}$ satisfying the condition $C^{\mathfrak{a}}$ belongs to $\mathcal{S}$. Also, if $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$, then the local cohomology module $H_{\mathfrak{a}}^i(M) \in \mathcal{S}$ for all $i \geq 0$.

1. Introduction

Throughout this paper, let R denote a commutative Noetherian ring (with identity), $\mathfrak{a}$ an ideal of R and M an R-module. Cuong and Nam in [4] defined the *i-th local homology module* $H_i^{\mathfrak{a}}(M)$ of an R-module M with respect to the ideal $\mathfrak{a}$ by

$$H_i^{\mathfrak{a}}(M) = \underleftarrow{lim}\mathrm{Tor}_{\mathrm{i}}^{\mathrm{R}}(R/\mathfrak{a}^n, M).$$

This definition slightly differs from the definition of Greenlees and May [7]. However, both difinitions are the same for the Artinian modules. It should be noticed that this definition of local homology modules is, in some sense, dual to the definition of local cohomology modules of Grothendieck [8]. For each $i \geq 0$, the *i-th local cohomology module*

MSC(2010): Primary: 13D45, 13M10; Secondary: 13E15.
Keywords: Local homology, local cohomology, Serre category, $C^{\mathfrak{a}}$ condition.
Received: 2 September 2018, Accepted: 9 July 2019.
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$H^i_{\mathfrak{a}}(M)$ with respect to the ideal $\mathfrak{a}$ is defined by

$$H^i_{\mathfrak{a}}(M) = \varinjlim \mathrm{Ext}^{\mathrm{i}}_{\mathrm{R}}(R/\mathfrak{a}^n, M).$$

The reader can refer to [4], [8] for some basic properties of local homology and local cohomology modules.

Vanishing, finiteness and artinianness of local cohomology and local homology modules are the main problems in commutative algebra. Also, these problems consist the core of almost all researches in the theory of local homology and local cohomology modules. It is well known that Noetherian and Artinian R-modules are Serre subcategory of the category of R-modules. Instead of the above problems, one can ask the following general question which is a new attitude to the above questions.

Question Under which conditions, local cohomology and local homology modules belong to a Serre category?

Recall that a subcategory $\mathcal{S}$ of the category of R-modules is called Serre subcategory if it is closed under taking submodules, quotients and extensions. This paper is to provide some conditions to answer the above question. The rest of the paper is organised as follows:

In Section 2, we would like to investigate the membership of the local homology modules $H^{\mathfrak{a}}_i(M)$ in $\mathcal{S}$ with the condition $\mathfrak{a} \subseteq \sqrt{0 : H^{\mathfrak{a}}_i(M)}$ for some integers i, where M is an Artinian R-module (Theorems 2.4 and 2.5). The Theorem 2.7 is one of the main results of this section which shows that $H^{\mathfrak{a}}_s(M)/\mathfrak{a}H^{\mathfrak{a}}_s(M) \in \mathcal{S}$ for Artinian R-module M when $H^{\mathfrak{a}}_i(M) \in \mathcal{S}$ for all $i < s$.

In Section 3, we define the condition $C^{\mathfrak{a}}$ on the Serre subcategory of the category of R-modules $\mathcal{S}$ as follows: If $\frac{M}{\mathfrak{a}M} \in \mathcal{S}$ and M is $\mathfrak{a}$-separated, then $M \in \mathcal{S}$. This definition is dual sense to the Melkersson condition $C_{\mathfrak{a}}$ in [1]. First we provide some results for local homology modules, when $(R, \mathfrak{m})$ is a local ring and $\mathcal{S}$ satisfies the condition $C^{\mathfrak{m}}$ (Lemma 3.8, Corollary 3.9 and Corrolary 3.10). Next, we show that if M is an Artinian R-module with $\mathrm{Ndim} M = d$ and $\mathcal{S}$ satisfies the condition $C^{\mathfrak{a}}$, then $H^{\mathfrak{a}}_d(M) \in \mathcal{S}$. Finally, as a main theorem of this section we show that if $\mathcal{S}$ is a Serre subcategory of the category of R-modules satisfying the condition $C^{\mathfrak{a}}$ and M is a minimax R-module of the category $\mathcal{S}$, then $H^{\mathfrak{a}}_i(M) \in \mathcal{S}$ for all $i \geq 0$ (Theorem 3.12).

In sections 4 and 5 we prove some results for local cohomology modules as a dual of sections 2 and 3. Specially in section 4, as a main theorem, we show that if M is a finitely generated R-module and $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$ for all $i > t$, then $H^t_{\mathfrak{a}}(M)/\mathfrak{a}H^t_{\mathfrak{a}}(M) \in \mathcal{S}$. In section 5, we assume that $\mathcal{S}$ is a Serre subcategory of the category of R-modules satisfying the condition $C_{\mathfrak{a}}$ and prove some results for local cohomology modules. As a main theorem of this section we show that if M is a minimax R-module of $\mathcal{S}$, then $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$ for all $i \geq 0$ (Theorem 5.5).

2. Local Homology and Serre Subcategory

Let $\mathfrak{a}$ be an ideal of R and M an R-module. It is well-known that the *i-th local homology module* $H^{\mathfrak{a}}_i(M)$ of M with respect to the ideal $\mathfrak{a}$ is defined in [4] by

$$H^{\mathfrak{a}}_i(M) = \varprojlim \mathrm{Tor}^{\mathrm{R}}_{\mathrm{i}}(R/\mathfrak{a}^n, M).$$

It is clear that $H^{\mathfrak{a}}_0(M) \cong \Lambda_{\mathfrak{a}}(M)$ where $\Lambda_{\mathfrak{a}}(M) = \varprojlim M/\mathfrak{a}^n M$ is the $\mathfrak{a}$-adic completion of M. Moreover, if M is a finitely generated R-module, then $H^{\mathfrak{a}}_i(M) = 0$ for all $i > 0$ according to [4, Remark 3.2].

Remark 2.1. Let M be an Artinian R-module. Then there exists a positive integer n such that $\mathfrak{a}^t M = \mathfrak{a}^n M$ for all $t \geqslant n$ and so $H^{\mathfrak{a}}_0(M) \cong \Lambda_{\mathfrak{a}}(M) \cong M/\mathfrak{a}^n M$. Additionally, we have the short exact sequence of Artinian R-modules

$$0 \to \bigcap_{j>0} \mathfrak{a}^j M \to M \to \Lambda_{\mathfrak{a}}(M) \to 0.$$

We put $K := \mathfrak{a}^n M$ and assume that $\mathfrak{a}K = K$. There is an element x in $\mathfrak{a}$ such that $xK = K$. Thus, we have the short exact sequence of Artinian modules as follows:

$$0 \to 0 :_K x \to K \xrightarrow{x} K \to 0.$$

Lemma 2.2. [4, Corollary 4.5] *Let M be an Artinian R-module and $K := \bigcap_{j>0} \mathfrak{a}^j M$. Then*

$$H^{\mathfrak{a}}_i(K) \cong \begin{cases} 0 & i = 0, \\ H^{\mathfrak{a}}_i(M) & i \geqslant 1. \end{cases}$$

Now, we recall the definition of the notation $\mathrm{Ndim_R M}$ by the approach of Kirby [10].

Definition 2.3. The noetherian dimension of M denoted by $\mathrm{Ndim_R} M$ is defined inductively as follows: when $M = 0$, put $\mathrm{Ndim} M = -1$. Then by induction, for any integer $d \geq 0$, we define $\mathrm{Ndim_R} M = d$ if $\mathrm{Ndim_R} M < d$ is false, and for every ascending sequence $M_0 \subseteq M_1 \subseteq \cdots$ of submodules of M, there exists a positive integer n_0 such that $\mathrm{Ndim}(M_{n+1}/M_n) < d$ for all $n \geq n_0$. Thus M is non-zero and finitely generated if and only if $\mathrm{Ndim_R} M = 0$. Also, if M is an Artinian module, then $\mathrm{Ndim_R} M < \infty$. If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of R-modules, then $\mathrm{Ndim}_{\mathrm{R}} M = \max\{\mathrm{Ndim}_{\mathrm{R}} M', \mathrm{Ndim}_{\mathrm{R}} M''\}$. For more details see [10, 15].

Theorem 2.4. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules, M be an Artinian R-module and s be a non-negative integer. If $\mathfrak{a} \subseteq \sqrt{0 : H_i^{\mathfrak{a}}(M)}$ for all $i > s$, then $H_i^{\mathfrak{a}}(M) \in \mathcal{S}$ for all $i > s$.*

Proof. We use induction on $d := \mathrm{Ndim} M$. For $d = 0$, $H_i^{\mathfrak{a}}(M) = 0$ for all $i > 0$ according to [4, Proposition 4.8]. Now, let $d > 0$ and assume that the claim holds for all R-modules with Noetherian dimension less than d. By Lemma 2.2 we have $H_i^{\mathfrak{a}}(M) \cong H_i^{\mathfrak{a}}(K)$ for all $i > 0$. Thus, the proof will be completed if we show that $H_i^{\mathfrak{a}}(K) \in \mathcal{S}$ for all $i > s$. From Remark 2.1 there is an element $x \in \mathfrak{a}$ such that $xK = K$. Also, there exists a positive integer r such that $x^r H_i^{\mathfrak{a}}(K) = 0$ for all $i > s$ by assumption. The short exact sequence

$$0 \to 0 :_K x^r \to K \xrightarrow{x^r} K \to 0$$

induces a short exact sequence of local homology modules

$$0 \to H_{i+1}^{\mathfrak{a}}(K) \to H_i^{\mathfrak{a}}(0 :_K x^r) \to H_i^{\mathfrak{a}}(K) \to 0,$$

for all $i > s$. It follows $\mathfrak{a} \subseteq \sqrt{0 : H_i^{\mathfrak{a}}(0 :_K x^r)}$ for all $i > s$. It should be noted by [5, Lemma 4.7] that $\mathrm{Ndim}(0 :_{\mathrm{K}} \mathrm{x}^{\mathrm{r}}) \leqslant \mathrm{d} - 1$. By the inductive hypothesis $H_i^{\mathfrak{a}}(0 :_K x^r) \in \mathcal{S}$ for all $i > s$. Therefore $H_i^{\mathfrak{a}}(K) \in \mathcal{S}$ for all $i > s$, as required. □

Theorem 2.5. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules, M be an Artinian R-module such that $M \in \mathcal{S}$ and s be a positive integer. If $\mathfrak{a} \subseteq \sqrt{0 : H_i^{\mathfrak{a}}(M)}$ for all $i < s$, then $H_i^{\mathfrak{a}}(M) \in \mathcal{S}$ for all $i < s$.*

Proof. We use induction on s. If $s = 1$, $H_0^{\mathfrak{a}}(M) \cong \frac{M}{K}$ so $H_0^{\mathfrak{a}}(M) \in \mathcal{S}$. Let $s > 1$. By Lemma 2.2 we have $H_i^{\mathfrak{a}}(M) \cong H_i^{\mathfrak{a}}(K)$ for all $i > 0$. Therefore, the proof will be completed if we show that $H_i^{\mathfrak{a}}(K) \in S$ for all $i < s$. From Remark 2.1 there is an element $x \in \mathfrak{a}$ such that $xK = K$. Moreover, there is a positive integer r such that $x^r H_i^{\mathfrak{a}}(K) = 0$ for all $i < s$, since $\mathfrak{a} \subseteq \sqrt{0 : H_i^{\mathfrak{a}}(K)}$ for all $i < s$. Now the short exact sequence

$$0 \to 0 :_K x^r \to K \xrightarrow{x^r} K \to 0$$

induces a short exact sequence of local homology modules

$$0 \to H_i^{\mathfrak{a}}(K) \to H_{i-1}^{\mathfrak{a}}(0 :_K x^r) \to H_{i-1}^{\mathfrak{a}}(K) \to 0,$$

for all $i < s$. It follows $\mathfrak{a} \subseteq \sqrt{0 : H_{i-1}^{\mathfrak{a}}(0 :_K x^r)}$ for all $i < s$ and by the inductive hypothesis $H_{i-1}^{\mathfrak{a}}(0 :_K x^r) \in \mathcal{S}$ for all $i < s$. Therefore, $H_i^{\mathfrak{a}}(K) \in \mathcal{S}$ for all $i < s$, as required. □

Corollary 2.6. [4, Proposition 4.7] *Let M be an Artinian R-module and s be a positive integer. Then the following statements are equivalent:*

(i) $\mathfrak{a} \subseteq \sqrt{0 : H_i^{\mathfrak{a}}(M)}$ *for all* $i < s$;

(ii) $H_i^{\mathfrak{a}}(M)$ *is an Artinian R-module for all* $i < s$.

Proof. $(i) \Rightarrow (ii)$ It is a direct result of what we explained in Theorem 2.5.

$(ii) \Rightarrow (i)$ Let $H_i^{\mathfrak{a}}(M)$ is an Artinian R-module for all $i < s$. Since $H_i^{\mathfrak{a}}(M)$ is $\mathfrak{a}$-separated, it follows that $\mathfrak{a}^t H_i^{\mathfrak{a}}(M) = 0$ for some positive integer t according to [4, Proposition 3.3(i)]. Therefore $\mathfrak{a} \subseteq \sqrt{0 : H_i^{\mathfrak{a}}(M)}$ for all $i < s$. ☐

Theorem 2.7. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules, M be an Artinian R-module such that $M \in \mathcal{S}$ and s be a non-negative integer. If $H_i^{\mathfrak{a}}(M) \in \mathcal{S}$ for all $i < s$, then $H_s^{\mathfrak{a}}(M)/\mathfrak{a}H_s^{\mathfrak{a}}(M) \in \mathcal{S}$.*

Proof. We use induction on s. Let $s = 0$. Then the short exact sequence

$$0 \to \bigcap_{t>0} \mathfrak{a}^t M \to M \to \Lambda_{\mathfrak{a}}(M) \to 0$$

induces an exact sequence

$$\bigcap_{t>0} \mathfrak{a}^t M/\mathfrak{a} \bigcap_{t>0} \mathfrak{a}^t M \to M/\mathfrak{a}M \to \Lambda_{\mathfrak{a}}(M)/\mathfrak{a}\Lambda_{\mathfrak{a}}(M) \to 0.$$

From Remark 2.1, we have $\bigcap_{t>0} \mathfrak{a}^t M/\mathfrak{a} \bigcap_{t>0} \mathfrak{a}^t M = 0$, therefore $\Lambda_{\mathfrak{a}}(M)/\mathfrak{a}\Lambda_{\mathfrak{a}}(M) \cong M/\mathfrak{a}M$, hence $\Lambda_{\mathfrak{a}}(M)/\mathfrak{a}\Lambda_{\mathfrak{a}}(M) \in \mathcal{S}$. Let $s > 0$. By Lemma 2.2, $H_i^{\mathfrak{a}}(M) \cong H_i^{\mathfrak{a}}(K)$ for all $i > 0$. Thus, the proof will be completed if we show that $H_s^{\mathfrak{a}}(K)/\mathfrak{a}H_s^{\mathfrak{a}}(K) \in \mathcal{S}$. By Remark 2.1, there is an element $x \in \mathfrak{a}$ such that $xK = K$. Now, the short exact sequence

$$0 \to 0 :_K x \to K \xrightarrow{x} K \to 0$$

induces a long exact sequence of local homology modules

$$\cdots \to H_i^{\mathfrak{a}}(K) \xrightarrow{x} H_i^{\mathfrak{a}}(K) \to H_{i-1}^{\mathfrak{a}}(0 :_K x) \to H_{i-1}^{\mathfrak{a}}(K) \to \cdots .$$

It follows from the hypothesis that $H_i^{\mathfrak{a}}(0 :_K x) \in \mathcal{S}$ for all $i < s - 1$, hence by the inductive hypothesis $H_{s-1}^{\mathfrak{a}}(0 :_K x)/\mathfrak{a}H_{s-1}^{\mathfrak{a}}(0 :_K x) \in \mathcal{S}$. Now consider the exact sequence

$$H_s^{\mathfrak{a}}(K) \xrightarrow{x} H_s^{\mathfrak{a}}(K) \xrightarrow{f} H_{s-1}^{\mathfrak{a}}(0 :_K x) \xrightarrow{g} H_{s-1}^{\mathfrak{a}}(K),$$

which induces the following exact sequences

$$0 \to \operatorname{im} f \to H_{s-1}^{\mathfrak{a}}(0 :_K x) \to \operatorname{im} g \to 0$$

and

$$H_s^{\mathfrak{a}}(K) \xrightarrow{x} H_s^{\mathfrak{a}}(K) \to \operatorname{im} f \to 0.$$

Therefore, the following two exact sequences can be obtained:

$$\operatorname{Tor}_1^{\mathrm{R}}(R/\mathfrak{a}, \operatorname{im} g) \to \operatorname{im} f/\mathfrak{a}(\operatorname{im} f) \to H_{s-1}^{\mathfrak{a}}(0 :_K x)/\mathfrak{a}H_{s-1}^{\mathfrak{a}}(0 :_K x)$$

and

$$H_s^{\mathfrak{a}}(K)/\mathfrak{a}H_s^{\mathfrak{a}}(K) \xrightarrow{x} H_s^{\mathfrak{a}}(K)/\mathfrak{a}H_s^{\mathfrak{a}}(K) \to \operatorname{im} f/\mathfrak{a}(\operatorname{im} f) \to 0.$$

As $x \in \mathfrak{a}$, one can get $H_s^{\mathfrak{a}}(K)/\mathfrak{a}H_s^{\mathfrak{a}}(K) \cong \operatorname{im} f/\mathfrak{a}(\operatorname{im} f)$. By [2, Lemma 2.1], $\operatorname{Tor}_1^{\mathrm{R}}(R/\mathfrak{a}, \operatorname{im} g) \in \mathcal{S}$. Thus $\operatorname{im} f/\mathfrak{a}(\operatorname{im} f) \in \mathcal{S}$. □

3. Local homology and Serre category with the condition $C^{\mathfrak{a}}$

In this section we introduce the concept of $C^{\mathfrak{a}}$ condition on a Serre category of R-modules which is dual to Melkersson condition $C_{\mathfrak{a}}$. Also, we will prove some results on local homology moduls under this condition. This section starts by reminding the concepts of linearly compact modules, co-associated prime and co-support.

In [11], Macdonald defined the linearly compact module M as follows:

Definition 3.1. A Hausdorf linearly topologized R-module M is said to be linearly compact if $\mathcal{F}$ is a family of closed cosets (i.e., cosets of closed submodules) in M which has the finite intersection property, then the cosets in $\mathcal{F}$ have a non-empty intersection.

It is clear that Artinian R-modules are linearly compact with the discrete topology. Moreover, if $(R, \mathfrak{m})$ is a complete ring, then the finitely generated R-modules are also linearly compact.

Definition 3.2. A prime ideal $\mathfrak{p}$ of R is called the co-associated prime of module M if there exists a cocyclic homomorphic image L of M such that $\mathfrak{p} = \operatorname{Ann}_{\mathrm{R}}(L)$.

Note that a module is cocyclic if it is a submodule of $E(R/\mathfrak{m})$ for some maximal ideal $\mathfrak{m} \in R$. The set of co-associated primes of M is denoted by $\operatorname{Coass}(M)$ (see [16]). Now, let T be a multiplicatively closed subset of R and M an R-module. In [12], Melkersson and Schenzel called the module ${}_TM = Hom_R(R_T, M)$ the co-localization of M with respect to T and $\operatorname{Cos}_{\mathrm{R}}(M) = \{\mathfrak{p} \in \mathit{Spec}R|_{\mathfrak{p}}M \neq 0\}$ the co-support of M. After that, Yassemi defined the co-support of an R-module M, denoted by $\operatorname{Cosupp}_{\mathrm{R}}(M)$, to be the set of primes $\mathfrak{p}$ such that there exists a cocyclic homomorphic image L of M with $\operatorname{Ann}(L) \subseteq \mathfrak{p}$ [16]. In general, we have $\operatorname{Coass}(M) \subseteq \operatorname{Cosupp}(M)$ and $\operatorname{Cos}_{\mathrm{R}}(M) \subseteq \operatorname{Cosupp}(M)$. Also Yassemi showed $\operatorname{Cos}_{\mathrm{R}}(M) = \operatorname{Cosupp}(M)$ in case M is an Artinian R-module [16].

Lemma 3.3. [14, Theorem 3.1] *Let $\mathfrak{a}$ be an ideal of R and M be a linearly compact R-module. Then $\bigcap_{n>0} \mathfrak{a}^n M = 0$ if and only if* $\mathrm{Cosuup}(M) \subseteq V(\mathfrak{a})$.

The magnitude of M is defined by

$$\mathrm{mag_R}(M) := sup\{dim R/\mathfrak{p} | \mathfrak{p} \in Coass_R M\}.$$

If $M = 0$, then we put $\mathrm{mag_R} M = -\infty$ (see [17]).

Lemma 3.4. *Let M be a linearly compact R-module. Then the following assertions are true.*
(i) If $\mathrm{Coaas}(M/\mathfrak{a}M)$ *is finite and $\bigcap_{n>0} \mathfrak{a}^n M = 0$, then* $\mathrm{Coass}(M)$ *is finite.*
(ii) If $\mathrm{mag_R}(M/\mathfrak{a}M) \leq s$ *and $\bigcap_{n>0} \mathfrak{a}^n M = 0$, then* $\mathrm{mag_R}(M) \leq s$.

Proof. (i) By [16, Theorem 1.21],

$$\mathrm{Coass}(M/\mathfrak{a}M) = \mathrm{Supp}(R/\mathfrak{a}) \cap \mathrm{Coass} M = V(\mathfrak{a}) \cap \mathrm{Coass} M.$$

Since $\mathrm{Coass} M \subseteq \mathrm{Cosupp} M \subseteq \mathrm{V}(\mathfrak{a})$, thus $\mathrm{Coass}(M/\mathfrak{a}M) = \mathrm{Coass} M$. (ii) follows from (i). □

Definition 3.5. Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules, $\mathfrak{a}$ be a finitely generated ideal of R and M be an R-module. We say that $\mathcal{S}$ satisfies the condition $(C^{\mathfrak{a}})$: if $\bigcap_{n>0} \mathfrak{a}^n M = 0$ and furthermore if $\frac{M}{\mathfrak{a}M} \in \mathcal{S}$, then M belongs to $\mathcal{S}$.

Example 3.6. The following module classes are Serre subcategories stisfying the condition $C^{\mathfrak{a}}$.

(i) The class of Noetherian R-modules satisfies the condition $C^{\mathfrak{a}}$ if R is a complete ring with respect to the $\mathfrak{a}$-adic completion [5, Lemma 5.1].

(ii) Let $\hat{R}$ denote the $\mathfrak{m}$-adic completion of $(R, \mathfrak{m})$. Then the class of Noetherian $\hat{R}$-module with finite co-support satisfies the condition $C^{\mathfrak{a}}$. It is reminded that for any Noetherian $\hat{R}$-module M, $\mathrm{Coass_R} M = \mathrm{Cos_R} M$ [6, Lemma 4.1]. Now, by Lemma 3.4, the result is achieved.

(iii) Let $\hat{R}$ denote the $\mathfrak{m}$-adic completion of $(R, \mathfrak{m})$ and s be a non-negative integer. Then the class of Noetherian $\hat{R}$-modules with co-dimension less than s satisfies on $C^{\mathfrak{a}}$ condition. It is recalled that for an R-module M, the co-dimension of M is defined as the integer $\mathrm{Cdim_R} M = sup\{\dim R/\mathfrak{p} | \mathfrak{p} \in \mathrm{Cos_R}(M)\}$(posibily infinite).

In [13], Nam defined $\mathfrak{a}$-coartinian module M as follows: An R-module M is said to be $\mathfrak{a}$-coartinian if $\mathrm{Cosupp}(M) \subseteq V(\mathfrak{a})$ and $\mathrm{Tor_i^R}(R/\mathfrak{a}, M)$ is an Artinian R-module for each i. This definition is in some sense dual to Hartshorne's concept of $\mathfrak{a}$-cofinite modules [9].

Example 3.7. Let $\mathcal{S}$ be the class of $\mathfrak{a}$-coartinian Noetherian modules on a complete local ring. Then $\mathcal{S}$ satisfies the condition $C^{\mathfrak{a}}$. Let $\bigcap_{n>0}\mathfrak{a}^n M = 0$ and $\frac{M}{\mathfrak{a}M} \in \mathcal{S}$. Then, $M/\mathfrak{a}M$ has finite length and hence M is finitely generated R-module by Example 3.6(i). Also, $\mathrm{Cosupp}M \subseteq V(\mathfrak{a})$ since $\bigcap_{n>0}\mathfrak{a}^n M = 0$. Now, M is $\mathfrak{a}$-coartinian according to [13, Theorem 4.8].

Lemma 3.8. *Let $(R, \mathfrak{m})$ be a local ring and $\mathcal{S}$ be a non-zero Serre subcategory of the category of R-modules satisfying the condition $C^{\mathfrak{m}}$. If M is a Noetherian R-module, then $M \in \mathcal{S}$.*

Proof. Since M is a Noetherian R-module, $\bigcap \mathfrak{m}^n M = 0$. Also $\frac{M}{\mathfrak{m}M}$ has finite length, and so $\frac{M}{\mathfrak{m}M} \in \mathcal{S}$ according to [2, Lemma 2.11]. Thus $M \in \mathcal{S}$, since $\mathcal{S}$ satisfies the condition $C^{\mathfrak{m}}$. □

Corollary 3.9. *Let $(R, \mathfrak{m})$ be a complete local ring, $\mathcal{S}$ be a non-zero Serre subcategory of the category of R-modules satisfying the condition $C^{\mathfrak{m}}$ and M be an Artinian R-module. Then $H_i^{\mathfrak{m}}(M) \in \mathcal{S}$ for all $i \geqslant 0$.*

Proof. By [5, Theorem 5.2] $H_i^{\mathfrak{m}}(M)$ is a Noetherian R-module for all $i \geqslant 0$. Thus, $H_i^{\mathfrak{m}}(M) \in \mathcal{S}$ for all $i \geqslant 0$ by Lemma 3.8. □

Corollary 3.10. *Let $(R, \mathfrak{m})$ be a complete local ring, $\mathcal{S}$ be a non-zero Serre subcategory of the category of R-modules satisfying the condition $C^{\mathfrak{m}}$ and M be an Artinian R-module with $\mathrm{Ndim}M = d$. Then $H_d^{\mathfrak{a}}(M) \in \mathcal{S}$.*

Proof. Using [5, Theorem 5.3] and Lemma 3.8, we have the result. □

Theorem 3.11. *Let $(R, \mathfrak{m})$ be a local ring, $\mathcal{S}$ be a non-zero Serre subcategory of the category of R-modules satisfying the condition $C^{\mathfrak{a}}$ and M be an Artinian R-module with $\mathrm{Ndim}M = d$. Then $H_d^{\mathfrak{a}}(M) \in \mathcal{S}$.*

Proof. We use induction on d. If $d = 0$, then M is a finitely generated R-module and therefore, is an $\mathfrak{a}$-separated R-module. By [5, Theorem 3.8], $H_0^{\mathfrak{a}}(M) \cong \Lambda_{\mathfrak{a}}(M) \cong M$ and therefore $H_0^{\mathfrak{a}}(M)$ has finite length and hence $H_0^{\mathfrak{a}}(M) \in \mathcal{S}$ according to [2, Lemma 2.11]. Let $d > 0$. From Lemma 2.2, we have $H_d^{\mathfrak{a}}(M) \cong H_d^{\mathfrak{a}}(\bigcap_{t>0}\mathfrak{a}^t M)$. If $\mathrm{Ndim}(\bigcap_{\mathrm{t}>0}\mathfrak{a}^{\mathrm{t}} M) < d$, then $H_d^{\mathfrak{a}}(M) = 0$ according to [5, Theorem4.8] and then there is nothing to prove. So, assume that $\mathrm{Ndim}(\bigcap_{\mathrm{t}>0}\mathfrak{a}^{\mathrm{t}} M) = d$, and without loss of generality M can replaced by $\bigcap_{t>0}\mathfrak{a}^t M$. Thus, there is an element $x \in \mathfrak{a}$ such that $xM = M$. The short exact sequence

$$0 \to 0 :_M x \to M \xrightarrow{x} M \to 0$$

induces an exact sequence of local homology modules

$$H_d^{\mathfrak{a}}(M) \xrightarrow{x} H_d^{\mathfrak{a}}(M) \xrightarrow{\delta} H_{d-1}^{\mathfrak{a}}(0 :_M x).$$

Note that according to [5, Lemma 4.7], $\mathrm{Ndim}(0 :_{\mathrm{M}} \mathrm{x}) \leq \mathrm{d}-1$. If $\mathrm{Ndim}(0 :_{\mathrm{M}} \mathrm{x}) < \mathrm{d}-1$, then $H^{\mathfrak{a}}_{d-1}(0 :_M x) = 0$ and hence according to [5, Lemma 3.2(ii)], we have

$$H^{\mathfrak{a}}_d(M) = xH^{\mathfrak{a}}_d(M) = \bigcap_{t>0} x^t H^{\mathfrak{a}}_d(M) = 0.$$

So it can be assumed that $\mathrm{Ndim}(0 :_{\mathrm{M}} \mathrm{x}) = \mathrm{d}-1$. It follows by the inductive hypothesis that $H^{\mathfrak{a}}_{d-1}(0 :_M x) \in \mathcal{S}$. On the other hand, we have $H^{\mathfrak{a}}_d(M)/xH^{\mathfrak{a}}_d(M) \cong \operatorname{im}\delta \subseteq H^{\mathfrak{a}}_{d-1}(0 :_M x)$. Thus $H^{\mathfrak{a}}_d(M)/xH^{\mathfrak{a}}_d(M) \in \mathcal{S}$, and hence $H^{\mathfrak{a}}_d(M)/\mathfrak{a}H^{\mathfrak{a}}_d(M) \in \mathcal{S}$, since $x \in \mathfrak{a}$. Therefore $H^{\mathfrak{a}}_d(M) \in \mathcal{S}$, since $\mathcal{S}$ satisfies the condition $C^{\mathfrak{a}}$. □

Theorem 3.12. *Let $(R, \mathfrak{m})$ be a local ring, $\mathcal{S}$ be a Serre subcategory of the category of R-modules satisfying the condition $C^{\mathfrak{a}}$ and $M \in \mathcal{S}$ be a minimax R-module. Then $H^{\mathfrak{a}}_i(M) \in \mathcal{S}$ for all $i \geqslant 0$.*

Proof. The proof will be divided into two steps:
Step 1. We assume that M is an Artinian R-module and prove the theorem using induction on i. If $i = 0$, then $H^{\mathfrak{a}}_0(M) \cong \Lambda_{\mathfrak{a}}(M) \cong M/\bigcap_{t>0} \mathfrak{a}^t M$ according to [14, Remark 2.2]. Thus $H^{\mathfrak{a}}_0(M) \in \mathcal{S}$, since $M \in \mathcal{S}$. Let $i > 0$. By Remark 2.1, the short exact sequence of Artinian modules is obtained

$$0 \to 0 :_K x \to K \xrightarrow{x} K \to 0,$$

which induces the following exact sequence of local homology modules

$$H^{\mathfrak{a}}_i(K) \xrightarrow{x} H^{\mathfrak{a}}_i(K) \xrightarrow{\delta} H^{\mathfrak{a}}_{i-1}(0 :_K x).$$

We have $H^{\mathfrak{a}}_i(K)/xH^{\mathfrak{a}}_i(K) \cong \operatorname{im}\delta \subseteq H^{\mathfrak{a}}_{i-1}(0 :_K x)$. By the inductive hypothesis, $H^{\mathfrak{a}}_{i-1}(0 :_K x) \in \mathcal{S}$. Therefore $H^{\mathfrak{a}}_i(K)/xH^{\mathfrak{a}}_i(K) \in \mathcal{S}$ and hence $H^{\mathfrak{a}}_i(K)/\mathfrak{a}H^{\mathfrak{a}}_i(K) \in \mathcal{S}$. Now $H^{\mathfrak{a}}_i(K) \in \mathcal{S}$, since $\mathcal{S}$ satisfies the condition $C^{\mathfrak{a}}$.
Step 2. We assume that M is a minimax R-module. Thus there is a short exact sequence

$$0 \to N \to M \to A \to 0,$$

where N is finitely generated and A is Artinian. Now, the long exact sequence of local homology modules is obtained

$$\cdots \to H^{\mathfrak{a}}_i(N) \to H^{\mathfrak{a}}_i(M) \to H^{\mathfrak{a}}_i(A) \to \cdots .$$

As it is proved in step 1, $H^{\mathfrak{a}}_i(A) \in \mathcal{S}$ for all $i \geq 0$. Also $H^{\mathfrak{a}}_i(N) = 0$ for $i > 0$ by [5, Lemma 3.2] and $H^{\mathfrak{a}}_0(N) \cong \Lambda_{\mathfrak{a}}(N) \cong N \in \mathcal{S}$. Thus $H^{\mathfrak{a}}_i(M) \in \mathcal{S}$ for all $i \geq 0$. □

4. Local cohomology and Serre category

In this section, we prove some results for local cohomology modules as a dual of section 2. Let $\mathfrak{a}$ be an ideal of R and M be an R-module. It is well-known that the *i-th local cohomology module* $H^i_{\mathfrak{a}}(M)$ of M with respect to the ideal $\mathfrak{a}$ can be defined by

$$H^i_{\mathfrak{a}}(M) = \varinjlim \mathrm{Ext}^{\mathrm{i}}_{\mathrm{R}}(R/\mathfrak{a}^n, M).$$

When $i = 0$, we have $H^0_{\mathfrak{a}}(M) \cong \cup_{n>0}(0 :_M \mathfrak{a}^n) = \Gamma_{\mathfrak{a}}(M)$.

Theorem 4.1. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules, M a finitely generated R-module and t a non-negative integer. If $\mathfrak{a} \subseteq \sqrt{0 : H^i_{\mathfrak{a}}(M)}$ for all $i > t$, then $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$ for all $i > t$.*

Proof. We use induction on $d = \dim M$. For $d = 0$, $H^i_{\mathfrak{a}}(M) = 0$ for all $i > 0$ according to [3, Theorem 6.1.2]. Now, let $d > 0$ and assume that the claim holds for all R-modules of dimension less than d. Without loss of generality, we can assume that M is an $\mathfrak{a}$-torsion free R-module. Then by assumption, there is an M-regular element $x \in \mathfrak{a}$ and a positive integer k such that $x^k H^i_{\mathfrak{a}}(M) = 0$. The short exact sequence

$$0 \to M \xrightarrow{x^k} M \to M/x^k M \to 0$$

gives rise to a long exact sequence

$$\cdots \to H^i_{\mathfrak{a}}(M) \xrightarrow{x^k} H^i_{\mathfrak{a}}(M) \to H^i_{\mathfrak{a}}(M/x^k M) \to H^{i+1}_{\mathfrak{a}}(M) \to \cdots,$$

for all $i \in \mathbb{N}_0$. From this exact sequence we have $\mathfrak{a} \subseteq \sqrt{0 : H^i_{\mathfrak{a}}(M/x^k M)}$ for all $i > t$. Since $\dim M/\mathrm{x}^{\mathrm{k}} M = \mathrm{d} - 1$, it follows from the inductive hypothesis that $H^i_{\mathfrak{a}}(M/x^k M) \in \mathcal{S}$ for all $i > t$. Since $x^k H^i_{\mathfrak{a}}(M) = 0$ for all $i > t$, the above long exact sequence implies that

$$0 \to H^i_{\mathfrak{a}}(M) \to H^i_{\mathfrak{a}}(M/x^k M) \to H^{i+1}_{\mathfrak{a}}(M) \to 0$$

is an exact sequence for all $i > t$. Hence $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$ for all $i > t$. □

Theorem 4.2. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules, M be a finitely generated R-module such that $M \in \mathcal{S}$ and t be a positive integer. If $\mathfrak{a} \subseteq \sqrt{0 : H^i_{\mathfrak{a}}(M)}$ for all $i < t$, then $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$ for all $i < t$.*

Proof. We use induction on t. When $t = 1$, we have $H^0_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{a}}(M) \in \mathcal{S}$. Now, let $t > 1$ and the result has been proved for smaller value of t. Without loss of generality, we can assume that M is an $\mathfrak{a}$-torsion free R-module. Then by assumption, there is an M-regular element $x \in \mathfrak{a}$

and a positive integer k such that $x^k H^i_{\mathfrak{a}}(M) = 0$ for all $i < t$. The short exact sequence

$$0 \to M \xrightarrow{x^k} M \to M/x^k M \to 0$$

gives us the long exact sequence

$$\cdots \to H^i_{\mathfrak{a}}(M) \xrightarrow{x^k} H^i_{\mathfrak{a}}(M) \to H^i_{\mathfrak{a}}(M/x^k M) \to H^{i+1}_{\mathfrak{a}}(M) \to \cdots .$$

Thus, we have an induced short exact sequence

$$0 \to H^{i-1}_{\mathfrak{a}}(M) \to H^{i-1}_{\mathfrak{a}}(M/x^k M) \to H^i_{\mathfrak{a}}(M) \to 0,$$

for all $i < t$. It follows that $\mathfrak{a} \subseteq \sqrt{0 : H^i_{\mathfrak{a}}(M/x^k M)}$ for all $i < t-1$ and by the inductive hypothesis $H^i_{\mathfrak{a}}(M/x^k M) \in \mathcal{S}$ for all $i < t-1$. Hence $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$ for all $i < t$. □

Corollary 4.3. [3, Proposition 9.1.2] *Let M be a finitely generated R-module and t be a positive integer. Then the following statements are equivalent:*

(i) $\mathfrak{a} \subseteq \sqrt{0 : H^i_{\mathfrak{a}}(M)}$ *for all $i < t$;*
(ii) $H^i_{\mathfrak{a}}(M)$ *is finitely generated for all $i < t$.*

Proof. $(i) \Rightarrow (ii)$ It is a direct result of what we explained in Theorem 4.2.
$(ii) \Rightarrow (i)$ $H^i_{\mathfrak{a}}(M)$ is finitely generated for all $i < t$, then $\mathfrak{a}^u H^i_{\mathfrak{a}}(M) = 0$ for some positive integer u, and so $\mathfrak{a} \subseteq \sqrt{0 : H^i_{\mathfrak{a}}(M)}$. □

Theorem 4.4. *Let $\mathcal{S}$ be a non-zero Serre subcategory of the category of R-modules, M be a finitely generated R-module and t be a non-negative integer. If $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$ for all $i > t$, then $H^t_{\mathfrak{a}}(M)/\mathfrak{a}H^t_{\mathfrak{a}}(M) \in \mathcal{S}$.*

Proof. We use induction on $d = \dim M$. If $d = 0$, then $H^i_{\mathfrak{a}}(M) = 0$ for all $i > 0$ according to [3, Theorem 6.1.2]. It is clear that $H^0_{\mathfrak{a}}(M)/\mathfrak{a}H^0_{\mathfrak{a}}(M)$ has finite length and thus $H^0_{\mathfrak{a}}(M)/\mathfrak{a}H^0_{\mathfrak{a}}(M) \in \mathcal{S}$ by [2, Lemma 2.11]. Now, let $d > 0$ and assume that the claim holds for all R-modules of dimension smaller than d. Without loss of generality, we can assume that M is an $\mathfrak{a}$-torsion free R-module. Then there is an M-regular element $x \in \mathfrak{a}$. The short exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

results the long exact sequence

$$\cdots \to H^i_{\mathfrak{a}}(M) \xrightarrow{x} H^i_{\mathfrak{a}}(M) \xrightarrow{f_i} H^i_{\mathfrak{a}}(M/xM) \xrightarrow{g_i} H^{i+1}_{\mathfrak{a}}(M) \to \cdots , \quad (4.1)$$

for all $i \in \mathbb{N}$. Since $\dim M/\mathrm{x}M = \mathrm{d} - 1$, it follows from the inductive hypothesis that $\frac{H^t_{\mathfrak{a}}(M/xM)}{\mathfrak{a}H^t_{\mathfrak{a}}(M/xM)} \in \mathcal{S}$. The exact sequence (4.1) induces the following exact sequence

$$0 \to \operatorname{im}(f_t) \to H^t_{\mathfrak{a}}(M/xM) \to \operatorname{im}(g_t) \to 0.$$

Now the exact sequence

$$\operatorname{Tor}_1^R(R/\mathfrak{a}, \operatorname{im}(g_t)) \to \operatorname{im}(f_t) \otimes_R R/\mathfrak{a} \to \frac{H^t_{\mathfrak{a}}(M/xM)}{\mathfrak{a}H^t_{\mathfrak{a}}(M/xM)} \to \frac{\operatorname{im}(g_t)}{\mathfrak{a}\operatorname{im}(g_t)} \to 0$$

implies that $\operatorname{im}(f_t) \otimes_R \frac{R}{\mathfrak{a}} \in \mathcal{S}$. Also, the long exact sequence (4.1) yields the isomorphism

$$H^t_{\mathfrak{a}}(M)/xH^t_{\mathfrak{a}}(M) \otimes_R \frac{R}{\mathfrak{a}} \cong \operatorname{im}(f_t) \otimes_R \frac{R}{\mathfrak{a}}.$$

As $x \in \mathfrak{a}$, $H^t_{\mathfrak{a}}(M)/xH^t_{\mathfrak{a}}(M) \otimes_R \frac{R}{\mathfrak{a}} \cong H^t_{\mathfrak{a}}(M)/\mathfrak{a}H^t_{\mathfrak{a}}(M)$. Hence we have $H^t_{\mathfrak{a}}(M)/\mathfrak{a}H^t_{\mathfrak{a}}(M) \in \mathcal{S}$. □

5. Local cohomolog and Serre category with $C^{\mathfrak{a}}$ condition

In this section, we remind the concept of $C^{\mathfrak{a}}$ condition on a Serre category induced by Melkersson in [1]. Concerning this condition some results about local cohomology modules similar to the section 3 are proved. It is reminded that a Serre subcategory $\mathcal{S}$ of the category of R-modules satisfies the condition:

($C_{\mathfrak{a}}$) If $M = \Gamma_{\mathfrak{a}}(M)$ and if $0 :_M \mathfrak{a} \in \mathcal{S}$ then $M \in \mathcal{S}$ (see [1]).

Lemma 5.1. *Let $(R, \mathfrak{m})$ be a local ring and $\mathcal{S}$ be a non-zero Serre subcategory of the category of R-modules satisfying the condition $C_{\mathfrak{m}}$. If M is an Artinian R-module, then $M \in \mathcal{S}$.*

Proof. Since M is an Artinian R-module, we put $M = \Gamma_{\mathfrak{m}}(M)$. Also, using $\mathfrak{m}(0 :_M \mathfrak{m}) = 0$, one can get $0 :_M \mathfrak{m}$ has finite length. Thus $0 :_M \mathfrak{m} \in \mathcal{S}$ and hence $M \in \mathcal{S}$, since $\mathcal{S}$ satisfies the condition $C_{\mathfrak{m}}$. □

Corollary 5.2. *Let $(R, \mathfrak{m})$ be a local ring, $\mathcal{S}$ be a non-zero Serre subcategory of the category of R-modules satisfying the condition $C_{\mathfrak{m}}$ and M be a finitely generated R-module. Then $H^i_{\mathfrak{m}}(M) \in \mathcal{S}$ for all $i \geqslant 0$.*

Proof. By [3, Theorem 7.1.3] and Lemma 5.1, we get the resut. □

Corollary 5.3. *Let $(R, \mathfrak{m})$ be a local ring, $\mathcal{S}$ be a non-zero Serre subcategory of the category of R-modules satisfying the condition $C_{\mathfrak{m}}$ and M be a finitely generated R-module of dimension d. Then $H^d_{\mathfrak{a}}(M) \in \mathcal{S}$.*

Proof. Using [3, Theorem 7.1.6] and Lemma 5.1, we have the result. □

Theorem 5.4. *Let $(R, \mathfrak{m})$ be a local ring, $\mathcal{S}$ be a non-zero Serre subcategory of the category of R-modules satisfying the condition $C_{\mathfrak{a}}$ and M be a finitely generated R-module of dimension d. Then $H^d_{\mathfrak{a}}(M) \in \mathcal{S}$.*

Proof. We use induction on d. If $d = 0$, then $\Gamma_{\mathfrak{a}}(M)$ is an Artinian R-module with finite length which results $H^0_{\mathfrak{a}}(M) \in \mathcal{S}$. Let $d > 0$. Then $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))$ for all $i \geq 1$. So, without loss of generality we can assume that M is an $\mathfrak{a}$-torsion-free R-module. Thus, there exists an element $x \in \mathfrak{a}$ which is non-zero divisor on M. The exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

induces an exact sequence

$$H^{d-1}_{\mathfrak{a}}(M/xM) \to H^d_{\mathfrak{a}}(M) \xrightarrow{x} H^d_{\mathfrak{a}}(M)$$

of local cohomology modules. Since $\dim M/xM = d-1$, it follows from the inductive hypothesis that $H^{d-1}_{\mathfrak{a}}(M/xM) \in \mathcal{S}$ and so $(0 :_{H^d_{\mathfrak{a}}(M)} x) \in \mathcal{S}$. Now, since $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$, we have $H^d_{\mathfrak{a}}(M) \in \mathcal{S}$. □

Theorem 5.5. *Let $\mathcal{S}$ be a non-zero Serre subcategory of the category of R-modules satisfying the $C_{\mathfrak{a}}$ condition and $M \in \mathcal{S}$ be a minimax R-module. Then $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$ for all $i \geq 0$.*

Proof. The proof will be divided into the following two steps:

Step 1. We assume that M is a Noetherian R-module and prove the theorem using induction on i. Assuming $i = 0$ results $\Gamma_{\mathfrak{a}}(M) \in \mathcal{S}$ since $M \in \mathcal{S}$. Let $i > 0$. Then $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))$. Moreover, without loss of generality, we can assume that M is an $\mathfrak{a}$-torsion-free R-module. Thus there exists an element $x \in \mathfrak{a}$ which is non-zero divisor on M. The exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

induces an exact sequence of local cohomology modules

$$H^{i-1}_{\mathfrak{a}}(M/xM) \to H^i_{\mathfrak{a}}(M) \xrightarrow{x} H^i_{\mathfrak{a}}(M).$$

It follows from the inductive hypothesis that $H^{i-1}_{\mathfrak{a}}(M/xM) \in \mathcal{S}$ and therefore $0 :_{H^i_{\mathfrak{a}}(M)} x \in \mathcal{S}$. Finally, the condition $C_{\mathfrak{a}}$ results what we would like to prove.

Step 2. We assume that M is a minimax R-module. So, there is a short exact sequence

$$0 \to N \to M \to A \to 0,$$

where N is a finitely generated R-module and A is an Artinian R-module. Now, we have the long exact sequence of local homology modules

$$\cdots \to H^i_{\mathfrak{a}}(N) \to H^i_{\mathfrak{a}}(M) \to H^i_{\mathfrak{a}}(A) \to \cdots .$$

According to the step 1, $H^i_{\mathfrak{a}}(N) \in \mathcal{S}$ for all $i \geq 0$. So, the properties of A result $H^i_{\mathfrak{a}}(A) = 0$ for $i > 0$ and $H^0_{\mathfrak{a}}(A) \in \mathcal{S}$. Now, the proof is completed. □

Acknowledgments

The authors thanks the referee for his/her valuable suggestions.

References

1. M. Aghapournahr and L. Melkersson, Local cohomology and Serre subcategories, *J. Algebra*, **320** (2008), 1275–1287.
2. M. Asgharzadeh and M. Tousi, A unified approach to local cohomology modules using Serre classes, *Canadian Math. Bull.*, **53**(1) (2010), 1–10.
3. M. Brodmann and R.Y. Sharp, *Local cohomology, an algebraic introduction with geometric applications*, Cambridge University Press, Cambridge, 1998.
4. N. T. Cuong and T. T. Nam, The I-adic completion and local homology for Artinian modules, *Math. Proc. Camb. Philos. Soc.*, **131** (2001), 61–72.
5. N. T. Cuong and T. T. Nam, A local homology theory for linearly compact module, *J. Algebra*, **319** (2008), 4712–4737.
6. M. Eghbali, On Artinianness of formal local cohomology, colocalization and coassociated primes, *Math. Scand.*, **113**(1) (2013), 5–19.
7. J. P. C. Greenlees and J. P. May, Derived functors of $\mathfrak{a}$-adic completion and local homology, *J. Algebra*, **149** (1992), 438–453.
8. A. Grothendieck, *Local Cohomology*, Springer-Verlag, New York, 1967.
9. R. Hartshorne, Affine duality and cofiniteness, *Invent. Math.*, **9** (1970), 145 –164.
10. D. Kirby, Dimension and length for Artinian modules,*Quart. J. Math.*, **41**(2) (1990), 419–429.
11. I. G. Macdonald, Duality over complete local rings, *Topology*, **1**(3) (1962), 213–235.
12. L. Melkersson and P. Schenzel, The co-localization of an Artinian module, *Proc. Edinburgh Math. Soc.*, **38** (1995), 121–131.
13. T. T. Nam, Cosupport and coartinian modules, *Algebra Colloq.*, **15**(1) (2008), 83 –96.
14. T. T. Nam, Minimax modules, local homology and local cohomology, *Int. J. Math.*, **26**(12) (2015), 96–111.
15. R. N. Roberts, Krull dimension for Artinian modules over quasi local commutative rings, *Quart. J. Math.*, **26**(3) (1975), 269–273.
16. S. Yassemi, Coassociated primes, *Comm. Algebra*, **23**(4) (1995), 1473–1498.
17. S. Yassemi, Magnitude of modules, *Comm. Algebra*, **23**(11) (1995), 3993–4008.
18. H. Zochinger, Minimax-modulen, *J. Algebra*, **102**(1) (1986), 1–32.

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Journal of Algebraic Systems

SERRE SUBCATEGORY, LOCAL HOMOLOGY AND LOCAL COHOMOLOGY

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زیررسته **سر**، همولوژی موضعی و کوهمولوژی موضعی

سعادت اله فرامرزی و زهرا برق‌سوز
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این مقاله به بررسی مدول‌های همولوژی موضعی و مدول‌های کوهمولوژی موضعی که در زیررسته **سر** از رسته R-مدول‌ها قرار می‌گیرند، می‌پردازد. برای ایده‌آل $\mathfrak{a}$ از R، مفهوم $C^{\mathfrak{a}}$ که در واقع دوگان شرط $C_{\mathfrak{a}}$ **ملکرسون** است را روی یک رسته **سر** تعریف می‌کنیم. به عنوان یک نتیجه اصلی نشان می‌دهیم که برای هر R-مدول مینی‌ماکس M از رسته **سر** $\mathcal{S}$ که در شرط $C^{\mathfrak{a}}$ صدق کند، مدول همولوژی موضعی $H_i^{\mathfrak{a}}(M)$ متعلق به $\mathcal{S}$ است. همچنین اگر $\mathcal{S}$ در شرط $C_{\mathfrak{a}}$ صدق کند، آنگاه برای هر $i \geq ۰$، مدول کوهمولوژی موضعی $H_{\mathfrak{a}}^i(M)$ متعلق به $\mathcal{S}$ است.

کلمات کلیدی: همولوژی موضعی، کوهمولوژی موضعی، زیررسته **سر**، شرط $C^{\mathfrak{a}}$.