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COFINITELY WEAK GENERALIZED δ-SUPPLEMENTED MODULES

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Abstract. We will study modules whose cofinite submodules have weak generalized-δ-supplements. We attempt to investigate some properties of cofinitely weak generalized δ-supplemented modules. We will prove for a module M and a semi-δ-hollow submodule N of M that, M is cofinitely weak generalized δ-supplemented if and only if $\frac{M}{N}$ is cofinitely weak generalized δ-supplemented. Also we show that any M-generated module is cofinitely weak generalized δ-supplemented module, where M is cofinitely weak generalized δ-supplemented. We obtain some other results about this kind of modules.

1. Introduction

Throughout the paper R will be an associative ring with identity and we will consider only left unital R-modules. All definition not given here can be found in [1, 3, 5, 10].

A submodule K of M is called *small* in M (denoted by $K \ll M$) if, $L + K \neq M$ for every proper submodule L of M. The sum of all small submodules of the module M is denoted by $Rad(M)$.

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A submodule N of M is called *cofinite* if $\frac{M}{N}$ is finitely generated.
For two submodules N and K of the module M, N is called a *supplement* of K in M if N is minimal with respect to the property $M = K + N$, equivalently $M = K + N$ and $N \cap K \ll N$. N is called a *weak supplement* of K in M if $N + K = M$ and $N \cap K \ll M$.
The module M is called *supplemented* if every submodule of M has a supplement in M. M is called *weakly supplemented* if every submodule of M has a weak supplement in M.

2. A background of δ-supplemented modules

In this section we introduce the δ-small submodule of a module and then some preliminary lemmas and propositions about the class of δ-supplemented modules are given. We develop to get some suitable results about the class cofinitely weak generalized δ-supplemented modules in the section 3.
The *singular* submodule of a module M (denoted by $Z(M)$) is $Z(M) = \{x \in M \mid Ix = 0$ for some ideal $I \trianglelefteq_e R\}$. A module M is called *singular* (*nonsingular*) if $Z(M) = M$ (resply. $Z(M) = 0$).
δ–*small* submodules were defined as a generalization of small submodules by Zhou in [11]. Let M be a module and $L \leq M$. Then L is called δ–*small* in M (denoted by $L \ll_\delta M$) if, for any submodule N of M with M/N singular, $M = N + L$ implies that $M = N$. The sum of all δ–small submodules of M is denoted by $\delta(M)$.

It is easy to see that every small submodule of a module M is δ–small in M, so $Rad(M) \subseteq \delta(M)$ and, if M is singular, all δ–small submodules of M are small and so $Rad(M) = \delta(M)$ in this case. Also any non-singular semisimple submodule of M is δ–small in M.

Example 2.1. Let R be a semisimple ring and $M = R_R$. Since R is the only essential ideal of R, so there is no nonzero singular factor module of M. Finally we conclude that all submodules of M (even M) are δ-small in M.
In the other hand since M is semisimple, 0 is the only small submodule of M. In this case $Rad(M) = 0$ and $\delta(M) = M$.
Especially let $R = M = \mathbb{Z}_6$. Then two non-trivial submodule of M, $M_1 = \{\bar{0}, \bar{3}\}$ and $M_2 = \{\bar{0}, \bar{2}, \bar{4}\}$ are δ-small in M, but neither M_1 nor M_2 is small in M. Moreover $M \ll_\delta M$. Finally we have $Rad(M) = 0$ but $\delta(M) = M$.

The above example also shows that the inclusion $Rad(M) \subseteq \delta(M)$ can be strict.

Let M be any module and $B \leq A$ be submodules of M. Then B is called a δ–*cosmall* submodule of A in M if $A/B \ll_\delta M/B$. A submodule N of M is called δ–*coclosed* in M if

N has no proper δ–cosmall submodule in M, that is, if $B \leq N$ such that $N/B \ll_\delta M/B$, then $N = B$. A submodule A of M is *weak δ–coclosed* in M if, given $B \leq A$ such that A/B is singular and $A/B \ll_\delta M/B$, then $A = B$. For a submodule N of M, $A \leq N$ is called a *δ–coclosure* of N in M if A is δ–coclosed in M and $N/A \ll_\delta M/A$ and A is called a *weak δ–coclosure* of N in M if A is weak δ–coclosed in M and $N/A \ll_\delta M/A$.(for more information see [6]).
Let K, N be submodules of module M. Then N is called a *δ–supplement* of K in M if $M = N + K$ and $N \cap K \ll_\delta N$. N is called a weak *δ–supplement* of K in M if $M = N + K$ and $N \cap K \ll_\delta M$. The module M is called *δ–supplemented* if every submodule of M has a δ–supplement in M. M is called weak *δ–supplemented* if every submodule of M has a weak δ–supplement in M.

Here we present two lemmas that state some properties of δ–small submodules which we will use throughout the section 3.

Lemma 2.2. *Let M and N be modules. Then*

(1) $\delta(M) = \sum\{L \leq M \mid L \ll_\delta M\} = \bigcap\{K \leq M \mid M/K$ *is singular simple* $\}$.

(2) *If $f : M \to N$ is an R-homomorphism, then $f(\delta(M)) \subseteq \delta(N)$. Therefore $\delta(M)$ is a fully invariant submodule of M. In particular, if $K \leq M$, then $\delta(K) \subseteq \delta(M)$.*

(3) *If $M = \oplus_{i\in I} M_i$, then $\delta(M) = \oplus_{i\in I}\delta(M_i)$.*

(4) *If every proper submodule of M is contained in a maximal submodule of M, then $\delta(M)$ is the unique largest δ–small submodule of M. In particular if M is finitely generated, then $\delta(M)$ is δ–small in M.*

Proof. See [11, Lemma 1.2]. □

Lemma 2.3. *Let M be a module and $\delta(M) \leq K \leq M$. Then the following hold:*

(1) *If $\delta(M)$ is δ–small in M and $\delta(M)$ is a δ–cosmall submodule of K in M, then K is δ-small in M.*

(2) $\delta(M/\delta(M)) = 0$.

Proof. See [11, Lemma 1.3]. □

3. Cofinitely weak generalized δ–supplemented modules

The submodules $Rad(M)$ and $\delta(M)$ of a module M in the category of modules play important roles. Many authors studied some generalizations of supplemented, weakly supplemented, δ-supplemented and weakly δ–supplemented modules by us of these two functors. We refer to [2, 7, 8] for some of them.
Here we study and investigate a generalization of weakly δ–supplemented modules.

Definition 3.1. Let M be a module and N, K two submodules of M. N is called a *generalized δ–supplement* of K in M if, $N + K = M$ and $N \cap K \subseteq \delta(N)$. N is called a *weak generalized δ–supplement* of K in M if, $N + K = M$ and $N \cap K \subseteq \delta(M)$.
The module M is called *(cofinitely) generalized δ–supplemented* (briefly (C)G-δ-S) if every (cofinite) submodules of M has a generalized δ–supplement in M. M is called *(cofinitely) weak generalized δ-supplemented* (briefly (C)WG-δ-S) if every (cofinite) submodule of M has a weak generalized δ–supplement in M.

G-δ-S modules are defined and investigated by Talebi and current author in [7]. Here a generalization of G-δ-S modules namely CWGδ-S modules and some other kind of modules related to these modules will be defined and investigated. First we present an elementary lemma.

Lemma 3.2. *Let M be a module and V, U submodules of M. If V is a weak generalized δ–supplement of U in M, then $\frac{V+L}{L}$ is a weak generalized δ–supplement of $\frac{U}{L}$ in $\frac{M}{L}$ for every $L \leq U$.*

Proof. We have $V + U = M$ and $V \cap U \leq \delta(M)$. So $\frac{M}{L} = \frac{V+L}{L} + \frac{U}{L}$. Let $\pi : M \longrightarrow \frac{M}{L}$ be the natural epimorphism. Then by Lemma 2.1 (2), $\pi(V \cap U) \subseteq \pi(\delta(M)) \subseteq \delta(\frac{M}{L})$, where $\pi(V \cap U) = \frac{V \cap U+L}{L} = \frac{V+L}{L} \cap \frac{U}{L}$ by modularity. Hence $\frac{V+L}{L}$ is a weak generalized δ–supplement of $\frac{U}{L}$ in $\frac{M}{L}$. □

Proposition 3.3. *Every homomorphic image of a CWG-δ-S module is again CWG-δ-S.*

Proof. Let M be a CWG-δ-S module and $\frac{U}{N}$ a cofinite submodule of $\frac{M}{N}$ where $N \leq U \leq M$. Then U is a cofinite submodule of M and so there exists a submodule V of M such that $V + U = M$ and $V \cap U \leq \delta(M)$. By Lemma 3.2, $\frac{V+N}{N}$ is a weak generalized δ–supplement of $\frac{U}{N}$ in $\frac{M}{N}$ and this completes the proof. □

Recall that a module M is called *semi–hollow* if every proper finitely generated submodule of M is small in M ([3, 2.12]). Here we call a module M, *semi-δ-hollow* if every proper finitely generated submodule of M is δ–small in M. It is clear that if M is semi-δ-hollow, then $\delta(M) = M$. $\frac{\mathbb{Z}}{p^n\mathbb{Z}}$ as $\mathbb{Z}$-modules, are semi-δ-hollow.

Proposition 3.4. *Let M be a module and N a semi-δ-hollow submodule of M. Then M is CWG-δ-S iff $\frac{M}{N}$ is.*

Proof. The necessity follows from Proposition 3.3.
For converse suppose that N is a semi-δ-hollow submodule of M and $\frac{M}{N}$ is a CWG-δ-S module. If U is a cofinite submodule of M, then $\frac{U+N}{N}$ is a cofinite submodule of $\frac{M}{N}$. Let $\frac{V}{N}$ be a weak generalized δ-supplement of $\frac{U+N}{N}$ in $\frac{M}{N}$. So

$$\frac{U+N}{N} + \frac{V}{N} = \frac{M}{N} \text{ and } \frac{U+N}{N} \cap \frac{V}{N} \subseteq \delta(\frac{M}{N})$$

This implies $U + V = M$ and $\frac{U \cap V + N}{N} \subseteq \delta(\frac{M}{N})$. Since N is semi-δ-hollow, we have $N = \delta(N) \subseteq \delta(M)$ and so $\frac{\delta(M)}{N}) = \delta(\frac{M}{N})$. Finally we get $U \cap V \subseteq \delta(M)$, as desired. □

Proposition 3.5. *Let M be a WG-δ-S module and N be a submodule of M such that $\delta(\frac{M}{N}) = 0$. Then $\frac{M}{N}$ is semisimple.*

Proof. Suppose that $N \leq K \leq M$. There exists a submodule V of M such that $K + V = M$ and $K \cap V \leq \delta(M)$. According to Lemma 3.2, $\frac{V+N}{N}$ is a weak generalized δ-supplement of $\frac{K}{N}$ in $\frac{M}{N}$, so that

$$\frac{K}{N} + \frac{V+N}{N} = \frac{M}{N} \text{ and } \frac{V+N}{N} \cap \frac{K}{N} \subseteq \delta(\frac{M}{N}) = 0$$

That is $\frac{M}{N}$ is semisimple. □

Corollary 3.6. *Let M be a CWG-δ-S module and $N \leq M$ such that $\delta(\frac{M}{N}) = 0$. Then every cofinite submodule of $\frac{M}{N}$ is a direct summand.*

Proof. If $\frac{K}{N}$ is a cofinite submodule of $\frac{M}{N}$, then K is a cofinite submodule of M. Now apply the proof of Proposition 3.5 to complete the proof. □

By Proposition 3.3 and Proposition 3.4 we have the next two corollaries.

Corollary 3.7. *If M is a WG-δ-S module, then $\frac{M}{\delta(M)}$ is semisimple.*

Corollary 3.8. *Let M be a CWG-δ-S module. Then every cofinite submodule of $\frac{M}{\delta(M)}$ is a direct summand.*

Proposition 3.9. *Let $f : M \longrightarrow N$ be a homomorphism and L a weak generalized δ-supplement submodule of M containing $\ker(f)$. Then $f(L)$ is a weak generalized δ-supplement of $f(M)$.*

Proof. Suppose that L is a weak generalized δ-supplement of K in M. Hence $M = L + K$ and $L \cap K \subseteq \delta(M)$. Then $f(M) = f(L) + f(K)$ and $f(L \cap K) \subseteq f(\delta(M)) \subseteq \delta(f(M))$ by Lemma 2.1. As $\ker(f) \subseteq L$, we have $f(L \cap K) = f(L) \cap f(K)$ and so $f(L)$ is a weak generalized δ-supplement of $f(K)$ in $f(M)$. □

Lemma 3.10. *Let M be a module, U a cofinite submodule of M and $N \leq M$ a weak generalized δ-supplemented module. If $N + U$ has a weak generalized δ-supplement in M, then U also has.*

Proof. Let X be a weak generalized δ-supplement of $N + U$ in M. We have $\frac{N}{N\cap(X+U)} \cong \frac{N+X+U}{X+U} = \frac{M}{X+U} \cong \frac{M/U}{(X+U)/U}$ is finitely generated. So $N \cap (X + U)$ has a weak generalized δ-supplement Y in N; i.e. $Y + [N \cap (X + U)] = N$ and $y \cap N \cap (X + U) = Y \cap (X + U) \subseteq \delta(N) \subseteq \delta(M)$. Now we have

$$M = U + X + N = U + X + Y + [N \cap (X + U)] = U + X + Y$$

and also

$$U \cap (X + Y) \subseteq X \cap (Y + U) + Y \cap (X + U) \subseteq X \cap (N + U) + Y \cap (X + U) \subseteq \delta(M)$$

That is $X + Y$ is a weak generalized δ-supplement of U in M. □

Proposition 3.11. *Any sum of CWG-δ-S modules is again CWG-δ-S.*

Proof. Let $\{M_i\}_{i\in I}$ be set of CWG-δ-S modules and $M = \sum_{i\in I} M_i$. Suppose that N is a copfinite submodule of M and $\frac{M}{N}$ is generated by $\{x_1 + N, x_2 + N, ..., x_k + N\}$. Thus $M = Rx_1 + Rx_2 + ... + Rx_k + N$. For every $i \in \{1, 2, ..., k\}$ We have $x_i \in \sum_{j\in F_i} M_j$ for some finite set $F_i \subseteq I$. Therefore $Rx_1 + Rx_2 + ... + Rx_k \leq \sum_{j\in F} M_j$ where $F = \bigcup_{i=1}^{k} F_i$. Suppose that $F = \{i_1, i_2, ..., i_r\}$. Then $M = N + \sum_{s=1}^{r} M_{i_s}$. Since $M = M_{i_r} + (N + \sum_{s=1}^{r-1} M_{i_s}$ has a trivial weak generalized δ-supplement 0 and M_{i_r} is a CWG-δ-S module, $N + \sum_{s=1}^{r-1} M_{i_s}$ has a weak generalized δ-supplement in M by Lemma 3.10. Similarly $N + \sum_{s=1}^{r-2} M_{i_s}$ has a weak generalized δ-supplement in M and so on. After we have used Lemma 3.10 r times, we will obtain that N has a weak generalized δ-supplement in M, as required. □

Corollary 3.12. *If M is a CWG-δ-S module, then every M-generated module is again CWG-δ-S.*

Proof. Follows immediately from Proposition 3.3 and Proposition 3.11. □

Lemma 3.13. *Let M be a module and X a cofinite (maximal) submodule of M. If Y is a weak generalized δ-supplement of X in M, then X has a finitely generated (cyclic) weak generalized δ-supplement in M contained in Y.*

Proof. We have $\frac{Y}{X\cap Y} \cong \frac{X+Y}{X} = \frac{M}{X}$ is finitely generated. Suppose that $\frac{Y}{X\cap Y} =< x_1 + X \cap Y, x_2 + X \cap Y, ..., x_n + X \cap Y >$. If $Z =< x_1, x_2, ..., x_n >$, then $Z \leq Y$ and $Z + X \cap Y = Y$. Now

$$Z + X = Z + X \cap Y + X = Y + X = M$$

and also $Z \cap X \leq Y \cap X \leq \delta(M)$. Therefore Z is a finitely generated weak generalized δ-supplement of X contained in Y.

Now if X is maximal, then $\frac{Y}{X\cap Y}$ is simple and especially cyclic and by the similar way there is a cyclic submodule W of Y which is a weak generalized δ-supplement of X in M. □

Recall that ([1, Proposition 10.4]) if M is a finitely generated module, then $Rad(M) \ll M$. Similarly we have the following lemma.

Lemma 3.14. *If M is a finitely generated module, then $\delta(M) \ll_\delta M$.*

In the next proposition we proceed with a weak condition to derive a strong property for such submodules. Then from this proposition we present some corollaries.

Proposition 3.15. *Let M be a module and X a cofinite submodule of M. Moreover suppose that Y is a weak generalized δ-supplement of X in M and every finitely generated submodule K of Y satisfies in condition that $\delta(K) = K \cap \delta(M)$. Then X has a finitely generated δ-supplement in M.*

Proof. We have $X + Y = M$ and $X \cap Y \subseteq \delta(M)$. Since $\frac{M}{X}$ is finitely generated, by Lemma 3.13 X has a finitely generated weak generalized δ-supplement K in M contained in Y. So $M = X + K$ and $X \cap K \subseteq \delta(M)$. Now $X \cap K \leq K \cap \delta(M) = \delta(K)$. By Lemma 3.14, $\delta(K) \ll_\delta K$ and hence K is a δ-supplement of X in M. □

The next corollary follows immediately from Proposition 3.15.

Corollary 3.16. *Let M be a module and X a cofinite submodule of M. Moreover suppose that Y is a weak generalized δ-supplement of X in M and every finitely generated submodule K of Y is a direct summand of M. Then X has a finitely generated generalized δ-supplement in M.*

Theorem 3.17. *Let M be a module such that every finitely generated submodule K of M satisfies in condition $K \cap \delta(M) = \delta(K)$. Then the following statements are equivalent:*

(1) *M is cofinitely δ-supplemented;*

(2) *M is CG-δ-S;*

(3) *M is cofinitely weak δ-supplemented;*

(4) *M is CWG-δ-S.*

Proof. $1 \Longrightarrow 2 \Longrightarrow 4$ and $1 \Longrightarrow 3 \Longrightarrow 4$ are clear.
So it is enough to prove $4 \Longrightarrow 1$. Suppose that X is a cofinite submodule of M. Since M is CWG-δ-S, X has a weak generalized δ-supplement in M. Now by Proposition 3.15, X has a δ-supplement in M and this completes the proof. □

Corollary 3.18. *Suppose that M is a module such that every finitely generated submodule of M is a direct summand. Then the following statements are equivalent*

(1) *M is cofinitely δ-supplemented;*

(2) *M is CG-δ-S;*

(3) *M is cofinitely weak δ-supplemented;*

(4) *M is CWG-δ-S.*

Proof. If K is a direct summand of M, then $K \cap \delta(M) = \delta(K)$. Now apply Theorem 3.17. □

Corollary 3.19. *Suppose that M is a finitely generated module such that for every finitely generated submodule K of M we have $K \cap \delta(M) = \delta(K)$. Then the following statements are equivalent*

(1) *M is δ-supplemented;*

(2) *M is generalized δ-supplemented;*

(3) *M is weak δ-supplemented;*

(4) *M is weak generalized δ-supplemented.*

Furthermore in this case every finitely generated submodule of M is a δ-supplement.

Proof. The first part follows from Theorem 3.17. To see the second part, suppose that K is a finitely generated submodule of M. Then K has a weak δ-supplement L in M. Therefore $L + K = M$ and $L \cap K \subseteq K \cap \delta(M) = \delta(K) \ll_\delta K$; i.e. K is a δ-supplement of L in M. □

Example 3.20. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}$. Then for every submodule $N = m\mathbb{Z}$ of M we have $0 = \delta(N) = N \cap \delta(M)$. Also M is not δ-supplemented, so M is not generalized δ-supplemented and especially M is not weak generalized δ-supplemented by Corollary 3.19.

Theorem 3.21. *For a module M the following are equivalent*

(1) *M is a CWG-δ-S module;*
(2) *$\frac{M}{\delta(M)}$ is a CWG-δ-S module;*
(3) *Every cofinite submodule of M is a direct summand.*

Proof. $1 \Longrightarrow 2$ follows from Proposition 3.3.
$2 \Longrightarrow 3$ is clear since $\delta(\frac{M}{\delta(M)}) = 0$.
To see $3 \Longrightarrow 1$ suppose that K is a cofinite submodule of M. Then $\frac{K+\delta(M)}{\delta(M)}$ is a cofinite submodule of $\frac{M}{\delta(M)}$. Hence there exists a submodule $\frac{L}{\delta(M)}$ of $\frac{M}{\delta(M)}$ such that

$$\frac{K+\delta(M)}{\delta(M)} + \frac{L}{\delta(M)} = \frac{M}{\delta(M)} \textit{ and } \frac{K+\delta(M)}{\delta(M)} \cap \frac{L}{\delta(M)} = 0$$

Therefore $K + L = M$ and $K \cap L \subseteq \delta(M)$ as desired. □

Corollary 3.22. *Let M be a CWG-δ-S module. Then*
(1) *Every maximal submodule of $\frac{M}{\delta(M)}$ is a direct summand.*
(2) *Every maximal submodule of $\frac{M}{\delta(M)}$ is a weak generalized δ-supplement.*
(3) *Every maximal submodule of M is weak generalized δ-supplement.*

Proof. It is clear that conditions 1, 2 and 3 are equivalent by Theorem 3.21. Now if M is CWG-δ-S, then 3 holds, since every maximal submodule is cofinite. □

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