Journal of Algebraic Structures and Their Applications
ISSN: 2382-9761
www.as.yazd.ac.ir

Algebraic Structures and Their Applications Vol. 7 No. 1 (2020) pp 21-28.

A NEW LOWER BOUND FOR COHOMOLOGICAL DIMENSION

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Abstract. Let $(R, \mathfrak{m})$ be a Noetherian local ring, M a finitely generated R-module, and $\mathfrak{a}$ an ideal of R. We define the $\mathfrak{a}$-minimum dimension $\mathrm{d}(\mathfrak{a}, M)$ of M by

$$\mathrm{d}(\mathfrak{a}, M) = \mathrm{Min}\{\dim \frac{R}{\mathfrak{p} + \mathfrak{a}} : \mathfrak{p} \in \mathrm{Assh}_R(M)\}.$$

In this paper, we show that $\mathrm{cd}(\mathfrak{a}, M) \geq \dim M - \mathrm{d}(\mathfrak{a}, M)$ and we give some sufficient conditions and characterization for the equality to hold true.

1. Introduction

Throughout this paper, let $(R, \mathfrak{m})$ be a commutative Noetherian local ring (with identity) and let M be a finitely generated R-module. For an R-module M, the i-th local cohomology module of M with respect to $\mathfrak{a}$ is defined as

$$\mathrm{H}^i_{\mathfrak{a}}(M) = \varinjlim_{n \geq 1} \mathrm{Ext}^i_R(\frac{R}{\mathfrak{a}^n}, M).$$

For the basic properties of local cohomology the reader can refer to [1] of Brodmann and Sharp.

DOI: 10.29252/as.2020.1621

MSC(2010): 13D45; 14B15.

Keywords: local cohomology, cohomological dimension, cofinite modules.

Received: 01 July 2019, Accepted: 07 Oct 2019.

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Recall that the cohomological dimension of M with respect to $\mathfrak{a}$ is defined as

$$\mathrm{cd}(\mathfrak{a}, M) := \max\{i \in \mathbb{Z} : \mathrm{H}^i_{\mathfrak{a}}(M) \neq 0\}.$$

The cohomological dimension has been studied by several authors; see, for example, Faltings [5], Hartshorne [6], Huneke-Lyubeznik [7] and Varbaro [10] . In particular in [5] and [7], several upper bounds for cohomological dimension were obtained. It follows from [1, Theorem 6.2.7] that $\mathrm{cd}(\mathfrak{a}, M)$ is greater than or equal to the $\mathrm{grade}(\mathfrak{a}, M)$. A natural question to ask is under what conditions one can obtain a better lower bound for $\mathrm{cd}(\mathfrak{a}, M)$. The main aim of this article is to establish a new lower bound for cohomological dimension of finitely generated modules over a local ring.

Throughout this article, we denote $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq \mathfrak{a}\}$ by $\mathrm{V}(\mathfrak{a})$, $\operatorname{Min} \mathrm{V}(\mathfrak{a})$ by $\operatorname{Min}(\mathfrak{a})$, and $\{\mathfrak{p} \in \operatorname{Ass}_R(M) : \dim \frac{R}{\mathfrak{p}} = \dim M\}$ by $\operatorname{Assh}_R(M)$. The radical of $\mathfrak{a}$, denoted by $\sqrt{\mathfrak{a}}$, is defined to be the set $\{x \in R : x^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}\}$. Recall that an R-module M is called $\mathfrak{a}$-cofinite if $\operatorname{Supp}(M) \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(\frac{R}{\mathfrak{a}}, M)$ is finitely generated for all $i \geq 0$. For any unexplained notation and terminology, we refer the reader to [1] and [8].

2. Main results

Definition 2.1. Let M be a finitely generated R-module, and let $\mathfrak{a}$ be an ideal of R. We define the $\mathfrak{a}$-minimum dimension $\mathrm{d}(\mathfrak{a}, M)$ of M by

$$\mathrm{d}(\mathfrak{a}, M) = \operatorname{Min}\{\dim \frac{R}{\mathfrak{p} + \mathfrak{a}} : \mathfrak{p} \in \operatorname{Assh}_R(M)\}.$$

To prove the main results of this paper, we need the following lemmas.

Lemma 2.2. (see [4, Lemma 2.5]) *Let M be a finitely generated R-module, and let $\mathfrak{a}$ be an ideal of R. Then*

$$\mathrm{cd}(\mathfrak{a} + Rx, M) \leq \mathrm{cd}(\mathfrak{a}, M) + 1$$

for any element $x \in \mathfrak{m}$.

Lemma 2.3. *Let M be a finitely generated R-module and $\mathfrak{a}$ be an ideal of R with* $\mathrm{d}(\mathfrak{a}, M) > 0$. *Then there exists an element $x \in \mathfrak{m}$ such that* $\dim \frac{M}{xM} = \dim M - 1$ *and* $\mathrm{d}(\mathfrak{a}, \frac{M}{xM}) \leq \mathrm{d}(\mathfrak{a}, M) - 1$.

Proof. Since $\mathrm{d}(\mathfrak{a}, M) > 0$, we have $\sqrt{\mathfrak{p} + \mathfrak{a}} \neq \mathfrak{m}$ for all $\mathfrak{p} \in \operatorname{Assh}_R(M)$, and so there exists

$$x \in \mathfrak{m} - \bigcup_{\mathfrak{q} \in \operatorname{Min}(\mathfrak{p}+\mathfrak{a}), \mathfrak{p} \in \operatorname{Assh}_R(M)} \mathfrak{q}.$$

By the definition, there exists $\mathfrak{p} \in \operatorname{Assh}_R(M)$ such that $\mathrm{d}(\mathfrak{a}, M) = \dim \frac{M}{(\mathfrak{p}+\mathfrak{a})M}$. Now let $\mathfrak{q} \in \operatorname{Assh}_R(\frac{M}{(\mathfrak{p}+Rx)M})$, then by the choice of x we have

$$\dim \frac{R}{\mathfrak{q}} = \dim \frac{M}{(\mathfrak{p} + Rx)M} = \dim M - 1 = \dim \frac{M}{xM}.$$

As $\mathrm{Assh}_R(\frac{M}{(\mathfrak{p}+Rx)M}) \subseteq \mathrm{Supp}\,\frac{M}{xM}$, we have $\mathfrak{q} \in \mathrm{Supp}\,\frac{M}{xM}$, and so by the above equalities we have $\mathfrak{q} \in \mathrm{Assh}_R(\frac{M}{xM})$. It follows that

$$\mathrm{d}(\mathfrak{a}, \frac{M}{xM}) \leq \dim \frac{M}{(\mathfrak{q}+\mathfrak{a})M} \leq \dim \frac{M}{(\mathfrak{p}+\mathfrak{a}+Rx)M} = \dim \frac{M}{(\mathfrak{p}+\mathfrak{a})M} - 1 = \mathrm{d}(\mathfrak{a}, M) - 1.$$

This element x has the requested properties. □

Theorem 2.4. *Let M be a finitely generated R-module, and let $\mathfrak{a}$ be an ideal of R. Then $\mathrm{cd}(\mathfrak{a}, M) \geq \dim M - \mathrm{d}(\mathfrak{a}, M)$.*

Proof. We prove this by induction on $n = \mathrm{d}(\mathfrak{a}, M)$. If $\mathrm{d}(\mathfrak{a}, M) = 0$ then we have $\dim \frac{M}{(\mathfrak{p}+\mathfrak{a})M} = 0$ for some $\mathfrak{p} \in \mathrm{Assh}_R(M)$ and so $\sqrt{\mathfrak{p}+\mathfrak{a}} = \mathfrak{m}$ for some $\mathfrak{p} \in \mathrm{Assh}_R(M)$. It follows from [1, Exercise 6.1.9] and Non-vanishing Theorem [1, 6.1.4] that

$$\mathrm{H}_{\mathfrak{a}}^{\dim M}(M) \otimes \frac{R}{\mathfrak{p}} \cong \mathrm{H}_{\mathfrak{a}}^{\dim M}(\frac{M}{\mathfrak{p}M}) \cong \mathrm{H}_{\mathfrak{a}+\mathfrak{p}}^{\dim M}(\frac{M}{\mathfrak{p}M}) \cong \mathrm{H}_{\mathfrak{m}}^{\dim \frac{M}{\mathfrak{p}M}}(\frac{M}{\mathfrak{p}M}) \neq 0,$$

and so $\mathrm{H}_{\mathfrak{a}}^{\dim M}(M) \neq 0$.

Now suppose, inductively, that $\mathrm{d}(\mathfrak{a}, M) > 0$, and the result has been proved for all finitely generated R-modules N with $\mathrm{d}(\mathfrak{a}, N) < \mathrm{d}(\mathfrak{a}, M)$. By Lemma 2.3, there exists an element $x \in \mathfrak{m}$ such that $\dim M = \dim \frac{M}{xM} + 1$ and $\mathrm{d}(\mathfrak{a}, M) \geq \mathrm{d}(\mathfrak{a}, \frac{M}{xM}) + 1$. So by induction hypothesis we have $\mathrm{cd}(\mathfrak{a}, M/xM) \geq \dim \frac{M}{xM} - \mathrm{d}(\mathfrak{a}, \frac{M}{xM})$. It follows that

$$\begin{array}{ll} \dim M - \mathrm{d}(\mathfrak{a}, M) & = \dim \frac{M}{xM} + 1 - \mathrm{d}(\mathfrak{a}, M) \\ & \leq \dim \frac{M}{xM} - \mathrm{d}(\mathfrak{a}, \frac{M}{xM}) \\ \text{[by induction hypothesis]} & \leq \mathrm{cd}(\mathfrak{a}, \frac{M}{xM}) \\ \text{[4, Theorem 2.2]} & \leq \mathrm{cd}(\mathfrak{a}, M). \end{array}$$

This completes the proof. □

The following examples shows that the equality does not hold in general.

Example 2.5. Let M be a finitely generated R-module such that $\bigcap_{\mathfrak{p} \in \mathrm{Assh}_R(M)} \mathfrak{p} \nsubseteq \mathfrak{q}$ for some $\mathfrak{q}$ in $\mathrm{Ass}_R(M)$. Then for $x \in \bigcap_{\mathfrak{p} \in \mathrm{Assh}_R(M)} \mathfrak{p} - \mathfrak{q}$ we have

$$\mathrm{cd}(Rx, M) = 1 > 0 = \dim(M) - \mathrm{d}(Rx, M).$$

For example, let $R = K[[X, Y, Z]]$, $M = \frac{K[[X,Y,Z]]}{\langle X \rangle \bigcap \langle Y,Z \rangle}$, and $x = X$, where K is a field and X, Y, Z are independent indeterminates.

Example 2.6. Let K be a field of characteristic 0. Let $R' := K[X_1, X_2, X_3]$, $\mathfrak{m}' := (X_1, X_2, X_3)$ and $\mathfrak{b} = (X_2^2 - X_1^2 - X_1^3)$. Set $R := (\frac{R'}{\mathfrak{b}})_{\frac{\mathfrak{m}'}{\mathfrak{b}}}$ and let $\mathfrak{p}$ be the extension of the ideal

$$(X_1 + X_2 - X_2X_3, (X_3 - 1)^2(X_1 + 1) - 1)$$

of R' to R. Then R is a 2-dimensional local domain, and $\mathfrak{p}$ is a prime ideal of R with $\dim \frac{R}{\mathfrak{p}} = 1$ (see [1, Exercise 8.2.9]), and we have

$$\operatorname{cd}(\mathfrak{p}, R) = 2 > 1 = \dim(R) - \operatorname{d}(\mathfrak{p}, R).$$

Therefore, it is natural to ask, under what conditions does the equality hold?

Our second aim is to find such conditions. The following theorem gives us a characterization for the equality $\operatorname{cd}(\mathfrak{a}, M) = \dim M - \operatorname{d}(\mathfrak{a}, M)$.

Theorem 2.7. *Let M be a finitely generated R-module, and let $\mathfrak{a}$ be an ideal of R. Then the following statements are equivalent:*

(i) $\operatorname{cd}(\mathfrak{a}, M) = \dim M - \operatorname{d}(\mathfrak{a}, M)$;

(ii) *There exists a sequence $x_1, x_2, ..., x_l$, where $l = \operatorname{d}(\mathfrak{a}, M)$, such that for each $i = 1, 2, ..., l$*

$$x_i \in \mathfrak{m} - \bigcup_{\substack{\mathfrak{q} \in \operatorname{Min}(\mathfrak{p}+\mathfrak{a}+Rx_1+\cdots+Rx_{i-1}) \\ \mathfrak{p} \in \operatorname{Assh}_R(M)}} \mathfrak{q}$$

and $\operatorname{H}^1_{Rx_i}(\operatorname{H}^{c+i-1}_{\mathfrak{a}+Rx_1+\cdots+Rx_{i-1}}(M)) \neq 0$, *where* $c = \operatorname{cd}(\mathfrak{a}, M)$.

Proof. (i)⇒(ii) We use induction on $l = \operatorname{d}(\mathfrak{a}, M)$. When $l = 0$, there is nothing to prove. So suppose that $\operatorname{d}(\mathfrak{a}, M) = l > 0$ and that the result has been proved for each ideal $\mathfrak{b}$ with $\operatorname{d}(\mathfrak{b}, M) < l$. Choose $x_1 \in \mathfrak{m} - \bigcup_{\substack{\mathfrak{q} \in \operatorname{Min}(\mathfrak{p}+\mathfrak{a}) \\ \mathfrak{p} \in \operatorname{Assh}_R(M)}} \mathfrak{q}$; then we have

$$\begin{aligned} \dim M - \operatorname{d}(\mathfrak{a}, M) &= \dim M - \operatorname{d}(\mathfrak{a} + Rx_1, M) - 1 \\ &\leq \operatorname{cd}(\mathfrak{a} + Rx_1, M) - 1 \\ \text{[by lemma 2.2]} \quad &\leq \operatorname{cd}(\mathfrak{a}, M). \end{aligned}$$

So $\operatorname{cd}(\mathfrak{a} + Rx_1, M) = \dim M - \operatorname{d}(\mathfrak{a} + Rx_1, M)$ and $\operatorname{d}(\mathfrak{a} + Rx_1, M) = l - 1$. Therefore, by the inductive hypothesis, there exists a sequence $x_2, x_3, ..., x_l \in \mathfrak{m}$ such that, for each $i = 2, 3, ..., l$,

$$x_i \in \mathfrak{m} - \bigcup_{\substack{\mathfrak{q} \in \operatorname{Min}(\mathfrak{p}+\mathfrak{a}+Rx_1+\cdots+Rx_{i-1}) \\ \mathfrak{p} \in \operatorname{Assh}_R(M)}} \mathfrak{q}$$

and $\operatorname{H}^1_{Rx_i}(\operatorname{H}^{c+i-1}_{\mathfrak{a}+Rx_1+\cdots+Rx_{i-1}}(M)) \neq 0$.

On the other hand, we have $\mathrm{cd}(\mathfrak{a}+Rx_1, M) = \mathrm{cd}(\mathfrak{a}, M)+1$ and so $\mathrm{H}^{c+1}_{\mathfrak{a}+Rx}(M) \neq 0$. By [1, Proposition 8.1.2 (i)], there is an exact sequence

$$\mathrm{H}^c_{\mathfrak{a}}(M) \longrightarrow \mathrm{H}^c_{\mathfrak{a}}(M_{x_1}) \longrightarrow \mathrm{H}^{c+1}_{\mathfrak{a}+Rx_1}(M) \longrightarrow 0.$$

It follows that the natural homomorphism $\mathrm{H}^c_{\mathfrak{a}}(M) \longrightarrow \mathrm{H}^c_{\mathfrak{a}}(M_{x_1})$ is not surjective. So $\mathrm{H}^1_{Rx_1}(\mathrm{H}^c_{\mathfrak{a}}(M)) \neq 0$ by [1, Remark 2.2.17]. This completes the proof of (i)⇒(ii).

(ii)⇒(i) For $\mathrm{d}(\mathfrak{a}, M) = 0$ the result is obvious. Now suppose, inductively, that $\mathrm{d}(\mathfrak{a}, M) = l > 0$ and the result has been proved for each ideal $\mathfrak{b}$ with $\mathrm{d}(\mathfrak{b}, M) < l$. Assume that there exists a sequence $x_1, x_2, ..., x_l \in \mathfrak{m}$ such that, for each $i = 1, 2, ..., l$,

$$x_i \in \mathfrak{m} - \bigcup_{\substack{\mathfrak{q}\in \mathrm{Min}(\mathfrak{p}+\mathfrak{a}+Rx_1+\cdots+Rx_{i-1}) \\ \mathfrak{p}\in \mathrm{Assh}_R(M)}} \mathfrak{q}$$

and $\mathrm{H}^1_{Rx_i}(\mathrm{H}^{c+i-1}_{\mathfrak{a}+Rx_1+\cdots+Rx_{i-1}}(M)) \neq 0$.

Note that $\mathrm{d}(\mathfrak{a}+Rx_1, M) = \mathrm{d}(\mathfrak{a}, M) - 1 = l - 1$, and so, by the inductive hypothesis, we have $\mathrm{cd}(\mathfrak{a}+Rx_1, M) = \dim(M) - \mathrm{d}(\mathfrak{a}+Rx_1, M)$. It follows that $\mathrm{cd}(\mathfrak{a}+Rx_1, M) - 1 = \dim(M) - \mathrm{d}(\mathfrak{a}, M)$. Since $\mathrm{H}^1_{Rx_1}(\mathrm{H}^c_{\mathfrak{a}}(M)) \neq 0$, the natural homomorphism $\mathrm{H}^c_{\mathfrak{a}}(M) \longrightarrow \mathrm{H}^c_{\mathfrak{a}}(M_{x_1})$ is not surjective by [1, Remark 2.2.17] and so $\mathrm{H}^{c+1}_{\mathfrak{a}+Rx_1}(M) \neq 0$ by [1, Proposition 8.1.2 (i)]. Hence $\mathrm{cd}(\mathfrak{a}+Rx_1, M) = \mathrm{cd}(\mathfrak{a}, M) + 1$ and the result follows. □

Recall that a sequence $x_1, x_2, ..., x_l \in \mathfrak{a}$ is called an $\mathfrak{a}$-filter regular sequence of M if $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}_R(\frac{M}{\langle x_1,x_2,...,x_{i-1}\rangle M}) - \mathrm{V}(\mathfrak{a})$ and all $i = 1, 2, ..., l$. For an R-module M, we shall denote $\frac{M}{\Gamma_{\mathfrak{a}}(M)}$ by $\overline{M}$.

Lemma 2.8. *Let M be a finitely generated R-module, and let $\mathfrak{a}$ be an ideal of R such that $\Gamma_{\mathfrak{a}}(M) \neq M$. If M is an equidimensional R-module, then*

(i) *$\overline{M}$ is an equidimensional R-module and we have $\dim M = \dim \overline{M}$, and $\mathrm{d}(\mathfrak{a}, M) = \mathrm{d}(\mathfrak{a}, \overline{M})$.*

(ii) *If R is a catenary ring then $\frac{\overline{M}}{x\overline{M}}$ is an equidimensional R-module for each $\mathfrak{a}$-filter regular element x of M.*

Proof. (i) This is immediate from the fact that

$$\mathrm{Min}\,\mathrm{Ass}_R(\overline{M}) = \mathrm{Min}\,\mathrm{Ass}_R(M) - \mathrm{V}(\mathfrak{a}) = \mathrm{Assh}_R(M) - \mathrm{V}(\mathfrak{a}) = \mathrm{Assh}_R(\overline{M}).$$

(ii) Let $\mathfrak{q} \in \mathrm{Min}\,\mathrm{Ass}_R(\frac{\overline{M}}{x\overline{M}})$. So we have $\mathfrak{q} \in \mathrm{Min}(\mathrm{Ann}_R(\overline{M}) + Rx)$. It follows that there exists $\mathfrak{p} \in \mathrm{Min}(\mathrm{Ann}_R(\overline{M})) = \mathrm{Assh}_R(\overline{M})$ such that $\mathfrak{q} \in \mathrm{Min}(\mathfrak{p} + Rx)$. As R is a catenary ring,

we have $\mathrm{h}(\mathfrak{q}) = \mathrm{h}(\mathfrak{p}) + 1$ and so $\dim(R/\mathfrak{q}) = \dim(R/\mathfrak{p}) - 1$.
It follows that

$$\dim(R/\mathfrak{q}) = \dim(R/\mathfrak{p}) - 1 = \dim(\overline{M}) - 1 = \dim(\frac{\overline{M}}{x\overline{M}}).$$

Hence $\mathfrak{q} \in \mathrm{Assh}_R(\frac{\overline{M}}{x\overline{M}})$ and so the claim follows. □

Theorem 2.9. *Let R be a catenary ring, M a finitely generated equidimensional R-module, and $l = \dim(M) - \mathrm{d}(\mathfrak{a}, M)$. If there exists an $\mathfrak{a}$-filter regular sequence $x_1, x_2, ..., x_l$ of M such that $\mathrm{d}(\mathfrak{a}, M_{i-1}) = \mathrm{d}(\mathfrak{a}, M_i)$, where $M_0 = M$ and $M_i = \frac{\overline{M_{i-1}}}{x_i\overline{M_{i-1}}}$, for all $i = 1, 2, ..., l$, then*

$$\mathrm{cd}(\mathfrak{a}, M) = \dim(M) - \mathrm{d}(\mathfrak{a}, M).$$

Proof. By Theorem 2.4, it is enough for us to show that $\mathrm{cd}(\mathfrak{a}, M) \leq l$. We argue by induction on l. When $l = 0$, since M is equidimensional, by the definition of $\mathrm{d}(\mathfrak{a}, M)$ we have $M = \Gamma_{\mathfrak{a}}(M)$ and so $\mathrm{cd}(\mathfrak{a}, M) = \dim(M) - \mathrm{d}(\mathfrak{a}, M)$.

Now suppose, inductively, that $l > 0$ and the result has been proved for smaller values of l. By the pervious lemma, in this case, $\overline{M}$ is an equidimensional R-module, and we have $\dim(M) = \dim(\overline{M})$, $\mathrm{d}(\mathfrak{a}, M) = \mathrm{d}(\mathfrak{a}, \overline{M})$, and $\mathrm{cd}(\mathfrak{a}, M) = \mathrm{cd}(\mathfrak{a}, \overline{M})$. So in view of the inductive hypothesis we can replace M by $\overline{M}$, and assume that M is $\mathfrak{a}$-torsion free. The exact sequence

$$0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow M_1 \longrightarrow 0$$

induces an exact sequence

$$\mathrm{H}^{i-1}_{\mathfrak{a}}(M_1) \longrightarrow \mathrm{H}^i_{\mathfrak{a}}(M) \xrightarrow{x_1} \mathrm{H}^i_{\mathfrak{a}}(M).$$

Since $\mathrm{d}(\mathfrak{a}, M) = \mathrm{d}(\mathfrak{a}, M_1)$ and $\dim(M_1) = \dim(M) - 1$, we obtain

$$\dim(M_1) - \mathrm{d}(\mathfrak{a}, M_1) = l - 1.$$

By the pervious lemma, M_1 is an equidimensional R-module, so by induction hypothesis $\mathrm{H}^i_{\mathfrak{a}}(M_1) = 0$ for all $i > l - 1$. Therefore, in view of the above exact sequence, $(0 \underset{\mathrm{H}^i_{\mathfrak{a}}(M)}{:} x_1) = 0$ for all $i > l$. But $x_1 \in \mathfrak{a}$ and $\mathrm{H}^i_{\mathfrak{a}}(M)$ is an $\mathfrak{a}$-torsion R–module, and so $\mathrm{H}^i_{\mathfrak{a}}(M) = 0$ for all $i > l$. This complete the inductive step, and the proof. □

The following is an example to illustrate Theorem 2.9.

Example 2.10. Let $R = K[[X_1, X_2, X_3, X_4, X_5]]$ denote the formal power series ring in five variables over a field K. Put $M = \frac{K[[X_1,X_2,X_3,X_4,X_5]]}{\langle X_2\rangle \bigcap \langle X_3\rangle}$ and $\mathfrak{a} = \langle X_1, X_2, X_3\rangle$. In this case we have $\dim(M) - \mathrm{d}(\mathfrak{a}, M) = 2$, and $x_1, x_2 + x_3$ is an $\mathfrak{a}$-filter regular sequence of M which has the property mentioned in Theorem 2.9. It follows that $\mathrm{cd}(\mathfrak{a}, M) = 2$ and $\mathrm{H}^3_{\langle X_1,X_2,X_3\rangle}(\frac{K[[X_1,X_2,X_3,X_4,X_5]]}{\langle X_2\rangle \bigcap \langle X_3\rangle}) = 0$.

Before proving Theorem 2.12, we need the following lemma which is proved in [2].

Lemma 2.11. (see [2, Lemma 4.3]) *Let M be a finitely generated R-module, and let* $\mathfrak{q} \in \mathrm{V}(\mathrm{Ann}_R(\mathrm{H}_{\mathfrak{m}}^{\dim M}(M)))$ *such that* $\dim M_{\mathfrak{q}} = \dim M - \dim \frac{R}{\mathfrak{q}}$*. Then* $\mathrm{Ann}_R(0 :_{\mathrm{H}_{\mathfrak{m}}^{\dim M}(M)} \mathfrak{q}) = \mathfrak{q}$.

Theorem 2.12. *Let R be a catenary ring, and let $\mathfrak{a}$ be an ideal of R such that* $\dim \frac{R}{\mathfrak{a}} = 1$. *Then the following statements are equivalent:*

(i) $\mathrm{cd}(\mathfrak{a}, M) = \dim(M) - \mathrm{d}(\mathfrak{a}, M)$ *for each finitely generated R-module M;*

(ii) $\mathrm{cd}(\mathfrak{a}, \frac{R}{\mathfrak{p}}) = \dim(\frac{R}{\mathfrak{p}}) - \mathrm{d}(\mathfrak{a}, \frac{R}{\mathfrak{p}})$ *for each prime ideal $\mathfrak{p}$ of R;*

(iii) $\mathrm{Ann}_R(0 :_{\mathrm{H}_{\mathfrak{a}}^{\dim \frac{R}{\mathfrak{p}}}(\frac{R}{\mathfrak{p}})} \mathfrak{q}) = \mathfrak{q}$ *for each prime ideal $\mathfrak{p}$ of R and each prime ideal* $\mathfrak{q} \in \mathrm{V}(\mathrm{Ann}_R(\mathrm{H}_{\mathfrak{a}}^{\dim \frac{R}{\mathfrak{p}}}(\frac{R}{\mathfrak{p}}))$ *with* $\dim \frac{R}{\mathfrak{q}} = 1$.

Proof. (i)⇒(ii) is clear.

(ii)⇒(iii) For $\mathfrak{p} \in \mathrm{Spec}(R)$, let $\mathfrak{q}$ be a prime ideal of R such that $\mathfrak{q} \supseteq \mathrm{Ann}_R(\mathrm{H}_{\mathfrak{a}}^{\dim \frac{R}{\mathfrak{p}}}(\frac{R}{\mathfrak{p}}))$ and $\dim \frac{R}{\mathfrak{q}} = 1$. Since $\mathrm{H}_{\mathfrak{a}}^{\dim \frac{R}{\mathfrak{p}}}(\frac{R}{\mathfrak{p}}) \neq 0$, it follows from statement (ii) that $\sqrt{\mathfrak{p} + \mathfrak{a}} = \mathfrak{m}$ and so $\mathrm{H}_{\mathfrak{a}}^{\dim \frac{R}{\mathfrak{p}}}(\frac{R}{\mathfrak{p}}) \cong \mathrm{H}_{\mathfrak{m}}^{\dim \frac{R}{\mathfrak{p}}}(\frac{R}{\mathfrak{p}})$. Therefore the proof is complete if we show that

$$\mathrm{Ann}_R(0 :_{\mathrm{H}_{\mathfrak{m}}^{\dim \frac{R}{\mathfrak{p}}}(\frac{R}{\mathfrak{p}})} \mathfrak{q}) = \mathfrak{q}.$$

Since R is catenary, we have

$$\dim(\frac{R}{\mathfrak{p}})_{\mathfrak{q}} = \dim \frac{R}{\mathfrak{p}} - 1 = \dim(\frac{R}{\mathfrak{p}}) - \dim \frac{R}{\mathfrak{q}}.$$

The result now follows from Lemma 2.11.

(iii)⇒(i) It is enough, in order to prove this part, to show that, if $\mathrm{cd}(\mathfrak{a}, M) = \dim(M)$, then there exists $\mathfrak{p} \in \mathrm{Assh}_R(M)$ such that $\sqrt{\mathfrak{p} + \mathfrak{a}} = \mathfrak{m}$. By [9, Corollary 2.2], there exists $\mathfrak{p} \in \mathrm{Assh}_R(M)$ such that $\mathrm{cd}(\mathfrak{a}, \frac{R}{\mathfrak{p}}) = \dim(\frac{R}{\mathfrak{p}})$. We show that for this $\mathfrak{p}$, we have $\sqrt{\mathfrak{p} + \mathfrak{a}} = \mathfrak{m}$. Suppose, on the contrary, that $\sqrt{\mathfrak{p} + \mathfrak{a}} \neq \mathfrak{m}$. Then there exists a prime ideal $\mathfrak{q}$ of R such that $\mathfrak{q} \supseteq \sqrt{\mathfrak{p} + \mathfrak{a}}$ and $\dim \frac{R}{\mathfrak{q}} = 1$. Since $\mathfrak{q} \supseteq \mathfrak{p} = \mathrm{Ann}_R(\mathrm{H}_{\mathfrak{a}}^{\dim \frac{R}{\mathfrak{p}}}(\frac{R}{\mathfrak{p}}))$, by assumption (iii), we have $\mathrm{Ann}_R(0 :_{\mathrm{H}_{\mathfrak{a}}^{\dim \frac{R}{\mathfrak{p}}}(\frac{R}{\mathfrak{p}})} \mathfrak{q}) = \mathfrak{q}$. It follows that $(0 :_{\mathrm{H}_{\mathfrak{a}}^{\dim \frac{R}{\mathfrak{p}}}(\frac{R}{\mathfrak{p}})} \mathfrak{a})$ is not finitely generated. But by [3, Theorem 3], Artinian local cohomology module $\mathrm{H}_{\mathfrak{a}}^{\dim \frac{R}{\mathfrak{p}}}(\frac{R}{\mathfrak{p}})$ is $\mathfrak{a}$-cofinite, and this is a contradiction. □

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