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ON SOME DESIGNS CONSTRUCTED FROM THE GROUPS $PSL_2(q)$, $q = 53, 61, 64$

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Abstract. In this paper, we use the primitive permutation representations of the simple groups $PSL_2(53)$, $PSL_2(61)$ and $PSL_2(64)$ and construct 1-designs by the Key-Moori Method 1. It is shown that the groups $PSL_2(53)$, $PSL_2(53)$:2, $PSL_2(61)$, $PSL_2(61)$:2, $PSL_2(64)$, $PSL_2(64)$:2, $PSL_2(64)$:3 and $PSL_2(64)$:6 appear as the full automorphism groups of these obtained designs.

1. Introduction

Combinatorics and Algebra interact in a substantial and nice fashion when we study combinatorial structures using algebraic methods. As it is seen, the combinatorics might be designs and the algebra could be group theory. Construction of Witt designs using Mathieu groups is a classical result which leads us to this interesting interaction. In 2002, a method for constructing 1-designs and regular graphs from the primitive representations of a group, known as the Key-Moori Method 1, is described in [10, 11]. Key and Moori [10] used the primitive

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actions of the Janko groups J_1 and J_2, and proved that J_1 and J_2 appear as the full automorphism group of these combinatorial structures. From a geometric viewpoint, the designs that admitting fairly large automorphism groups are generally most interesting. In [12], the authors applied the Key-Moori method 1 to the groups $PSp_n(q)$, A_6 and A_9. They wanted to obtain 1-designs from a group G such that their automorphism groups and $\mathrm{Aut}(G)$ have no containment relationship. In [6, 7], Darafsheh et al. considered all the primitive permutation representations of the groups $PSL_2(q)$, where $q \leq 35$ is a prime power, and found 1-designs and their automorphism groups. In [8], Darafsheh et al. considered the primitive actions of the groups $PSL_2(q)$, where $q = 37, 41, 43, 47, 49$, and constructed 1-designs and found their automorphism groups. Darafsheh [5] also considered the group $PSL_2(q)$, where q is a power of 2, and found two classes of 1-designs such that one of them is invariant under the full automorphism group S_{q+1}. Moreover, the author [9] examined 1-designs and their automorphism groups constructed from the primitive representations of $PSL_2(59)$.

In this paper, we employ the Key-Moori Method 1 and construct 1-designs from all the primitive permutation representations of the groups $PSL_2(53)$, $PSL_2(61)$ and $PSL_2(64)$. We obtain the parameters of the constructed 1-designs and find their automorphism groups. We show that $PSL_2(53)$, $PSL_2(53){:}2$, $PSL_2(61)$, $PSL_2(61){:}2$, $PSL_2(64)$, $PSL_2(64){:}2$, $PSL_2(64){:}3$ and $PSL_2(64){:}6$ appear as the full automorphism groups of these 1-designs.

2. Terminology and notation

For the structure of groups and their maximal subgroups, we follow the notation of the atlas of finite groups [4]. The groups $G.H$, $G{:}H$ and $G{\cdot}H$ denote a general extension, a split extension and a non-split extension, respectively. A cyclic group of order m is denoted by m. When p is prime, p^n indicates the elementary abelian group of that order.

The incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ consists of a point set $\mathcal{P}$, a block set $\mathcal{B}$ and an incidence relation $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$. The symbol $p\,\mathcal{I}\,B$ means that $(p, B) \in \mathcal{I}$. If $\mathcal{I}$ is the membership relation $\in$ then we can write $p \in B$ instead of $(p, B) \in \mathcal{I}$. The incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a t-(v, k, λ) design if $|\mathcal{P}| = v$, $|B| = k$ for any $B \in \mathcal{B}$ and every t points of $\mathcal{P}$ is incident with exactly λ blocks. Set $|\mathcal{B}| = b$. The design $\mathcal{D}$ is called symmetric if $v = b$. Set λ_s $(s \leq t)$ be the number of blocks through any set of s points. It is deduced that λ_s is independent of the set and equal to $\lambda\binom{\nu-s}{t-s}/\binom{k-s}{t-s}$. Also, $\mathcal{D}$ is an s-(v, k, λ_s) design as well. A t-(v, k, λ) design is trivial if any subset of $\mathcal{P}$ with cardinality k is a block. The dual of the incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is $\mathcal{S}^t = (\mathcal{B}, \mathcal{P}, \mathcal{I}^t)$, where $\mathcal{I}^t = \{(\mathcal{B}, \mathcal{P}) \mid (\mathcal{P}, \mathcal{B}) \in \mathcal{I}\}$. If $\mathcal{D}$ is a t-(v, k, λ) design then $\mathcal{D}^t$ is a design with b points and the block size λ_1. The incidence matrix of $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a $(0, 1)$-matrix M of size $|\mathcal{B}| \times |\mathcal{P}|$ whose rows and columns are labeled by blocks and points, respectively, such that entry (B, p) is 1 if p is incidence with B, and 0 otherwise. It is clear that the incidence matrix of $\mathcal{D}^t$ is M^t, the transpose of M. Two structures $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ and $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathcal{I}')$ are isomorphic if there is a one to one correspondence $\theta : \mathcal{P} \to \mathcal{P}'$ such that $p\ \mathcal{I}\ B \longleftrightarrow \theta(p)\ \mathcal{I}'\ \theta(B)$ for all $p \in \mathcal{P}$ and $B \in \mathcal{B}$. In this case, we write $\mathcal{S} \cong \mathcal{S}'$. The incidence structure $\mathcal{S}$ is called to be self-dual if $\mathcal{S} \cong \mathcal{S}^t$. An isomorphism of $\mathcal{S}$ onto itself is an automorphism of $\mathcal{S}$. The set of all the automorphisms of $\mathcal{S}$ is a group and denoted by $\mathrm{Aut}(\mathcal{S})$. See [1, 3] for further properties of designs.

Let F_q be the Galois field of order q, where $q = p^n$ is prime power, and let $F_q^* = F_q \setminus \{0\}$. Denote by $GL_2(q)$ the group of all the invertible 2×2 matrices over the finite field F_q and let $SL_2(q)$ be the subgroup of $GL_2(q)$ consisting of the matrices with determinant 1. We know that there is a natural action of $GL_2(q)$ on the 1-dimensional subspaces of the vector space F_q^2 such that its kernel is $N = \{\lambda I \mid \lambda \in F_q^*\}$. The projective general linear group $PGL_2(q)$ is the quotient $GL_2(q)/N$. Furthermore, $SL_2(q)$ acts on the same set with the kernel $N \cap SL_2(q)$ and the projective special linear group, denoted by $PSL_2(q)$, is defined to be $SL_2(q)/(N \cap SL_2(q))$. It is known that $|PGL_2(q)| = q(q^2 - 1)$ and $|PSL_2(q)| = q(q^2 - 1)/(2, q - 1)$.

Lemma 2.1. [16] *A maximal subgroup of $PSL_2(q)$ has one of the following shapes:*

- *A dihedral group of order $2(q - \varepsilon)/(2, q - 1)$ except $\varepsilon = 1$, $q = 3, 5, 7, 9, 11$ and $\varepsilon = -1$, $q = 2, 7, 9$.*
- *A solvable group of order $q(q - 1)/(2, q - 1)$.*
- *A_4 if $q > 3$ is prime and $q \equiv 3, 13, 27, 37 \pmod{40}$.*
- *S_4 if q is an odd prime number and $q \equiv \pm 1 \pmod 8$.*
- *A_5 if $q = 5^m, 4^m$ and m is prime, q is prime and $q \equiv \pm 1 \pmod 5$, or q is the square of an odd prime number and $q \equiv -1 \pmod 5$.*
- *$PSL(2, r)$ if $q = r^m$ and m is odd prime.*
- *$PGL(2, r)$ if $q = r^2$.*

For further properties of the linear groups, we refer the reader to [15, 16].

3. The construction method

Suppose that G is a permutation group acting on a set Ω of size n. The group G naturally acts on $\Omega \times \Omega$ by $(\alpha, \beta)^g = (\alpha^g, \beta^g)$ for all $g \in G$ and $\alpha, \beta \in \Omega$. An orbit of G on $\Omega \times \Omega$ is called an orbital. If O is an orbital then $O^* = \{(\alpha, \beta) \mid (\beta, \alpha) \in O\}$, called the paired orbital of O, is also an orbital. The orbital O is self-paired if $O = O^*$. Now, let $\alpha \in \Omega$ and $\Delta \neq \{\alpha\}$ be an orbit of the stabilizer G_α. It is easy to see that $\bar{\Delta} = \{(\alpha, \delta)^g \mid \delta \in \Delta, g \in G\}$ is an orbital. If $\bar{\Delta}$ is a self-paired orbital then Δ is called to be self-paired. In this paper, our construction for 1-designs is based on the following method:

Theorem 3.1. *(Key-Moori Method 1)*[10, 11] *Let G be a finite primitive permutation group acting on a set Ω of size n. Let $\alpha \in \Omega$ and $\Delta \neq \{\alpha\}$ be an orbit of the stabilizer G_α. Then, the incidence structure $\mathcal{D} = (\Omega, \Delta^G)$ is a symmetric 1-$(n, |\Delta|, |\Delta|)$ design. Moreover, if Δ is a self-paired orbit then $\mathcal{D}$ is self-dual and G acts as an automorphism group on $\mathcal{D}$ such that it is primitive on points and blocks of $\mathcal{D}$.*

If Δ be any union of the orbits of G_α, including the singleton orbit $\{\alpha\}$, then (Ω, Δ^G) is still a symmetric 1-design with the group operating. Conversely, if G acts primitively on the points and the blocks of a symmetric 1-design $\mathcal{D}$ then $\mathcal{D} = (\Omega, \Delta^G)$, where Δ is a union of the orbits of a stabilizer [13].

Theorem 3.2. [14] *If $\mathcal{D}$ is a 1-design constructed by the Key-Moori Method 1, then $G \leq$* Aut$(\mathcal{D})$.

4. 1-Designs from the groups $PSL_2(53)$, $PSL_2(61)$ and $PSL_2(64)$

Consider the projective special linear groups $PSL_2(53)$, $PSL_2(61)$ and $PSL_2(64)$. Magma shows that these groups have four, five and five maximal subgroups, up to conjugacy, respectively. Magma gives us the orders of the maximal subgroups and then, as it is noted in Section 2, we can determine their shapes. Now, by a computer proram in Magma [2], let G be one of these linear groups and M be a maximal subgroup of G. If Ω is the set of the right cosets of M in G then G acts primitively on Ω. Choose $\alpha \in \Omega$. Consider the action of G_α on Ω and let Δ be an orbit of the stabilizer such that $|\Delta| \geq 2$. Theorem 3.1 implies that $\mathcal{D} = (\Omega, \Delta^G)$ is a 1-$(n, |\Delta|, |\Delta|)$ symmetric design.

The information we obtain about all the primitive permutation representations of $PSL_2(53)$, $PSL_2(61)$ and $PSL_2(64)$ are listed in Tables 1, 2 and 3, respectively. The shape of a maximal subgroup and its index are written under the columns 'Max. Sub.' and 'Degree', respectively. The headings '#' and 'Length' indicate the number of non-singleton orbits of a stabilizer and their lengths, respectively, such that the entry '$m(n)$' shows that there are n orbits of length m. Moreover, the designs constructed by these orbits are denoted by $\mathcal{D}_{m(n)}$. For a constructed 1-design, the order of the automorphism group is shown in the last column. Computations with Magma show that $PSL_2(53)$, $PSL_2(61)$ and $PSL_2(64)$ have maximal subgroups of orders 1378, 1830 and 4032, respectively, such that they are solvable. For these maximal subgroups, the action of the group on Ω is 2-transitive and so, the constructed 1-design is trivial and will not be considered. Moreover, Magma shows that $PSL_2(61)$ have two maximal subgroups, up to conjugacy, isomorphic to A_5 such that they give us the same results. Therefore, their results are written in a same row.

Max. Sub.	*Degree*	*#*	*Length*	$\lvert \mathrm{Aut}(\mathcal{D}) \rvert$
A_4	*6201*	*532*	*4(16)*	*74412*
			4(1)	*148824*
			6(6)	*148824*
			12(496)	*74412*
			12(12)	*148824*
D_{52}	*1431*	*42*	*13(2)*	*74412*
			26(24)	*74412*
			52(2)	*74412*
			52(13)	*148824*
D_{54}	*1378*	*39*	*27(24)*	*74412*
			27(1)	*148824*
			54(13)	*148824*

Table 1: 1-Designs from the group $PSL_2(53)$

Max. Sub.	*Degree*	*#*	*Length*	$\lvert \mathrm{Aut}(\mathcal{D}) \rvert$
D_{62}	*1830*	*45*	*31(28)*	*113460*
			31(1)	*226920*
			62(15)	*226920*
A_5	*1891*	*41*	*6(1)*	*113460*
			10(1)	*113460*
			12(2)	*113460*
			20(4)	*113460*
			30(5)	*113460*
			60(27)	*113460*
D_{60}	*1891*	*48*	*15(2)*	*113460*
			30(28)	*113460*
			60(2)	*113460*
			60(15)	*226920*

Table 2: 1-Designs from the group $PSL_2(61)$

Max. Sub.	*Degree*	#	*Length*	$\lvert\mathrm{Aut}(\mathcal{D})\rvert$
D_{130}	*2016*	*32*	*65(24)*	*262080*
			65(6)	*524160*
			65(1)	*1572480*
A_5	*4368*	*89*	*12(6)*	*262080*
			15(3)	*524160*
			15(2)	*786240*
			20(6)	*262080*
			20(3)	*524160*
			20(1)	*1572480*
			60(54)	*262080*
			60(9)	*524160*
			60(4)	*786240*
D_{126}	*2080*	*33*	*63(24)*	*262080*
			63(6)	*524160*
			63(1)	*1572480*
			126(1)	*65*!
$PGL_2(8)$	*520*	*9*	*56(3)*	*524160*
			63(1)	*1572480*
			72(3)	*524160*
			72(1)	*1572480*

Table 3: 1-Designs from the group $PSL_2(64)$

Theorem 4.1. *(i) For* $PSL_2(53)$ *of degree 6201, the designs* $\mathcal{D}_{4(16)}$, $\mathcal{D}_{4(1)}$, $\mathcal{D}_{6(6)}$, $\mathcal{D}_{12(496)}$ *and* $\mathcal{D}_{12(12)}$ *with parameters* 1-$(6201, 4, 4)$, 1-$(6201, 4, 4)$, 1-$(6201, 6, 6)$ 1-$(6201, 12, 12)$ *and* 1-$(6201, 12, 12)$ *are obtained, respectively, such that* $\mathrm{Aut}(\mathcal{D}_{4(16)}) = \mathrm{Aut}(\mathcal{D}_{12(496)}) = PSL_2(53)$ *and* $\mathrm{Aut}(\mathcal{D}_{4(1)}) \cong \mathrm{Aut}(\mathcal{D}_{6(6)}) \cong \mathrm{Aut}(\mathcal{D}_{12(12)}) \cong PSL_2(53){:}2$.

(ii) For $PSL_2(53)$ *of degree 1431, the designs* $\mathcal{D}_{13(2)}$, $\mathcal{D}_{26(24)}$, $\mathcal{D}_{52(2)}$ *and* $\mathcal{D}_{52(13)}$ *with parameters* 1-$(1431, 13, 13)$, 1-$(1431, 26, 26)$, 1-$(1431, 52, 52)$ *and* 1-$(1431, 52, 52)$ *are constructed, respectively, such that* $\mathrm{Aut}(\mathcal{D}_{13(2)}) = \mathrm{Aut}(\mathcal{D}_{26(24)}) = \mathrm{Aut}(\mathcal{D}_{52(2)}) = PSL_2(53)$ *and* $\mathrm{Aut}(\mathcal{D}_{52(13)}) \cong PSL_2(53){:}2$.

(iii) For $PSL_2(53)$ *of degree 1378, the three designs* $\mathcal{D}_{27(1)}$*,* $\mathcal{D}_{27(24)}$ *and* $\mathcal{D}_{54(13)}$ *with parameters* 1-$(1378, 27, 27)$*,* 1-$(1378, 27, 27)$ *and* 1-$(1378, 54, 54)$ *are obtained, respectively. Moreover,* $\mathrm{Aut}(\mathcal{D}_{27(24)}) = PSL_2(53)$ *and* $\mathrm{Aut}(\mathcal{D}_{27(1)}) \cong \mathrm{Aut}(\mathcal{D}_{54(13)}) \cong PSL_2(53){:}2$*.*

Proof. By Magma computations and Theorem 3.1, the designs with the above parameters are constructed.

For $\mathcal{D}_{4(16)}$, $\mathcal{D}_{12(496)}$, $\mathcal{D}_{13(2)}$, $\mathcal{D}_{26(24)}$, $\mathcal{D}_{52(2)}$ and $\mathcal{D}_{27(24)}$, Magma shows that their automorphism groups have order 74412. Since $|PSL_2(53)| = 74412$, Theorem 3.2 implies that the automorphism groups of these 1-designs are the same and equal to $PSL_2(53)$.

Computations with Magma show that the automorphism groups of $\mathcal{D}_{4(1)}$, $\mathcal{D}_{6(6)}$, $\mathcal{D}_{12(12)}$, $\mathcal{D}_{52(13)}$, $\mathcal{D}_{27(1)}$ and $\mathcal{D}_{54(13)}$ are isomorphic to each other. Thus, consider the 1-design $\mathcal{D}_{4(1)}$. By Magma, $|\mathrm{Aut}(\mathcal{D}_{4(1)})| = 148824$. Theorem 3.2 implies that there exists a subgroup N of $\mathrm{Aut}(\mathcal{D}_{4(1)})$ isomorphic to $PSL_2(53)$. Since $|\mathrm{Aut}(\mathcal{D}_{4(1)})/N| = 148824/74412 = 2$, $N \trianglelefteq \mathrm{Aut}(\mathcal{D}_{4(1)})$. Moreover, we find such a subgroup N and an involution with the cycle type $2^{675}1^{28}$ in $\mathrm{Aut}(\mathcal{D}_{4(1)}) \setminus N$. So, $\mathrm{Aut}(\mathcal{D}_{4(1)}) \cong PSL_2(53){:}2$. □

Similarly, the following theorem is deduced. Note that $|PSL_2(61)| = 113460$.

Theorem 4.2. *(i) For* $PSL_2(61)$ *of degree 1830, the designs* $\mathcal{D}_{31(28)}$*,* $\mathcal{D}_{31(1)}$ *and* $\mathcal{D}_{62(15)}$ *with parameters* 1-$(1830, 31, 31)$*,* 1-$(1830, 31, 31)$ *and* 1-$(1830, 62, 62)$ *are constructed, respectively, such that* $\mathrm{Aut}(\mathcal{D}_{31(28)}) = PSL_2(61)$ *and* $\mathrm{Aut}(\mathcal{D}_{31(1)}) \cong \mathrm{Aut}(\mathcal{D}_{62(15)}) \cong PSL_2(61){:}2$*.*

(ii) For $PSL_2(61)$ *of degree 1891, the designs* $\mathcal{D}_{6(1)}$*,* $\mathcal{D}_{10(1)}$*,* $\mathcal{D}_{12(12)}$*,* $\mathcal{D}_{20(4)}$*,* $\mathcal{D}_{30(5)}$ *and* $\mathcal{D}_{60(27)}$ *with parameters* 1-$(1891, 6, 6)$*,* 1-$(1891, 10, 10)$*,* 1-$(1891, 12, 12)$*,* 1-$(1891, 20, 20)$*,* 1-$(1891, 30, 30)$ *and* 1-$(1891, 60, 60)$ *are obtained, respectively, such that their automorphism groups are the same and equal to* $PSL_2(61)$*.*

(iii) For $PSL_2(61)$ *of degree 1891, the four designs* $\mathcal{D}_{15(2)}$*,* $\mathcal{D}_{30(28)}$*,* $\mathcal{D}_{60(2)}$ *and* $\mathcal{D}_{60(15)}$ *with parameters* 1-$(1891, 15, 15)$*,* 1-$(1891, 30, 30)$*,* 1-$(1891, 60, 60)$ *and* 1-$(1891, 60, 60)$ *are obtained, respectively, such that* $\mathrm{Aut}(\mathcal{D}_{15(2)}) = \mathrm{Aut}(\mathcal{D}_{30(28)}) = \mathrm{Aut}(\mathcal{D}_{60(2)}) = PSL_2(61)$ *and* $\mathrm{Aut}(\mathcal{D}_{60(15)}) \cong PSL_2(61){:}2$*.*

Theorem 4.3. *(i) For* $PSL_2(64)$ *of degree 2016, the designs* $\mathcal{D}_{65(24)}$*,* $\mathcal{D}_{65(6)}$ *and* $\mathcal{D}_{65(1)}$ *with the same parameters* 1-$(2016, 65, 65)$ *are obtained such that* $\mathrm{Aut}(\mathcal{D}_{65(24)}) = PSL_2(64)$*,* $\mathrm{Aut}(\mathcal{D}_{65(6)}) \cong PSL_2(64){:}2$ *and* $\mathrm{Aut}(\mathcal{D}_{65(1)}) \cong PSL_2(64){:}6$*.*

(ii) For $PSL_2(64)$ *of degree 4368, the nine designs* $\mathcal{D}_{12(6)}$*,* $\mathcal{D}_{15(3)}$*,* $\mathcal{D}_{15(2)}$*,* $\mathcal{D}_{20(6)}$*,* $\mathcal{D}_{20(3)}$*,* $\mathcal{D}_{20(1)}$*,* $\mathcal{D}_{60(54)}$*,* $\mathcal{D}_{60(9)}$ *and* $\mathcal{D}_{60(4)}$ *with parameters* 1-$(4368, 12, 12)$*,* 1-$(4368, 15, 15)$*,* 1-$(4368, 15, 15)$*,* 1-$(4368, 20, 20)$*,* 1-$(4368, 20, 20)$*,* 1-$(4368, 20, 20)$*,* 1-$(4368, 60, 60)$*,* 1-$(4368, 60, 60)$ *and* 1-$(4368, 60, 60)$ *are constructed, respectively, such that* $\mathrm{Aut}(\mathcal{D}_{12(6)}) =$

$\mathrm{Aut}(\mathcal{D}_{20(6)}) = \mathrm{Aut}(\mathcal{D}_{60(54)}) = PSL_2(64)$, $\mathrm{Aut}(\mathcal{D}_{15(3)}) \cong \mathrm{Aut}(\mathcal{D}_{20(3)}) \cong \mathrm{Aut}(\mathcal{D}_{60(9)}) \cong PSL_2(64){:}2$, $\mathrm{Aut}(\mathcal{D}_{15(2)}) \cong \mathrm{Aut}(\mathcal{D}_{60(4)}) \cong PSL_2(64){:}3$ *and* $\mathrm{Aut}(\mathcal{D}_{20(1)}) \cong PSL_2(64){:}6$.

(iii) For $PSL_2(64)$ *of degree 2080, the designs* $\mathcal{D}_{63(24)}$, $\mathcal{D}_{63(6)}$, $\mathcal{D}_{63(1),1}$ *and* $\mathcal{D}_{126(1)}$ *with parameters* 1-$(2080, 63, 63)$, 1-$(2080, 63, 63)$, 1-$(2080, 63, 63)$ *and* 1-$(2080, 126, 126)$ *are obtained, respectively, such that* $\mathrm{Aut}(\mathcal{D}_{63(24)}) = PSL_2(64)$, $\mathrm{Aut}(\mathcal{D}_{63(6)}) \cong PSL_2(64){:}2$, $\mathrm{Aut}(\mathcal{D}_{63(1),1}) \cong PSL_2(64){:}6$ *and* $\mathrm{Aut}(\mathcal{D}_{126(1)}) \cong S_{65}$.

(iv) For $PSL_2(64)$ *of degree 520, the four designs* $\mathcal{D}_{56(3)}$, $\mathcal{D}_{63(1),2}$, $\mathcal{D}_{72(3)}$ *and* $\mathcal{D}_{72(1)}$ *with parameters* 1-$(520, 56, 56)$, 1-$(520, 63, 63)$, 1-$(520, 72, 72)$ *and* 1-$(520, 72, 72)$ *are obtained, respectively, such that* $\mathrm{Aut}(\mathcal{D}_{56(3)}) \cong \mathrm{Aut}(\mathcal{D}_{72(3)}) \cong PSL_2(64){:}2$ *and* $\mathrm{Aut}(\mathcal{D}_{63(1),2}) \cong \mathrm{Aut}(\mathcal{D}_{72(1)}) \cong PSL_2(64){:}6$.

Proof. Computations with Magma and Theorem 3.1 give us the designs with the above parameters.

Magma shows that the orders of the automorphism groups of $\mathcal{D}_{65(24)}$, $\mathcal{D}_{12(6)}$, $\mathcal{D}_{20(6)}$, $\mathcal{D}_{60(54)}$ and $\mathcal{D}_{63(24)}$ are 262080. By Theorem 3.2, the automorphism groups of these 1-designs are the same and equal to $PSL_2(64)$ since $|PSL_2(64)| = 262080$.

Magma computations show that the automorphism groups of $\mathcal{D}_{65(6)}$, $\mathcal{D}_{15(3)}$, $\mathcal{D}_{20(3)}$, $\mathcal{D}_{60(9)}$, $\mathcal{D}_{63(6)}$, $\mathcal{D}_{56(3)}$ and $\mathcal{D}_{72(3)}$ are isomorphic to each other. Consider the 1-design $\mathcal{D}_{65(6)}$. By Theorem 3.2 and Magma, we have $PSL_2(64) \leq \mathrm{Aut}(\mathcal{D}_{65(6)})$ and $|\mathrm{Aut}(\mathcal{D}_{65(6)})| = 524160 = 2|PSL_2(64)|$. Furthermore, we find a subgroup $N \leq \mathrm{Aut}(\mathcal{D}_{65(6)})$ such that $N \cong PSL_2(64)$ and an involution with the cycle type $2^{2142}1^{84}$ in $\mathrm{Aut}(\mathcal{D}_{65(6)}) \setminus N$. Therefore, $\mathrm{Aut}(\mathcal{D}_{65(6)}) \cong PSL_2(64){:}2$.

By Magma, $\mathrm{Aut}(\mathcal{D}_{15(2)}) \cong \mathrm{Aut}(\mathcal{D}_{60(4)})$ and $|\mathrm{Aut}(\mathcal{D}_{15(2)})| = 786240 = 3|PSL_2(64)|$. Magma computations show that there exists a normal subgroup N of $\mathrm{Aut}(\mathcal{D}_{15(2)})$ such that $N \cong PSL_2(64)$. Also, we find a permutation with the cycle type $3^{1449}1^{21}$ in $\mathrm{Aut}(\mathcal{D}_{65(6)}) \setminus N$. This implies that $\mathrm{Aut}(\mathcal{D}_{15(2)}) \cong PSL_2(64){:}3$.

We can see that the automorphism groups of $\mathcal{D}_{65(1)}$, $\mathcal{D}_{20(1)}$, $\mathcal{D}_{63(1),1}$, $\mathcal{D}_{63(1),2}$ and $\mathcal{D}_{72(1)}$ are isomorphic to each other and $|\mathrm{Aut}(\mathcal{D}_{65(1)})| = 1572480 = 6|PSL_2(64)|$. By computations with Magma, we find a normal subgroup N of $\mathrm{Aut}(\mathcal{D}_{65(1)})$ such that $N \cong PSL_2(64)$. Moreover, we find a cyclic subgroup H of $\mathrm{Aut}(\mathcal{D}_{65(1)})$ such that $|H| = 6$ and $N \cap H = 1$. So, $\mathrm{Aut}(\mathcal{D}_{65(1)}) \cong PSL_2(64){:}6$.

Finally, consider the 1-design $\mathcal{D}_{126(1)}$. By Magma, $|\mathrm{Aut}(\mathcal{D}_{126(1)})| = 65!$ and a composition series for $\mathrm{Aut}(\mathcal{D}_{126(1)})$ is $1 \leq A_{65} \leq \mathrm{Aut}(\mathcal{D}_{126(1)})$. This implies that $\mathrm{Aut}(\mathcal{D}_{126(1)}) \cong A_{65}{:}2 \cong S_{65}$. □

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References

[1] E. F. Assmus, Jr. and J. D. Key, *Designs and Their Codes*, Cambridge University Press, Cambridge, (1992).

[2] W. Bosma and J. Cannon, *Handbook of Magma Functions*, Department of Mathematics, University of Sydney, (1994), `http://www.magma.maths.usyd.edu.au/magma/`.

[3] T. Beth, D. Jungnickel and H. Lenz, *Design Theory* Vol. 1, (second edition), Cambridge University Press, Cambridge, (1999).

[4] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups*, Oxford University Press, Oxford, (1985).

[5] M. R. Darafsheh, *Designs from the Group $PSL_2(q)$, q Even*, Des. Codes Cryptogr. 39 (2006), pp. 311–316.

[6] M. R. Darafsheh, A.R. Ashrafi and M. Khademi, *Some Designs Related to Group Actions*, Ars. Combin. 86 (2008), pp. 65–75.

[7] M. R. Darafsheh, A.R. Ashrafi and M. Khademi, *On Designs Constructed by Group Actions*, J. Combin. Math. Combin. Comput. 70 (2009), pp. 235–245.

[8] M. R. Darafsheh, A. Iranmanesh and R. Kahkeshani, *Designs from the Groups $PSL_2(q)$ for Certain q*, Quaest. Math. 32 (2009), pp. 1–10.

[9] R. Kahkeshani, *1-Designs from the Group $PSL_2(59)$ and Their Automorphism Groups*, Math. Interdisc. Res. 3 (2018), pp. 147–158.

[10] J. D. Key and J. Moori, *Designs, Dodes and Graphs from the Janko Groups J_1 and J_2*, J. Combin. Math. Combin. Comput. 40 (2002), pp. 143–159.

[11] J. D. Key and J. Moori, *Correction to: Codes, Designs and Graphs from the Janko Groups J_1 and J_2, J. D. Key and J. Moori, JCMCC 40 (2002), pp. 143–159*, J. Combin. Math. Combin. Comput. 64 (2008), pp. 153.

[12] J. D. Key, J. Moori and B.G. Rodrigues, *On some Designs and Codes from Primitive Representations of some Finite Simple Groups*, J. Combin. Math. Combin. Comput. 45 (2002), pp. 3-19.

[13] J. Moori, *Finite Groups, Designs and Codes*, Information Security, Coding Theory and Related Combinatorics, NATO Sci. Peace Secur. Ser. D Inf. Commun. Secur. Vol. 29 (2011), pp. 202-230.

[14] B. Rodrigues, *Codes of Designs and Graphs from Finite Simple Groups*, Ph.D. Thesis, University of Natal, South Africa, (2003).

[15] J. J. Rotman, *An Introduction to the Theory of Groups*, (fourth edition), Springer-Verlag, Berlin, (1995).

[16] M. Suzuki, *Group Theory I*, Springer-Verlag, Berlin, (1982).

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