Journal of Algebraic Structures and Their Applications
ISSN: 2382-9761
www.as.yazd.ac.ir

Algebraic Structures and Their Applications Vol. 7 No. 1 (2020) pp 101-115.

SOME CATEGORICAL STRUCTURES OF GENERALIZED TOPOLOGIES IN TERMS OF MONOTONE OPERATORS

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Abstract. In this paper, we give some generalized categories of topological spaces in terms of monotone operators and investigate some categorical properties of them. In particular, we present some equivalent categories of generalized topological spaces in terms of closure and interior operators. Also, we study the properties of some classes of morphisms as final, initial, closed and open morphisms in these categories.

1. Introduction and preliminaries

General topology is important in many fields of applied sciences as well as branches of mathematics. In reality it is used in data mining, computational topology for geometric design and molecular design, computer-aided design, computer-aided geometric design, digital topology, information system, particle physics, quantum physics and etc. The theory of generalized topological spaces, which was founded by Á. Császár [3], is one of the most important developments of general topology. He used monotone mappings from the power set of a nonempty set X to itself and introduced the notions of generalized neighborhood systems and generalized topological spaces. He also introduced the notions of generalized continuous maps and

DOI:10.29252/as.2020.1734

MSC(2010): Primary: 65F05, 46L05, 11Y50.

Keywords: Generalized topology, generalized category, monotone operator, closure and interior operator.

Received: 17 Nov 2019, Accepted: 21 Feb 2020.

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associated interior and closure operators on generalized topological spaces and other concepts [3–8]. Then, many authors studied some topological notions in such spaces, such as separation axioms [14], weak continuity [10, 11] and other notions [9].

In this paper, we give some generalized and modification categories of topological spaces by monotone operators and investigate some categorical properties of them. In particular, we study the properties of some classes of morphisms, such as final, initial, closed and open morphisms in these categories.

In the following, we recall some notions and notations defined in [3]. A mapping $\gamma : P(X) \to P(X)$ defined on the power set $P(X)$ of a set X is said to be *monotone* provided that $A \subseteq B \subseteq X$ implies $\gamma A \subseteq \gamma B$, where we write γA for $\gamma(A)$. The pair (X, γ) is called a Γ-*space*. A set $A \subseteq X$ is said to be γ-*open* provided that $A \subseteq \gamma A$; γ-*closed* provided that $\gamma A \subseteq A$ and the collection μ_γ of all γ-open sets is a generalized topology in the sense of [3], where a subset μ of $P(X)$ is called a *generalized topology* (briefly GT) on X and the pair (X, μ) is called a *generalized topological space* (briefly GTS) if $\emptyset \in \mu$ and any union of elements of μ belongs to μ. A GTS (X, μ) is called *strong* if $X \in \mu$. Also the collection $\mu^*_\gamma = \{A \mid \gamma(X - A) \subseteq X - A\}$ is a GT on X.

A monotone map $\gamma : P(X) \to P(X)$ is said to be:

(1) *idempotent* if $\gamma^2 A = \gamma\gamma A = \gamma A$ for $A \subseteq X$;
(2) *restricting* if $\gamma A \subseteq A$ for $A \subseteq X$;
(3) *enlarging* if $A \subseteq \gamma A$ for $A \subseteq X$;
(4) $\vee$-*additive* if $\gamma(A \cup B) = \gamma A \cup \gamma B$ for $A, B \subseteq X$;
(5) $\wedge$-*additive* if $\gamma(A \cap B) = \gamma A \cap \gamma B$ for $A, B \subseteq X$.

The conjugate of a monotone map γ is defined by $\gamma^* A = X - \gamma(X - A)$ for $A \subseteq X$. Clearly (X, γ^*) is a Γ-space. If μ is a GT on X, then the interior operator $i_\mu : P(X) \to P(X)$ defined by $i_\mu A = \bigcup\{M \in \mu \mid M \subseteq A\}$ is monotone, idempotent and restricting; and the closure operator $c_\mu : P(X) \to P(X)$ defined by $c_\mu A = \bigcap\{N \mid A \subseteq N, X - N \in \mu\}$ is monotone, idempotent and enlarging.

A mapping $f : (X, \mu_X) \to (Y, \mu_Y)$ between GTS's is said to be *g-continuous* if $f^{-1}(B) \in \mu_X$ whenever $B \in \mu_Y$ [3, 10]. We denote by **Top** and **GenTop** the category of all topological spaces with continuous maps; and the category of all generalized topological spaces with g-continuous maps, respectively. In the following sections, readers are suggested to refer to [1] for some categorical notions.

2. Generalized categories of **Top**

In this section, we present some generalized and modification categories of topological spaces in terms of closure and interior operators. Recall that every monotone and restricting operator is called an interior operator, and every monotone and enlarging operator is called a closure operator.

Definition 2.1. Let $f:(X,\gamma)\to(Y,\delta)$ be a mapping between Γ-spaces. We say that f is *i-continuous* if $f^{-1}(\delta B)\subseteq\gamma f^{-1}(B)$ for all subset B of Y; and *c-continuous* if $\gamma f^{-1}(B)\subseteq f^{-1}(\delta B)$ for all subset B of Y, or equivalently, $f(\gamma A)\subseteq\delta f(A)$ for all subset A of X.

We denote by:

(1) Γ and Γ^* the category of all Γ-spaces and i-continuous maps, and the category of all Γ-spaces and c-continuous maps, respectively;

(2) Γ_r and Γ^*_e the full subcategories of Γ and Γ^* of all restricting maps, and of all enlarging maps, respectively;

(3) Γ_{ir} and Γ^*_{ie} the full subcategories of Γ_r and Γ^*_e of all idempotent maps, respectively;

(4) $\Gamma_{\wedge ir}$ and $\Gamma^*_{\vee ie}$ the full subcategories of Γ_{ir} and Γ^*_{ie} of all $\wedge$-additive maps; and of all $\vee$-additive maps, respectively.

Recall that Γ^*_{ie} is often called the category of closure spaces [13].
The following lemma is an immediate consequence of the definitions of monotone operators.

Lemma 2.2. *Let γ be a monotone map on $P(X)$. Then the following statements hold:*

(1) $(\gamma^*)^*=\gamma$, $\mu^*_{\gamma^*}=\mu_\gamma$ *and* $\mu^*_\gamma=\mu_{\gamma^*}$.

(2) $(\gamma^*)^2=\gamma^*$ *if and only if* $\gamma^2=\gamma$.

(3) γ *is restricting if and only if* γ^* *is enlarging.*

(4) $A\subseteq X$ *is* γ*-open if and only if* $X-A$ *is* γ^**-closed.*

(5) γ *is* $\wedge$*-additive if and only if* γ^* *is* $\vee$*-additive.*

(6) $f:(X,\gamma)\to(Y,\delta)$ *is* i*-continuous if and only if* $f:(X,\gamma^*)\to(Y,\delta^*)$ *is* c*-continuous.*

Theorem 2.3. *We have the following isomorphisms of categories:*

$$\Gamma\cong\Gamma^*,\quad \Gamma_r\cong\Gamma^*_e,\quad \Gamma_{ir}\cong\Gamma^*_{ie},\quad \Gamma_{\wedge ir}\cong\Gamma^*_{\vee ie}.$$

Proof. Define the functor F by $F((X,\gamma)\overset{f}{\to}(Y,\delta))=(X,\gamma^*)\overset{f}{\to}(Y,\delta^*)$. By Lemma 2.2, it is easy to show that F is an isomorphism in any of the four parts. □

Remark 2.4. It is well know that a mapping $f:(X,\mu)\to(Y,\lambda)$ between topological spaces (GTS's) is continuous (g-continuous) if and only if $f:(X,i_\mu)\to(Y,i_\lambda)$ is i-continuous if and only if $f:(X,c_\mu)\to(Y,c_\lambda)$ is c-continuous. Now, if $f:(X,\gamma)\to(Y,\delta)$ between Γ-spaces is i-continuous, then $f:(X,\mu_\gamma)\to(Y,\mu_\delta)$ is g-continuous. Conversely, if $f:(X,\mu_\gamma)\to(Y,\mu_\delta)$ is g-continuous such that γ and δ are both idempotent and restricting, then for every subset B of Y, $\delta B\in\mu_\delta$, so $f^{-1}(\delta B)\subseteq\gamma f^{-1}(\delta B)\subseteq\gamma f^{-1}(B)$. Thus $f:(X,\gamma)\to(Y,\delta)$ is i-continuous.

Theorem 2.5. (1) *The categories* Γ_{ir}, Γ^*_{ie} *and* **GenTop** *are isomorphic.*

(2) *The categories* $\Gamma_{\wedge ir}$, $\Gamma^*_{\vee ie}$ *and* **Top** *are isomorphic.*

Proof. Define the functors $F : \Gamma_{ir} \to \mathbf{GenTop}$ and $G : \mathbf{GenTop} \to \Gamma_{ir}$ as follows:

$$F((X,\gamma)\xrightarrow{f}(Y,\delta)) = (X,\mu_\gamma)\xrightarrow{f}(Y,\mu_\delta) \quad \text{and} \quad G((X,\mu)\xrightarrow{f}(Y,\lambda)) = (X,i_\mu)\xrightarrow{f}(Y,i_\lambda).$$

If (X,μ) is a GTS, then $\mu_{i_\mu} = \{A \subseteq X \mid A \subseteq i_\mu(A)\} = \{A \subseteq X \mid A = i_\mu(A)\} = \mu$. So $F \circ G(X,\mu) = F(X,i_\mu) = (X,\mu_{i_\mu}) = (X,\mu)$. On the other hand, if $(X,\gamma) \in \Gamma_{ir}$, then it is easy to show that $i_{\mu_\gamma} = \gamma$. So $G \circ F(X,\gamma) = (X,i_{\mu_\gamma}) = (X,\gamma)$. Thus F is an isomorphism. Finally, if $(X,\gamma) \in \Gamma_{\wedge ir}$, then $F(X,\gamma) \in \mathbf{Top}$, which shows that F is an isomorphism from $\Gamma_{\wedge ir}$ to **Top**. Now, by Theorem 2.3, the proof is complete. □

Theorem 2.6. GenTop *is fully embeddable into any of the categories* Γ, Γ_r, Γ^* *and* Γ^*_e *as a reflective subcategory.*

Proof. Let the functor $G : \mathbf{GenTop} \to \Gamma(\Gamma_r)$ be defined by:

$$G((X,\mu)\xrightarrow{f}(Y,\lambda)) = (X,i_\mu)\xrightarrow{f}(Y,i_\lambda).$$

It is clear that G is faithful. If $G(X,\mu) = G(Y,\lambda)$, then $X = Y$ and $i_\mu = i_\lambda$, so $\mu = \lambda$, which shows that G is injective on objects. If $f : (X,i_\mu) \to (Y,i_\lambda)$ is i-continuous, then by Remark 2.4, $f : (X,\mu) \to (Y,\lambda)$ is g-continuous, so G is full. Hence G is a full embedding. Now, we show that G is an adjoint functor. Let (X,γ) be an object in Γ or Γ_r. Since $i_{\mu_\gamma}A \subseteq \gamma(i_{\mu_\gamma}A) \subseteq \gamma A$ for every $A \subseteq X$, it follows that the identity map $id : (X,\gamma) \to (X,i_{\mu_\gamma})$ is i-continuous. Suppose that (Y,λ) is a GTS and $f : (X,\gamma) \to G(Y,\lambda) = (Y,i_\lambda)$ is an i-continuous map. By remark 2.4, $\bar{f} = f : (X,\mu_\gamma) \to (Y,\mu_{i_\lambda} = \lambda)$ is g-continuous, so the unique map satisfying the condition $G(\bar{f}) \circ id = f$. Thus id is a G-universal arrow for (X,γ). Now, by Theorem 2.3, the proof is complete. □

The following diagram summarizes the previous results, where we use the notations $\cong$, $\uparrow$ and $\hookrightarrow$ for isomorphic, full subcategory and reflective full subcategory, respectively.

$$\begin{array}{ccccc}
\Gamma & \hookleftarrow & \mathbf{GenTop} & \hookrightarrow & \Gamma^* \\
\uparrow & & \shortparallel & & \uparrow \\
\Gamma_r & \hookleftarrow & \mathbf{GenTop} & \hookrightarrow & \Gamma^*_e \\
\uparrow & & \shortparallel & & \uparrow \\
\Gamma_{ir} & \cong & \mathbf{GenTop} & \cong & \Gamma^*_{ie} \\
\uparrow & & \uparrow & & \uparrow \\
\Gamma_{\wedge ir} & \cong & \mathbf{Top} & \cong & \Gamma^*_{\vee ie}
\end{array}$$

3. Initial and final morphisms

In this section, we study the notions of initial and final morphisms with respect to closure and interior operators. Let $(\mathbf{A}, |-|)$ be a concrete category over a category $\mathbf{X}$. An $\mathbf{A}$-morphism $f : A \to B$ is called *initial* provided that for any $\mathbf{A}$-object C an $\mathbf{X}$-morphism $g : |C| \to |A|$ is an $\mathbf{A}$-morphism whenever $f \circ g : |C| \to |B|$ is an $\mathbf{A}$-morphism. An initial morphism $f : A \to B$ that has a monomorphic underlying $\mathbf{X}$-morphism $f : |A| \to |B|$ is called an *embedding*. The concepts of *final morphism* and *quotient morphism* are dual to the concepts of initial morphism and embedding, respectively. A concrete category over the category **Set** of sets is called a construct [1].

Remark 3.1. In the construct **Top** (**GenTop**) a continuous (g-continuous) map $f : (X, \tau) \to (Y, \sigma)$ is initial if and only if τ is the initial topology (initial generalized topology) with respect to f and σ, i.e., $\tau = \{f^{-1}(S) \mid S \in \sigma\}$. Thus embeddings are precisely the topological embeddings, i.e., homeomorphisms onto subspaces. Dually, f is final if and only if σ is the final topology (final generalized topology) with respect to f and τ, i.e., $\sigma = \{A \subseteq Y \mid f^{-1}(A) \in \tau\}$. Thus the quotient morphisms are the topological (generalized topological) quotient maps [1].

In an arbitrary category with a subject structure and a closure operator, the notions of initial and final morphisms and some properties of them were introduced in [2, 12]. Similarly, we have the following definition.

Definition 3.2. A mapping $f : (X, \gamma) \to (Y, \delta)$ between Γ-spaces is called:

(1) *c-final* if $\delta B = f(\gamma f^{-1}(B))$ for all subset B of Y;
(2) *c-initial* if $\gamma A = f^{-1}(\delta f(A))$ for all subset A of X;
(3) *i-final* if $\delta^* B = f(\gamma^* f^{-1}(B))$ for all subset B of Y;
(4) *i-initial* if $\gamma^* A = f^{-1}(\delta^* f(A))$ for all subset A of X.

It is easy to show that c-final and c-initial maps are c-continuous and i-final and i-initial maps are i-continuous. Thus we study the properties of c-final and c-initial maps in Γ^* and its full subcategories (i.e., Γ^*_e, Γ^*_{ie}, $\Gamma^*_{\vee ie}$); and the properties of i-final and i-initial maps in Γ and its full subcategories (i.e., Γ_r, Γ_{ir}, $\Gamma^*_{\wedge ir}$).

Theorem 3.3. *In the construct Γ^* or any of its full subcategories, a mapping f is c-initial if and only if it is an initial morphism.*

Proof. Suppose that $f : (X, \gamma) \to (Y, \delta)$ is c-initial in one of the constructs Γ^*, Γ^*_e, Γ^*_{ie}, $\Gamma^*_{\vee ie}$, (Z, α) a Γ-space and $h : Z \to X$ is a function such that $f \circ h$ is c-continuous. Then, we have $h^{-1}(\gamma A) = h^{-1} f^{-1}(\delta f(A)) \supseteq \alpha(h^{-1} f^{-1}(f(A))) \supseteq \alpha(h^{-1}(A))$ for every $A \subseteq X$. Thus $h : (Z, \alpha) \to (X, \gamma)$ is c-continuous. Hence f is an initial morphism.

Conversely, let f be an initial morphism in the construct Γ^*. We define $\gamma' : P(X) \to P(X)$ by $\gamma'(A) = f^{-1}(\delta f(A))$ for every subset A of X. Clearly, (X, γ') is a Γ-space. Consider the identity map $i : (X, \gamma') \to (X, \gamma)$, we have $f \circ i(\gamma' A) = f \circ i(f^{-1}(\delta f(A))) = f(f^{-1}(\delta f(A))) \subseteq \delta f(A) = \delta(f \circ i(A))$. This implies that $f \circ i$ is c-continuous and since f is initial, so i is c-continuous. Thus $\gamma A = i^{-1}(\gamma(A)) \supseteq \gamma'(i^{-1}A) = \gamma' A = f^{-1}(\delta f(A)) \supseteq \gamma f^{-1} f(A) \supseteq \gamma A$. Hence $\gamma A = f^{-1}(\delta f(A))$. If δ is enlarging, then $\gamma'(A) = f^{-1}(\delta f(A)) \supseteq f^{-1} f(A) \supseteq A$ for every subset A of X. Thus γ' is enlarging. If δ is also idempotent, then ${\gamma'}^2(A) = f^{-1}\delta f(f^{-1}\delta f(A)) \subseteq f^{-1}(\delta^2 f(A)) = f^{-1}(\delta f(A)) = \gamma' A$ for every subset A of X. Hence $\gamma'^2 = \gamma'$. Finally, if δ is $\vee$-additive, then $\gamma'(A \cup B) = f^{-1}(\delta f(A \cup B)) = f^{-1}(\delta f(A)) \cup f^{-1}(\delta f(B)) = \gamma'(A) \cup \gamma'(B)$ for any subsets A and B of X. Thus, in the constructs Γ^*_e, Γ^*_{ie} and $\Gamma^*_{\vee ie}$ the result holds. □

Dually, similar to the proof of Theorem 3.3 and by Lemma 2.2, we have the following theorem.

Theorem 3.4. *In the construct Γ or any of its full subcategories, a mapping f is i-initial if and only if it is an initial morphism.*

Since every isomorphism functor between concrete categories preserves initial morphisms, by Theorems 2.5 and 3.3 we have the following result.

Corollary 3.5. *In the construct* **Top** (**GenTop**) *a continuous (g-continuous) map $f : (X, \tau) \to (Y, \sigma)$ is initial if and only if $c_\tau A = f^{-1}(c_\sigma f(A))$ for every subset A of X.*

Lemma 3.6. *In the construct Γ^* or any of its full subcategories, every c-final mapping is a final morphism.*

Proof. Suppose that $f : (X, \gamma) \to (Y, \delta)$ is c-final, (Z, α) a Γ-space and $g : Y \to Z$ is a function such that $g \circ f$ is c-continuous. Then, we have $g(\delta B) = gof(\gamma f^{-1}(B)) \subseteq \alpha(gof(f^{-1}(B))) \subseteq \alpha(g(B))$ for every $B \subseteq Y$. Thus $g : (Y, \delta) \to (Z, \alpha)$ is c-continuous. Hence f is a final morphism. □

Remark 3.7. We point out that in the constructs Γ^*_e, Γ^*_{ie} and $\Gamma^*_{\vee ie}$ finality does not characterize c-final maps. For example, let $X = Y = \{1, 2\}$, $\tau = \{\emptyset, \{2\}, X\}$, $\sigma = P(Y)$ and $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(1) = f(2) = 1$. Then, by Remark 3.1, f is a final morphism in **Top** and hence in **GenTop**. Since every isomorphism functor between concrete categories preserves final morphisms, it follows that $f : (X, c_\tau) \to (Y, c_\sigma)$ is a final morphism in $\Gamma^*_{\vee ie}$ and Γ^*_{ie}. But $c_\sigma\{2\} = \{2\}$ and $f(c_\tau f^{-1}(\{2\}) = \emptyset$. Thus f is not c-final.

Theorem 3.8. *(1) In the construct Γ^* a mapping f is c-final if and only if it is a final morphism.*
(2) In the construct Γ^*_e a mapping f is c-final if and only if it is a surjective final morphism.
(3) In any of the constructs Γ^*_{ie} and $\Gamma^*_{\vee ie}$ a mapping f is c-final if it is a bijective final morphism.

Proof. (1): By Lemma 3.6, every c-final mapping is a final morphism. Conversely, let $f : (X,\gamma) \to (Y,\delta)$ be a final morphism in Γ^*. We define $\delta' : P(Y) \to P(Y)$ by $\delta'(B) = f(\gamma f^{-1}(B))$ for every subset B of Y. Clearly, (Y,δ') is a Γ-space. Consider the identity map $i : (Y,\delta) \to (Y,\delta')$, we have $\delta'(iof(A)) = f(\gamma f^{-1}(iof(A))) = f(\gamma f^{-1}(f(A))) \supseteq f(\gamma A) = iof(\gamma A)$. This implies that iof is c-continuous and since f is final, so i is c-continuous. Thus $\delta B = i(\delta B) \subseteq \delta'(i(B)) = \delta'(B) = f(\gamma f^{-1}(B)) \subseteq f(f^{-1}(\delta B)) \subseteq \delta B$. Hence $\delta B = f(\gamma f^{-1}(B))$. Duality, let f be an initial morphism. We define $\gamma' : P(X) \to P(X)$ by $\gamma'(A) = f^{-1}(\delta f(A))$ for every subset A of X. Clearly, (X,γ') is a Γ-space. Consider the identity map $i : (X,\gamma') \to (X,\gamma)$, we have $f \circ i(\gamma' A) = f \circ i(f^{-1}(\delta f(A))) = f(f^{-1}(\delta f(A))) \subseteq \delta f(A) = \delta f \circ i(A)$. This implies that $f \circ i$ is c-continuous and since f is initial, so i is c-continuous. Thus $\gamma A = i^{-1}(\gamma(A)) \supseteq \gamma'(i^{-1}A) = \gamma' A = f^{-1}(\delta f(A)) \supseteq \gamma f^{-1} f(A) \supseteq \gamma A$. Hence $\gamma A = f^{-1}(\delta f(A))$.
(2): Let $f : (X,\gamma) \to (Y,\delta)$ be a c-final morphism in Γ^*_e. Then, $f(X) = f(\gamma X) = f(\gamma f^{-1} Y) = \delta Y = Y$. Thus f is surjective and hence by Lemma 3.6, the result holds. Conversely, similar to the proof of part (1), it is enough to show that the monotone operator δ' is enlarging. Since f is surjective, it follows that $\delta' B = f(\gamma f^{-1}(B)) \supseteq f f^{-1}(B) = B$ for every subset B of Y. Thus δ' is enlarging.
(3): Let $f : (X,\gamma) \to (Y,\delta)$ be a bijective final morphism in Γ^*_{ie}. Then, by part (2), we show that the monotone operator δ' is idempotent. For every subset B of Y we have $\delta'^2 B = f(\gamma f^{-1} f(\gamma f^{-1}(B))) = f(\gamma^2 f^{-1}(B)) = f(\gamma f^{-1}(B)) = \delta' B$. Finally, if γ is $\vee$-additive, then $\delta'(A \cup B) = f(\gamma f^{-1}(A \cup B)) = f(\gamma f^{-1}(A) \cup f(\gamma f^{-1}(B) = \delta'(A) \cup \delta'(B)$ for any two subsets A and B of X. Thus, in the constructs Γ^*_{ie} and $\Gamma^*_{\vee ie}$ the result holds. □

Lemma 3.9. *In the construct Γ or any of its full subcategories, every i-final mapping is a final morphism.*

Proof. Suppose that $f : (X,\gamma) \to (Y,\delta)$ is i-final, (Z,α) a Γ-space and $g : Y \to Z$ is a function such that $g \circ f$ is i-continuous. Then, we have $\delta g^{-1}(B) = Y - \delta^*(Y - g^{-1}(B)) = Y - f(\gamma^* f^{-1}(g^{-1}(Z - B))) \supseteq Y - f((f^{-1} g^{-1}(\alpha^*(Z - B))) \supseteq Y - g^{-1}(\alpha^*(Z - B)) = g^{-1}(\alpha(B))$ for every $B \subseteq Z$. Thus $g : (Y,\delta) \to (Z,\alpha)$ is i-continuous. Hence f is a final morphism. □

Dually, similar to the proof of Theorem 3.8 and by Lemmas 2.2 and 3.9, we have the following theorem.

Theorem 3.10. *(1) In the construct* Γ *a mapping* f *is i-final if and only if it is a final morphism.*
(2) In the construct Γ_r *a mapping* f *is i-final if and only if it is a surjective final morphism.*
(3) In any of the constructs Γ_{ir} *and* $\Gamma_{\wedge ir}$ *a mapping* f *is i-final if it is a bijective final morphism.*

By Theorems 3.8 and 3.10, the following result holds.

Corollary 3.11. *In the construct* Γ_r*, i-final maps and in the construct* Γ^*_e*, c-final maps are precisely quotient morphisms.*

Proposition 3.12. *In the construct* Γ^* *or any of its full subcategories the following statements hold:*

(1) *Every section (retraction) is c-initial (c-final).*
(2) *Every c-initial epimorphism is c-final.*
(3) *Every c-final monomorphism is c-initial.*

Proof. Let $f : (X, \gamma) \to (Y, \delta)$ be a mapping in the construct Γ^* or any of its full subcategories. Then:
(1): For a retraction g with $gof = 1_X$, we have $f^{-1}(\delta f(A) = g(f(f^{-1}(\delta f(A)))) \subseteq g(\delta f(A)) \subseteq \gamma g(f(A)) = \gamma(A)$. For a section g with $fog = 1_Y$, we have $\delta B = f(g(\gamma B)) \subseteq f(\gamma g(B)) \subseteq f(\gamma f^{-1}(f(g(B)))) = f(\gamma f^{-1}(B))$.
(2): For all $B \subseteq Y$, we have $\delta B = f(f^{-1}(\delta f(f^{-1}(B)))) = f(\gamma f^{-1}(B))$.
(3): For all $A \subseteq X$, we have $f^{-1}(\delta f(A)) = f^{-1}(f(\gamma f^{-1}(f(A)))) = \gamma A$. □

The following composition-cancellation rules are true in the construct Γ^* and any of its full subcategories and their proofs are straightforward:

(1) If each of the composable morphisms f and g is c-initial (c-final), then gof is c-initial (c-final).
(2) If gof is a c-initial, then f is c-initial, and also g is c-initial provided that f is surjective.
(3) If gof is a c-final, then g is c-final, and also f is c-final provided that g is injective.

In the following, we are mostly interested in the pullback behaviour of final and initial maps. A square

$$\text{(P)} \qquad \begin{array}{ccc} (P,\alpha) & \xrightarrow{f'} & (Z,\beta) \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ (X,\gamma) & \xrightarrow[f]{} & (Y,\delta) \end{array}$$

is a pullback diagram in the construct Γ^* or any of its full subcategories if and only if P is a pullback of f along g in **Set** and α is the c-initial operator with respect to f' and g', i.e., $\alpha A = f'^{-1}(\beta f'(A)) \cap g'^{-1}(\gamma g'(A))$ for every $A \subseteq P$ [2]. Similarly, the square (P) is a pullback diagram in the construct Γ or any of its full subcategories if and only if P is a pullback of f along g in **Set** and α^* is the i-initial operator with respect to f' and g', i.e., $\alpha^* A = f'^{-1}(\beta^* f'(A)) \cap g'^{-1}(\gamma^* g'(A))$ for every $A \subseteq P$.

Now, let the square (P) be a pullback diagram in any case. Then we have the following results:

Theorem 3.13. (1) *If f is c-final and g' c-initial, then f' is c-final and g c-initial.*

(2) *If f is i-final and g' i-initial, then f' is i-final and g i-initial.*

Proof. (1): For all $L \subseteq Z$ we have:

$$\begin{aligned} \beta L \subseteq g^{-1}(g(\beta L) &\subseteq g^{-1}(\delta g(L)) \qquad (g \text{ is c-continuous}) \\ &= g^{-1}(f(\gamma f^{-1}(g(L)))) \quad (f \text{ is c-final}) \\ &= f'((g')^{-1}(\gamma g'(f')^{-1}(L))) \\ &= f'(\alpha(f')^{-1}(L) \qquad (g' \text{ is c-initial}) \\ &\subseteq \beta f'((f')^{-1}(L)) \subseteq \beta(L) \qquad (f' \text{ is c-continuous}), \end{aligned}$$

which shows both, c-finality of f' and c-initiality of g.
(2): The proof is similar to part (1). □

Definition 3.14. A mapping $f : (X, \gamma) \to (Y, \delta)$ between Γ-spaces is called *c-quotient* if f is a surjective map which reflects closedness, i.e., $B \subseteq Y$ is δ-closed whenever $f^{-1}(B)$ is γ-closed.

The following result is an immediate consequence of the previous definition.

Proposition 3.15. *In any of the constructs Γ^*_e, Γ^*_{ie}, $\Gamma^*_{\vee ie}$ we have:*

1. *Every c-final map is c-quotient.*
2. *If each of the composable morphisms f and g is c-quotient, then gof is also c-quotient.*
3. *If gof is c-quotient, then g is c-quotient, and also f is c-quotient provided that g is monic.*

Remark 3.16. Let $f : X \to Y$ be a surjective map and (X, γ) be a Γ-space. If we define a monotonic map δ on Y by $\delta B = f(\gamma f^{-1}(B))$ for every subset B of Y, then $f : (X, \gamma) \to (Y, \delta)$ is c-final.

Let $f:(X,\gamma)\to(Y,\delta)$ be a c-continuous map. We define $\delta_f: P(Y)\to P(Y)$ by $\delta_f(B)=f(\gamma f^{-1}(B))$ for every subset B of Y. Clearly, δ_f is monotone and f is c-final if $\delta B\subseteq\delta_f(B)$ for every subset B of Y. Now we define an ascending chain of functions on $P(Y)$ by putting $\delta_f^0=id, \delta_f^{\alpha}=\delta_f\circ\delta_f^{\alpha-1}$ and $\delta_f^{\beta}=\bigcup_{\alpha<\beta}\delta_f^{\alpha}$ for every successor ordinal α and for every limit ordinal β. Thus, for every subset B of Y we obtain a sequence of subsets of Y, $\delta_f^0(B),\delta_f^1(B),\dots$. The sequence stabilizes, so we have $\delta_f^{\sigma}=\delta_f^{\sigma+1}$ for some ordinal σ. Now we introduce the notation $\delta_f^{\infty}=\delta_f^{\sigma}$ and we say that f is c^{∞}-final if $\delta B\subseteq\delta_f^{\infty}B$ and also we have a characterization of c-quotient maps which was proved in [13] (see Theorem 3.7).

Theorem 3.17. *Let $f:(X,\gamma)\to(Y,\delta)$ be a mapping in Γ_e^*. Then f is c-quotient if and only if it is c^{∞}-final.*

Theorem 3.18. *Let $f:(X,\gamma)\to(Y,\delta)$ be a mapping in Γ_{ie}^*.*

(1) *If f is c^{∞}-final, then it is a quotient morphism.*
(2) *If f is an injective quotient morphism, then it is c^{∞}-final.*

Proof. (1): Let f be c^{∞}-final. By definition of δ_f^{α}, we have $\delta_f(Y)=f(\gamma f^{-1}(Y))=f(\gamma(X))=f(X)$, $\delta_f^2(Y)=\delta_f(f(X))\subseteq\delta_f(Y)=f(X),\dots,\delta_f^{\alpha}(Y)\subseteq f(X)$. Thus we have $Y=\delta Y\subseteq\delta_f^{\infty}(Y)=\delta_f^{\sigma}(Y)\subseteq f(X)$, and hence $f(X)=Y$, so that f is surjective. Now let (Z,α) be a Γ-space and also $g:Y\to Z$ be a function for which gof is c-continuous. For all $C\subseteq Z$, we have

$\delta_f(g^{-1}(C))=f(\gamma f^{-1}(g^{-1}(C)))\subseteq f(f^{-1}(g^{-1}(\alpha C)))=g^{-1}(\alpha C)$.
$\delta_f^2(g^{-1}(C))\subseteq\delta_f(g^{-1}(\alpha C))=f(\gamma f^{-1}(g^{-1}(\alpha C)))\subseteq f(f^{-1}(g^{-1}(\alpha^2C)))=$
$g^{-1}(\alpha C),\dots,\delta_f^{\alpha}(g^{-1}(C))\subseteq g^{-1}(\alpha C)$.
Thus we have $\delta(g^{-1}(C))\subseteq\delta_f^{\infty}(g^{-1}(C))=\delta_f^{\sigma}(g^{-1}(C))\subseteq g^{-1}(\alpha C)$, which shows that $g:(Y,\delta)\to(Z,\alpha)$ is c-continuous. Thus f is a final morphism.
(2): Since f is injective, it follows that δ_f^{α} is idempotent for every ordinal α. Thus by Theorem 3.17, the result holds. □

4. Closed and open morphisms

In this section, we study the notions of closed and open morphisms with respect to closure and interior operators.

Definition 4.1. A mapping $f:(X,\gamma)\to(Y,\delta)$ between Γ-spaces is called:

(1) *c-closed* or Γ-*preserving* if $f(\gamma A)=\delta f(A)$ for all subset A of X;
(2) *c-open* or Γ-*reflecting* if $f^{-1}(\delta B)=\gamma f^{-1}(B)$ for all subset B of Y;
(3) *i-closed* or Γ^*-*preserving* if $f(\gamma^*A)=\delta^*f(A)$ for all subset A of X;
(4) *i-open* or Γ^*-*reflecting* if $f^{-1}(\delta^*B)=\gamma^*f^{-1}(B)$ for all subset B of Y.

It is easy to show that c-open and c-closed maps are c-continuous and i-open and i-closed maps are i-continuous. The following lemma shows that c-open and i-open maps are both c-continuous and i-continuous.

Lemma 4.2. *A mapping $f : (X, \gamma) \to (Y, \delta)$ between Γ-spaces is c-open if and only if it is i-open.*

Proof. For every $B \subseteq Y$ we have:

$$\begin{aligned} f^{-1}(\delta^* B) = \gamma^* f^{-1}(B) &\Leftrightarrow f^{-1}(Y - \delta(Y - B)) = X - \gamma(X - f^{-1}(B)) \\ &\Leftrightarrow f^{-1}(\delta(Y - B)) = \gamma(X - f^{-1}(B) \\ &\Leftrightarrow f^{-1}(\delta(Y - B)) = \gamma(f^{-1}(Y - B)) \\ &\Leftrightarrow f^{-1}(\delta B) = \gamma f^{-1}(B). \end{aligned}$$

□

Theorem 4.3. (1) *If $f : (X, \gamma) \to (Y, \delta)$ is c-closed in Γ^* or any of its full subcategories, then f maps γ-closed subsets to δ-closed subsets.*

(2) *Let $f : (X, \gamma) \to (Y, \delta)$ be a c-continuous mapping in any of the categories Γ^*_{ie} and $\Gamma^*_{\vee ie}$. Then f is c-closed if and only if f maps γ-closed subsets to δ-closed subsets.*

(3) *If $f : (X, \gamma) \to (Y, \delta)$ is i-closed in Γ or any of its full subcategories, then f maps γ^*-closed subsets to δ^*-closed subsets.*

(4) *Let $f : (X, \gamma) \to (Y, \delta)$ be an i-continuous mapping in any of the categories Γ_{ir} and $\Gamma_{\wedge ir}$. Then f is i-closed if and only if f maps γ^*-closed subsets to δ^*-closed subsets.*

Proof. (1): Let A be a γ-closed subset of X. Then $\delta f(A) = f(\gamma A) \subseteq f(A)$. Thus $f(A)$ is a δ-closed subset of Y.

(2): Let $f : (X, \gamma) \to (Y, \delta)$ be a c-continuous mapping such that f maps γ-closed subsets to δ-closed subsets. Then for every subset A of X, γA is γ-closed, so $f(\gamma A) = \delta f(\gamma A)$. Since γ is enlarging, it follows that $f(\gamma A) \supseteq \delta f(A)$. On the other hand, by c-continuity $f(\gamma A) \subseteq \delta f(A)$. Hence $f(\gamma A) = \delta f(A)$ for every subset A of X. Thus by part one the result holds.

The proof of parts (3) and (4) is similar to parts (1) and (2). □

Now, by Theorem 4.3, the following result holds.

Corollary 4.4. *Let $f : (X, \tau) \to (Y, \sigma)$ be a continuous mapping in any of the categories* **Top** *and* **GenTop**. *Then $f : (X, c_\tau) \to (Y, c_\sigma)$ is c-closed if and only if f maps c_τ-closed subsets to c_σ-closed subsets; and $f : (X, i_\tau) \to (Y, i_\sigma)$ is i-closed if and only if f maps $(i^*_\tau = c_\tau)$-closed* *subsets to* $(i^*_\sigma = c_\sigma)$*-closed subsets. Thus* f *is c-closed or i-closed if and only if* f *is a closed map.*

Theorem 4.5. (1) *If* $f : (X, \gamma) \to (Y, \delta)$ *is c-open in* Γ^* *or any of its full subcategories, then* f *maps* γ^**-open subsets to* δ^**-open subsets.*

(2) *Let* $f : (X, \gamma) \to (Y, \delta)$ *be a c-continuous mapping in any of the categories* Γ^*_{ie} *and* $\Gamma^*_{\vee ie}$*. Then* f *is c-open if and only if* f *maps* γ^**-open subsets to* δ^**-open subsets.*

(3) *If* $f : (X, \gamma) \to (Y, \delta)$ *is i-open in* Γ *or any of its full subcategories, then* f *maps* γ*-open subsets to* δ*-open subsets.*

(4) *Let* $f : (X, \gamma) \to (Y, \delta)$ *be an i-continuous mapping in any of the categories* Γ_{ir} *and* $\Gamma_{\wedge ir}$*. Then* f *is i-open if and only if* f *maps* γ*-open subsets to* δ*-open subsets.*

Proof. (1): Let A be a γ^*-open subset of X. By Lemmas 2.2 and 4.2, $f : (X, \gamma^*) \to (Y, \delta^*)$ is c-continuous, so $f(A) \subseteq f(\gamma^* A) \subseteq \delta^* f(A)$. Thus $f(A)$ is a δ^*-open subset of Y.
(2): Let $f : (X, \gamma) \to (Y, \delta)$ be a c-continuous mapping such that f maps γ^*-open subsets to δ^*-open subsets. Then for every subset B of Y, $\gamma^* f^{-1} B$ is γ^*-open, so $\gamma^* f^{-1}(B) \subseteq f^{-1} f(\gamma^* f^{-1}(B)) = f^{-1}(\delta^* f(\gamma^* f^{-1}(B))) \subseteq f^{-1}(\delta^* B)$. On the other hand, by i-continuity $f : (X, \gamma^*) \to (Y, \delta^*)$, we have $f^{-1}(\delta^* B) \subseteq \gamma^* f^{-1}(B)$. Hence $f^{-1}(\delta^* B) = \gamma^* f^{-1}(B)$ for every subset B of Y. By Lemma 4.2, f is c-open. Thus by part one the result holds.
The proof of parts (3) and (4) is similar to parts (1) and (2). □

Corollary 4.6. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a continuous mapping in any of the categories* **Top** *and* **GenTop***. Then* $f : (X, c_\tau) \to (Y, c_\sigma)$ *is c-open if and only if* f *maps* $(c^*_\tau = i_\tau)$*-open subsets to* $(c^*_\sigma = i_\sigma)$*-open subsets; and* $f : (X, i_\tau) \to (Y, i_\sigma)$ *is i-open if and only if* f *maps* i_τ*-open subsets to* i_σ*-open subsets. Thus* f *is c-open or i-open if and only if* f *is an open map.*

The following propositions hold in Γ^* and any of its full subcategories and proofs are straightforward:

Proposition 4.7. (1) *Every c-closed and c-open monomorphism is c-initial.*

(2) *Every c-closed and c-open epimorphism is c-final.*

(3) *Every c-initial epimorphism is c-closed and c-open, equivalently, every c-final monomorphism has this properties.*

Proposition 4.8. (1) *If each of the composable morphisms* f *and* g *is c-closed (c-open), then* gof *is c-closed (c-open).*

(2) *If* gof *is c-closed (c-open), then* g *is c-closed (c-open) provided that* f *is surjective.*

(3) *If* gof *is c-closed (c-open), then* f *is c-closed (c-open) provided that* g *is monic.*

In the following, we study the pullback behaviour of c-closed and c-open maps. Hence, let the square (P) be a pullback diagram in Γ^* or any of its full subcategories. Then we have the following results:

Theorem 4.9. (1) *If f is c-closed (c-open) and g' c-initial, then f' is c-closed (c-open).*
(2) *If f' is c-closed (c-open) and g c-final, then f is c-closed (c-open).*

Proof. (1. c-closed): For all $K \subseteq P$ we have:

$$\begin{aligned}
\beta f'(K) \subseteq \beta g^{-1}(g(f'(K)) &\subseteq g^{-1}(\delta g(f'(K))) \qquad (f \text{ is c-continuous})\\
&= g^{-1}(\delta f((g'(K))))\\
&= g^{-1}(f(\gamma g'(K))) \qquad (f \text{ is c-closed})\\
&= f'(g')^{-1}(\gamma g'(K))\\
&= f'(\alpha K) \qquad (g' \text{ is c-initial}).
\end{aligned}$$

(2. c-closed): For all $A \subseteq X$ we have:

$$\begin{aligned}
\delta f(A) &= g(\beta g^{-1}(f(A)) \qquad (g \text{ is c-final})\\
&= g(\beta f'(g')^{-1}(A))\\
&= g(f'(\alpha(g')^{-1}(A))) \quad (f' \text{ is c-closed})\\
&= f(g'(\alpha(g')^{-1}(A)))\\
&\subseteq f(\gamma g'(g')^{-1}(A)) \quad (g' \text{ is c-continuous})\\
&\subseteq f(\gamma A).
\end{aligned}$$

(1. c-open): For all $L \subseteq Z$ we have:

$$\begin{aligned}
(f')^{-1}(\beta L) &\subseteq (f')^{-1}(\beta g^{-1}(g(L))\\
&\subseteq (f')^{-1}(g^{-1}(\delta g(L)) \quad (g \text{ is c-continuous})\\
&= (g')^{-1}(f^{-1}(\delta g(L))\\
&= (g')^{-1}(\gamma f^{-1}(g(L)) \quad (f \text{ is c-open})\\
&= (g')^{-1}(\gamma g'(f')^{-1}(L))\\
&= \alpha(f')^{-1}(L) \qquad (g' \text{ is c-initial}).
\end{aligned}$$

(2. c-open): For all $B \subseteq Y$ we have:

$$\begin{aligned} f^{-1}(\delta B) &= f^{-1}(g(\beta g^{-1}(B))) \qquad (g \text{ is c-final}) \\ &= g'(f')^{-1}(\beta g^{-1}(B)) \\ &= g'(\alpha(f')^{-1}(g^{-1}(B))) \quad (f' \text{ is c-open}) \\ &= g'(\alpha(g')^{-1}(f^{-1}(B))) \\ &\subseteq g'((g')^{-1}(\gamma f^{-1}(B))) \quad (g' \text{ is c-continuous}) \\ &\subseteq \gamma f^{-1}(B). \end{aligned}$$

□

5. Conclusion

In this paper, we have given some isomorphic and generalized categories of the category **Top** of topological spaces and the category **GenTop** of generalized topological spaces in terms of closure operators as Γ^*, Γ^*_e, Γ^*_{ie}, $\Gamma^*_{\vee ie}$; and in terms of interior operators as Γ, Γ_r, Γ_{ir}, $\Gamma_{\wedge ir}$. We have studied the properties of some classes of morphisms such as final, initial, closed and open morphisms with respect to closure operators by defining c-final, c-initial, c-closed, c-open maps; and with respect to interior operators by defining i-final, i-initial, i-closed and i-open maps, respectively. It is shown that the categories $\Gamma^*_{\vee ie}$ and $\Gamma_{\wedge ir}$ are convenient isomorphic categories of **Top**; and the categories Γ^*_{ie} and Γ_{ir} are convenient isomorphic categories of **GenTop**.

Acknowledgement. The authors are very grateful to the referee(s) for the careful reading of the paper and for the useful comments.

References

[1] J. Adamek, H. Herrlich, and G.E. Strecker, *Abstract and concrete categories*, John Wiely and Sons Inc., New York, 1990.

[2] M.M. Clementino, E. Giuli and W. Tholen, *what is a quotient map with respect to a closure operator?* Appl. Categ. structures, **9** (2001), 139–151.

[3] Á. Császár, *Generalized topology, generalized continuity*, Acta Math. Hungar., **96** (2002), 351–357.

[4] Á. Császár, *Generalized open sets*, Acta Math. Hungar., **57 (1-2)** (1997), 65–87.

[5] Á. Császár, *Mixed constructions for generalized topologies*, Acta Math. Hungar., **122** (2009), 153-159.

[6] Á. Császár and J.E. Makai, *Further remarks on δ- and θ-modificatons*, Acta Math. Hungar., **123** (2009), 223-228.

[7] Á. Császár, *Further remarks on the formula for γ-interior*, Acta Math. Hungar., **113** (2006), 325-332.

[8] Á. Császár, *Separation axioms for generalized topologies*, Acta Math. Hungar., **104** (2004), 63–69.

[9] J.E. Makai, E. Peyghan and B. Samadi, *Weak and strong structures and the $T_{3.5}$ property for generalized topological spaces*, Acta Math. Hungar., **150** (2016), 1–35.

[10] W.K. Min, *Generalized continuous functions defined by generalized open sets on Generalized topological spaces*, Acta Math. Hungar., **128** (2010), 299–306.

[11] W.K. Min, *Weak continuity on generalized topological spaces*, Acta Math. Hungar., **124** (2009), 73-81.

[12] E. Giuli and W. Tholen, *Openess with respect to a closure operator*, Appl. Categ. structures, **8** (2000), 487–502.

[13] Gh. Mirhosseinkhani, *On some classes of quotient maps in closure spaces*, Int. Math. Forum, **6** (2011), 1155–1161.

[14] G. Xun and G. Ying, *μ-Separations in generalized topological spaces*, Appl. Math. J. Chinese Univ., **25(2)** (2010), 243–252.

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