Journal of Systems Thinking in Practice
JSTINP, *QUARTERLY*
Homepage: jstinp.um.ac.ir
Online ISSN: 2821-1669
Research Article
DOI: 10.22067/JSTINP.2023.81194.1037
JSTINP 2023; Vol. 2, No. 1

Optimal Control Problem: A Case Study on Production Planning in the Reverse Logistics System

Mohammad Eshaghnezhad[a], Amin Mansoori [b]*, Sohrab Effati [b]

[a] *Department of Applied Mathematics, Faculty of Basic Sciences, Shahid Sattari University of Aeronautical Science and Technology, Tehran, Iran.*
[b] *Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran.*

How to cite this article

Eshaghnezhad, M., Mansoori, A., Effati, S., 2023. Optimal control problem: A case study on production planning in the reverse logistics system. *Journal of Systems Thinking in Practice*, *2*(1), pp.28-38. doi: 10.22067/JSTINP.2023.81194.1037.
URL: https://jstinp.um.ac.ir/article_43617.html.

ABSTRACT

Finished products and manufacturing plants are some elements of the production system in the supply chain, and there are other manufacturing plants. They produce work in process and finished products and hold them in warehouses. So, they need to plan and control production and inventories. Isolated planning and control by different manufacturers increase inventories in them, and then they must plan and control integratory. This paper presents an iterative approach for solving the optimal control problem with bounded control variables. The projection function constructs the iterative method to approximate the control law. Employing the approximation of control law, the approximation of state and the co-state variables are obtained. For this purpose, we apply the Hamiltonian of the optimal control problem. From the Hamiltonian, the approximation of control law and then the approximation of state law is obtained. A simple example is given to compare the results with another published paper. Also, a case study on production planning in a three-stock reverse logistics system with deteriorating items is derived to show the method's performance.

Keywords

Optimal control problem, Projection method, Production planning system, Reverse logistics system.

Article history

Received: 2023-02-18
Revised: 2023-03-03
Accepted: 2023-03-08
Published (Online): 2023-03-15

Number of Figures: 2 **Number of Tables: 1** **Number of Pages: 11** **Number of References: 22**

***Corresponding author**
Email: a-mansoori@mail.um.ac.ir

1. Introduction

The study of the linear quadratic optimal control problem (OCP) with linear systems has a history of over fifty years. Many attempts have been made to obtain a satisfactory solution based on different approaches. The application of Pontryagin's maximum principle to OCP, as outlined by Naidu (2003) and Pontryagin et al. (1962), results in a system of coupled two-point boundary-value (TPBV) problems. Within the Dynamic Programming approach, the sufficient conditions for an optimal controller and the functional with prescribed derivative proposed in Kharatishvili (1961) lead to a set of partial differential equations called the Riccati Equation for the systems. Neural networks are also another approach that is desirable to use for researchers (Pooya et al., 2021; Effati et al., 2021). In these methods, the OCP changes to a system of equations and then by using some known neural networks such as Perceptron, the problem is solved.

In optimal control problems, it is sometimes the case that control is restricted to be between a lower and an upper bound, called a bounded optimal control problem. Bang-bang optimal control problems are also in which the optimal control switches from one extreme to another (i.e., strictly never in between the bounds). Bounded optimal control problems also have many applications, such as modelling infected diseases (Sweilam and AL-Mekhlafi, 2021; Liu et al.,2022; Ojo et al., 2022; Kovacevic et al., 2022; Sweilam et al., 2020), tank reactor systems (Göllmann et al., 2009), production planning systems (Hedjar et al., 2015; Pooya and Pakdaman, 2019; 2017 and 2018), etc. One major hurdle in the path of bounded optimal control problems discovery is the solution approach which is not similar to the methods without control restriction.

Motivated by the former discussion, we will present a novel method to solve delay and bounded optimal control problems. In this way, we applied the projection function to tackle the challenge of bounded control variables. We test the method on a case study to show our technique's performance. The case study is on production planning in a three-stock reverse logistics system with deteriorating items. The motivation of the paper can be summarized as follows:

1. Use the projection method to solve the OCP.
2. Solve a production planning problem modelled by an OCP.

The paper is organized as follows. The next section dedicates to the problem formulation and optimality conditions for OCP. The iterative method is proposed in Section 3. The case study is presented in Section 4, and the paper is concluded in section 5.

DOI: 10.22067/JSINP.2023.81194.1037

2. Problem formulation and optimality conditions

In this section, the problem formulation and the optimality conditions of the problem are stated in (1). Consider the OCP in the following form.

$$\min J = \frac{1}{2}x^T(t_f)Sx^T(t_f) + \frac{1}{2}\int_0^{t_f}(x^T(t)Qx(t) + u^T(t)Ru(t))\,dt$$
$$\dot{x} = Ax(t) + Bu(t)$$
$$x(0) = x_0 \tag{1}$$
$$u(t) \in K, \qquad t \in [0, t_f]$$

where $x(t)$ and $u(t)$ are piecewise continuous the state and the control vectors, respectively. Also, A and B are two matrices of appropriate dimensions and x_0 is the initial state. Moreover, $K \subseteq \mathbb{R}^m$ is a close set. The initial condition $x(t = 0) = x_0$ is given. The terminal time t_f is specified, and the final state $x(t_f)$ is not specified. Furthermore, $Q, S \in \mathbb{R}^{n\times n}$ is positive semi-definite and $R \in \mathbb{R}^{m\times m}$ is positive definite.

Now, we will state the optimality conditions of equation (1). Consider the following Hamiltonian equation for (1):

$$H(x(t), \lambda(t), u(t), t) = \frac{1}{2}x^T(t)Qx(t) + \frac{1}{2}u^T(t)Ru(t) + \lambda^T[Ax(t) + Bu(t)]. \tag{2}$$

Where $\lambda(t)$ is the state variable. Based on equation (2), the optimality conditions can be stated as follows:

$$\dot{x} = \frac{\partial H}{\partial \lambda(t)} = Ax(t) + BR^{-1}(t)B^T(t)\lambda(t) \tag{3}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial \lambda(t)} = -Qx(t) - A^T\lambda(t), \tag{4}$$

$$u(t) = \arg\min_{\{u\in K\}} H(x(t), \lambda(t), u(t), t), \qquad 0 \le t \le t_f \tag{5}$$

$$\lambda(t_f) = Sx(t_f), \qquad x(0) = x_0 \tag{6}$$

Equations of (3)-(6) are known as a TPBV problem. The initial value of $x(t)$ is $x(0) = x_0$ and the initial value of $\lambda(t)$ is $\lambda(t_f) = Sx(t_f)$.

3. Projection method for solving OCP

Here, the projection method for solving the OCP is studied.

Consider the optimality conditions of OCP (1) stated in equations (3)-(6). Assume that the equation (7) instead of equation (5) in optimality conditions:

$$u(t) - P_K\left[u(t) - Z(u(t))\right] = 0, \quad 0 \le t \le t_f \tag{7}$$

where $P_K(.)$ is a projection map and is defined as: (Eshaghnezhad et al., 2022; Mansoori and Effati, 2019).

$$P_K(u) = \arg\min_{v \in K} \| u - v \|$$

Also, $Z(u(t)) = -\frac{\partial H}{\partial u(t)}$. Note that, $P_K(.)$ is a piecewise function. Here, some results about the $Z(u(t))$ are investigated.

Lemma 1. $Z(u(.))$ satisfies the Lipschitz condition.

Proof. As $Z(u(t)) = -\frac{\partial H}{\partial u(t)} = R(t)u(t) + B^T(t)\lambda(t)$, so the proof is obvious.

Remark 2. According to the equations (3)-(6), when we want to obtain the solution to the problem, we should at first find $u(t)$ form equation (5) and then substitute in equations (4) and (3) the co-state vector $\lambda(t)$ and state vector $x(t)$ are obtained.

Now, in the previous discussion, we are going to settle down some iterative schemes to find the solution to the problem.

The projection method gives an iteration sequence of controls by the rule in equation (8):

$$u^{k+1}(t) = P_K\left[u^k(t) - Z\left(u^k(t)\right)\right], \qquad k = 0, 1, \dots . \tag{8}$$

We use the notation $Z(u^k(t)) = -H_u(x^k(t), u^k(t), \lambda^k(t), t)$ where $x^k(t)$ and $\lambda^k(t)$ are the solutions of the state and co-state equations, respectively, related to the control function $u^k(.)$ and u_0 is an initial control approximation. We consider the grid points $t_i = ih$, $i = 0, 1, \dots, N$ for $N = \frac{t_f}{h}$, the initial approximation $u_i^0 = u_0(t_i)$, $i = 0, 1, \dots, N-1$, and the definition of the $(k+1)$ approximation is given in equation (9):

$$u^{k+1}(t) = P_K\left[u^k(t) - \bar{Z}\left(u^k(t)\right)\right], \qquad k = 0, 1, \dots, \tag{9}$$

where $\bar{Z}(u_i^k) = -H_u\left(x_i^k, u_i^k, \lambda_i^k, t_i\right)$ and x_i^k, λ_i^k are obtained after applying the Euler method to the state and co-state equations using the control approximations u_i^k on the intervals $[t_i, t_{i+1}]$, $i = 0, 1, \dots, N-1$, i. e.,

$$x_{i+1}^k = x_i^k + h\left(Ax_i^k(t_i) + Bu_i^k(t_i)\right), \qquad x_0^k = x_0, \tag{10}$$

$$\lambda_i^k = \lambda_{i+1}^k + hH_x(x_{i+1}^k, u_{i+1}^k, \lambda_{i+1}^k, t_{i+1}), \qquad \lambda_N^k = S^k x_N^k. \tag{11}$$

Note that, from the above equations, the state and co-state vectors are computed forward and backward, respectively.

Remark 3 Based on Remark 2, u^k is obtained from equation (9) and then x^k and λ^k are provided in equations (10) and (11). Finally, by applying the obtained u^k and x^k the quadratic performance index can be calculated according to the equation (1):

$$J^k = \frac{1}{2}(x^k)^T(t_f)Sx^k(t_f) + \frac{1}{2}\int_0^{t_f}\left((x^k)^T(t)Q(t)x^k(t) + (u^k)^T(t)R(t)u^k(t)\,dt\right) \tag{12}$$

For accuracy analysis, we consider the following criterion (equation (13)). The optimal control (9) has the desirable accuracy when for a given positive constant ε, the following condition holds:

$$\left|\frac{J^k - J^{k-1}}{J^{k-1}}\right| < \varepsilon. \tag{13}$$

If the tolerance error bound $\varepsilon > 0$ is chosen small enough, then the kth order optimal control law will be very close to the optimal control law $u^*(t)$, the value of the quadratic performance index in equation (12) will be very close to its optimal value J^*, and the boundary state conditions will be satisfied tightly.

The convergence analysis of the projection method is given in the following theorem. The proof was derived in Pulova (2009).

Theorem 4. Let the sequence $u^k = \left(u_0^k, u_1^k, \dots, u_{N-1}^k\right)$, $u_i^k \epsilon K$, $K \subseteq \mathbb{R}^m$, $k = 0,1,\dots$, is obtained from applying the projection method. There exists an accumulation point $\tilde{u}$ of this sequence and a piecewise constant function defined by $\tilde{u}(t) \equiv \tilde{u}_i$ for $t \in [t_i, t_{i+1})$. Also, for $u^*(t) \in T^*$ where $T^* = \{u(.)| < Z\left(u(.)\right), v(.) - u(.) >\geq 0, v(.)\epsilon K\}$ we have:

$$||u^* - \tilde{u}\,||^2 \leq O(h), \tag{14}$$

where $||u - v|| = \max_{0\leq i\leq N-1} |u_i - v_i|$.

4. Simulation results

This section will test the method on an example and a case study.

4.1. An example

Consider the following OCP Pulova (2009):

$$\begin{aligned} &\min \quad \int_0^1 [x^2(t) + u^2(t)]dt \\ &s.t. \quad \dot{x} = -ax(t) + Bu(t) \\ &\qquad\quad x(0) = 1 \\ &\qquad |u| < 1 \end{aligned} \tag{15}$$

The analytical optimal solution to this problem is

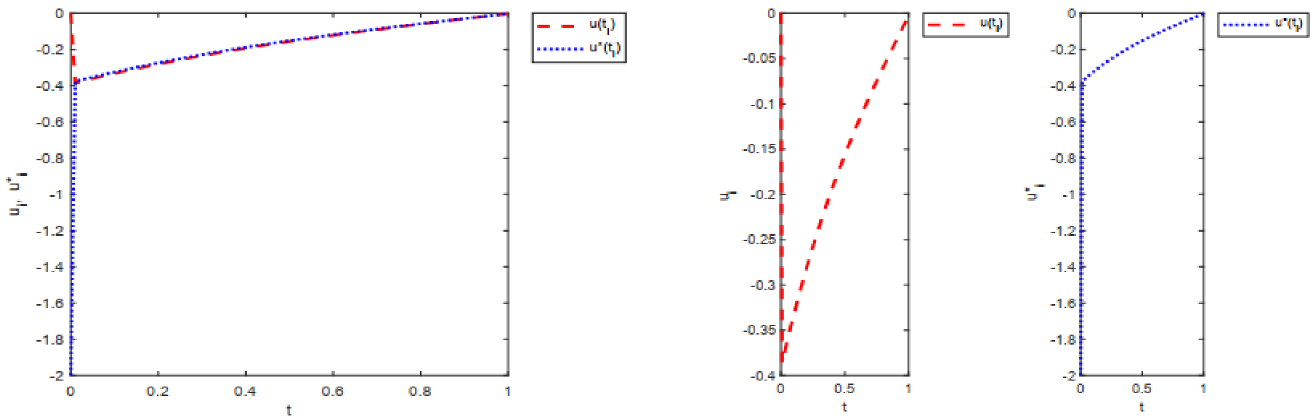
$$u^* = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

where,

$$r_1 = \sqrt{a^2 + 1}, \qquad r_2 = -\sqrt{a^2 + 1},$$

$$c_1 = \frac{1}{r_1 - a - (r_2 - a)e^{r_1 - r_2}}, \qquad c_2 = \frac{1}{r_2 - a - (r_1 - a)e^{r_1 - r_2}}.$$

We solve the problem by setting $a = 1$, $N = 100$, $h = 0.01$, and $t_i = ih$ for $i = 0,1,\ldots,N$. The transient behaviour of the optimal solution of the control variable is given in Figure 1. As you can see we choose the initial value from out of the feasible region ($u_0 = -2$) and the solution converges to the optimal solution. This is the advantage of using the projection method.



(a) Trajectory of $u(t_i)$ and u^*. (b) Trajectory of $u(t_i)$ and u^*.

Figure 1. Trajectories of control vector

4.2. Case study: production planning in reverse logistics system

Finished products and manufacturing plants are some elements of the production system in SC, and there are other manufacturing plants. They produce work in process and finished products and hold them in warehouses. So, they need to plan and control production and inventories. Isolated planning and control by different manufacturers increase inventories in them, and then they must plan and control integratory. The application in management science consists of the control of dynamics, i.e., continuous or discrete-time systems are such systems. The difference between these systems depends on whether time varies continuously or discretely. These systems are an important research area in management (Sethi and Thompson, 2000; Kistner and Dobos, 2000; Tang and et al., 2021; Vicil, 2021). The exciting topic in this area is the application of optimal control theory to the product inventory system.

Here, we are going to solve the OCP with the proposed method. The OCP was modelled based on production planning in a three-stock reverse logistics system with deteriorating items (11). Assume some definitions from Hedjar et al. (2015) as follows:

$I_r(t)$: Inventory of remanufacturing at time t.
$I_m(t)$: Inventory of manufacturing at time t.
$I_t(t)$: Inventory of returned items at time t.
$u_r(t)$: Level of remanufacturing at time t.
$u_m(t)$: Level of manufacturing at time t.
$u_d(t)$: Level of disposal at time t.

From Hedjar et al. (2015), the control and the state vectors are as $u(t) = \left(\Delta\, u_m(t).\ \Delta u_r(t). \Delta u_d(t)\right)^T$ and $x(t) = \left(\Delta\, I_m(t).\ \Delta I_r(t). \Delta I_t(t)\right)^T$, respectively, where

$$\Delta\, I_m(t) = I_m(t) - \widehat{I_m(t)}$$
$$\Delta\, I_r(t) = I_r(t) - \widehat{I_r(t)}$$
$$\Delta\, I_t(t) = I_t(t) - \widehat{I_t(t)}$$
$$\Delta\, u_m(t) = u_m(t) - \widehat{u_m(t)}$$
$$\Delta\, u_r(t) = u_r(t) - \widehat{u_r(t)}$$
$$\Delta\, u_d(t) = u_d(t) - \widehat{u_d(t)}$$

Also, "$\hat{:}$" shows the target value of the variables. The following OCP is given in Hedjar et al. (2015):

$$\min \mathrm{J} = \frac{1}{2}\int_0^{t_f} [q_m\Delta\, I_m(t) + q_r\Delta\, I_r(t) + q_t\Delta I_t(t) + r_m\Delta u_m(t) + r_r\Delta u_r(t) + r_d\Delta u_d(t)]$$

$$s.t.\ \frac{d(\Delta I_m(t))}{dt} = \Delta\, u_m(t) - \theta_m\Delta I_m(t)$$

$$\frac{d(\Delta I_r(t))}{dt} = \Delta\, u_r(t) - \theta_r\Delta I_r(t)$$

$$\frac{d(\Delta I_t(t))}{dt} = -\Delta\, u_r(t) - \Delta u_d(t)$$

$$\Delta I_m(0) = I_m^0\,, \quad \Delta I_r(0) = I_r^0\,, \ \Delta I_t(0) = I_t^0.$$

The OCP can be restated as the following matrix form:

$$\min \quad J = \frac{1}{2}\int_0^{t_f} (x^T(t)Qx(t) + u^T(t)Ru(t))dt$$

$$s.t. \quad \dot{x} = A(t)x(t) + B(t)u(t)$$

$$x(0) = x_0$$

$$|u| \leq 10.$$

where,

$$Q = \begin{bmatrix} q_m & 0 & 0 \\ 0 & q_r & 0 \\ 0 & 0 & q_t \end{bmatrix}, \quad R = \begin{bmatrix} r_m & 0 & 0 \\ 0 & r_r & 0 \\ 0 & 0 & r_d \end{bmatrix}, \quad A = \begin{bmatrix} -\theta_m & 0 & 0 \\ 0 & -\theta_r & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} \Delta I_m(0) \\ \Delta I_r(0) \\ \Delta I_t(0) \end{bmatrix}$$

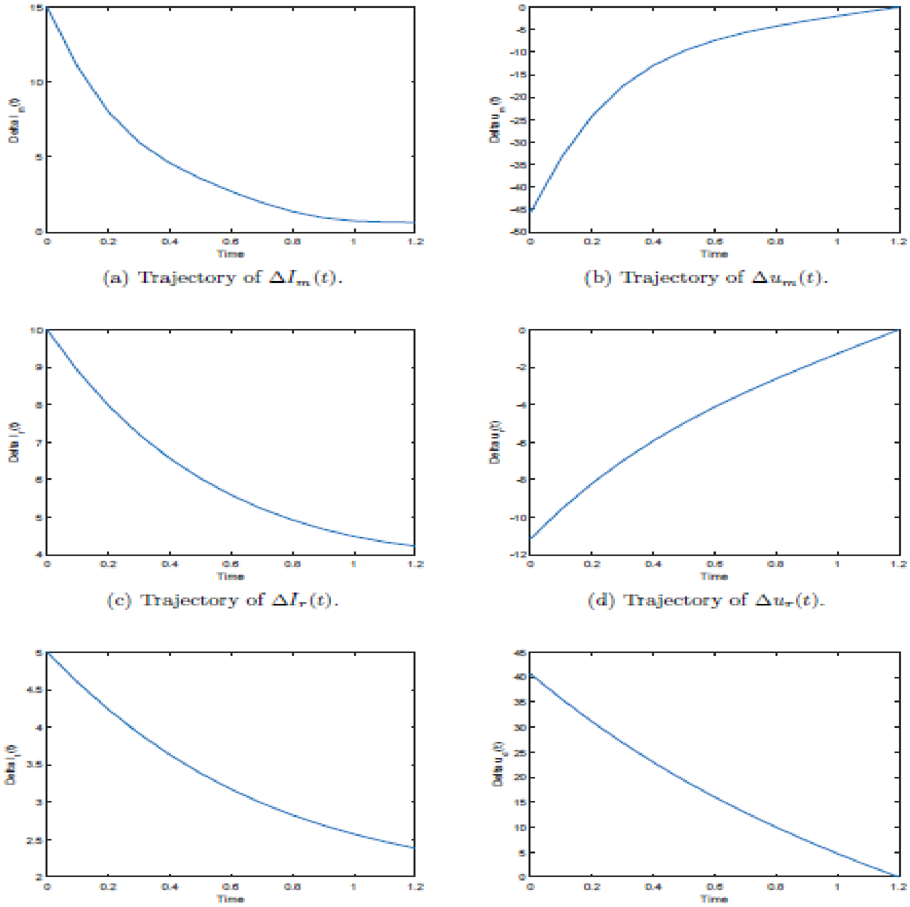
Now, assume the given values in Table 1 from Hedjar et al. (2015).

Table 1. The given parameters and initial states

Parameter	value	Parameter	value	Parameter	value	Parameter	value	Parameter	value
$\Delta I_m(0)$	15	q_m	1	θ_m	0.01	r_m	0.1	r_d	0.3
$\Delta I_r(0)$	10	q_r	2	θ_r	0.02	r_r	0.2	t_f	1.2
$\Delta I_t(0)$	5	q_t	3						

Employing the proposed method gives Figure 2 depicting the optimal control and state variables trajectories.

DOI: 10.22067/JSINP.2023.81194.1037



(a) Trajectory of $\Delta I_m(t)$. (b) Trajectory of $\Delta u_m(t)$.

(c) Trajectory of $\Delta I_r(t)$. (d) Trajectory of $\Delta u_r(t)$.

(e) Trajectory of $\Delta I_t(t)$. (f) Trajectory of $\Delta u_d(t)$.

Figure 2. Trajectories of state and control vectors

The solutions tend to be zero, similar to the obtained results in Hedjar et al. (2015). Hedjar et al. (2015) used the predictive control approach for solving the presented OCP.

5. Conclusion

This article presented an iterative approach to solving the linear quadratic optimal control problem with bounded control variables. The challenges of the optimal control problems were the bounded control variables so that conventional techniques could not be applied. The iterative approach presented in this paper guaranteed the uniform convergence of the solution for the problem. We applied the projection function to construct the approximation method. Employing the projection function had other advantages: we could select the initial value from out of the region. Finally, a case study on production planning in a reverse logistics system with deteriorating items was given and solved based on the proposed method.

Disclosure statement

No potential conflict of interest was reported by the author(s).

References

Effati, S., Mansoori, A. and Eshaghnezhad, M., 2021. Linear quadratic optimal control problem with fuzzy variables via neural network. *Journal of Experimental & Theoretical Artificial Intelligence*, *33*(2), pp.283-296. https://doi.org/10.1080/0952813X.2020.1737245.

Eshaghnezhad, M., Effati, S. and Mansoori, A., 2022. A compact MLCP-based projection recurrent neural network model to solve shortest path problem. *Journal of Experimental & Theoretical Artificial Intelligence*, pp.1-19. https://doi.org/10.1080/0952813X.2022.2067247.

Göllmann, L., Kern, D. and Maurer, H., 2009. Optimal control problems with delays in state and control variables subject to mixed control–state constraints. *Optimal Control Applications and Methods*, *30*(4), pp.341-365. https://doi.org/10.1002/oca.843.

Hedjar, R., Garg, A.K. and Tadj, L., 2015. Model predictive production planning in a three-stock reverse-logistics system with deteriorating items. *International Journal of Systems Science: Operations & Logistics*, *2*(4), pp.187-198. https://doi.org/10.1080/23302674.2015.1015661.

Kharatishvili, G.L., 1961. The maximum principle in the theory of optimal processes involving delay. In *Doklady Akademii Nauk* (Vol. 136, No. 1, pp. 39-42). Russian Academy of Sciences.

Kistner, K.P. and Dobos, I., 2000. Optimal production-inventory strategies for a reverse logistics system. *Optimization, Dynamics, and Economic Analysis: Essays in Honor of Gustav Feichtinger*, pp.246-258. https://doi.org/10.1007/978-3-642-57684-3_21.

Kovacevic, R.M., Stilianakis, N.I. and Veliov, V.M., 2022. A distributed optimal control model applied to COVID-19 pandemic. *SIAM Journal on Control and Optimization*, *60*(2), pp.S221-S245. https://doi.org/10.1137/20M1373840.

Liu, Y., Jian, S. and Gao, J., 2022. Dynamics analysis and optimal control of SIVR epidemic model with incomplete immunity. *Advances in Continuous and Discrete Models*, *2022*(1), pp.1-22. https://doi.org/10.1186/s13662-022-03723-7.

Mansoori, A. and Effati, S., 2019. Parametric NCP-based recurrent neural network model: A new strategy to solve fuzzy nonconvex optimization problems. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, *51*(4), pp.2592-2601. https://doi.org/10.1109/TSMC.2019.2916750.

Naidu, D.S., 2003. Optimal control systems, by CRC Press LLC.

Ojo, M.M., Benson, T.O., Peter, O.J. and Goufo, E.F.D., 2022. Nonlinear optimal control strategies for a mathematical model of COVID-19 and influenza co-infection. *Physica A: Statistical Mechanics and its Applications*, *607*, p.128173. https://doi.org/10.1016/j.physa.2022.128173.

Pontryagin, L.S., Boltyanskii, V.G., Gamkrelidze, R.V. and Mishchenko, E.F., 1962. The maximum principle. *The Mathematical Theory of Optimal Processes. New York: John Wiley and Sons*.

Pooya, A. and Pakdaman, M., 2017. Analysing the solution of production-inventory optimal control systems by neural networks. *RAIRO-Operations Research*, *51*(3), pp.577-590. https://doi.org/10.1051/ro/2016044.

Pooya, A. and Pakdaman, M., 2018. A delayed optimal control model for multi-stage production-inventory system with production lead times. *The International Journal of Advanced Manufacturing Technology*, *94*, pp.751-761. https://doi.org/10.1007/s00170-017-0942-5.

Pooya, A. and Pakdaman, M., 2019. Optimal control model for finite capacity continuous MRP with deteriorating items. *Journal of Intelligent Manufacturing*, *30*, pp.2203-2215. https://doi.org/10.1007/s10845-017-1383-6.

Pooya, A., Mansoori, A., Eshaghnezhad, M. and Ebrahimpour, S.M., 2021. Neural network for a novel disturbance optimal control model for inventory and production planning in a four-echelon supply chain with reverse logistic. *Neural Processing Letters*, *53*(6), pp.4549-4570. https://doi.org/10.1007/s11063-021-10612-9.

Pulova, N.V., 2009. A pointwise projected gradient method applied to an optimal control problem. *Journal of computational and applied mathematics*, *226*(2), pp.331-335. https://doi.org/10.1016/j.cam.2008.08.007.

Sethi, P., Thompson, G. L., 2000. *Optimal Control Theory, Applications to Management Science and Economics.* 2nd Edn., Springer, USA., Inc., (2000).

Sweilam, N.H., Al-Mekhlafi, S.M. and Shatta, S.A., 2021. Optimal bang-bang control for variable-order dengue virus; numerical studies. *Journal of Advanced Research*, *32*, pp.37-44. https://doi.org/10.1016/j.jare.2021.03.010.

Sweilam, N.H., Al-Mekhlafi, S.M., Albalawi, A.O. and Machado, J.T., 2021. Optimal control of variable-order fractional model for delay cancer treatments. *Applied Mathematical Modelling*, *89*, pp.1557-1574. https://doi.org/10.1016/j.apm.2020.08.012.

Tang, L., Yang, T., Tu, Y. and Ma, Y., 2021. Supply chain information sharing under consideration of bullwhip effect and system robustness. *Flexible Services and Manufacturing Journal*, *33*, pp.337-380. https://doi.org/10.1007/s10696-020-09384-6.

Vicil, O., 2021. Optimizing stock levels for service-differentiated demand classes with inventory rationing and demand lead times. *Flexible Services and Manufacturing Journal*, *33*(2), pp.381-424. https://doi.org/10.1007/s10696-020-09378-4.