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# Regularity Conditions for Non-Differentiable Infinite Programming Problems using Michel-Penot Subdifferential

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**Abstract.** In this paper we study optimization problems with infinite many inequality constraints on a Banach space where the objective function and the binding constraints are locally Lipschitz. Necessary optimality conditions and regularity conditions are given. Our approach are based on the Michel-Penot subdifferential.

**Keywords.** Programming problem, Regularity conditions, Optimality condition, Michel-Penot subdifferential.

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## 1 Introduction

Infinite Programming problem is an optimization problem on a feasible set described by infinitely many inequalities in a Banach space  $\mathcal{B}$ .

In the classical nonlinear programming, the necessary conditions of Fritz-John type can be viewed as being degenerate when the multiplier corresponding to the objective function vanishes, since then the function being minimized is not involved. Various supplementary conditions have been proposed under which it is possible to assert that the multiplier rule holds in the Karush-Kuhn-Tucker (KKT, briefly) form (i.e., the multiplier corresponding to objective function is equal to one). These conditions are called *regularity conditions*.

The theory of optimality conditions for infinite problems can be seen as a natural extension the classical KKT theory for ordinary nonlinear programming. Several papers studied infinite problems and gave the KKT necessary conditions (See e.g., [1, 6, 8] and their references). In these papers, two kinds of regularity conditions (RC in brief) are usually considered including "Farkas-Minkowski RC" and "closedness RC", using basic/limiting subdifferential or convex ones.

This paper focuses mainly on some kinds of RCs for infinite problem which are based on Michel-Penot subdifferential, their interrelations, and their applications to KKT necessary optimality conditions.

The remainder of the present paper is organized as follows. In Section 2, basic notations and preliminary results are reviewed. In Section 3, we present our main results.

## 2 Notations and Preliminaries

Let  $\mathcal{B}^*$  be the (continuous) dual space of  $\mathcal{B}$  and let  $\langle x^*, x \rangle$  denotes the value of the function  $x^* \in \mathcal{B}^*$  at  $x \in \mathcal{B}$ . If  $A^* \subseteq \mathcal{B}^*$ , set  $\langle A^*, x \rangle := \{\langle a^*, x \rangle \mid a^* \in A^*\}$ . The symbols  $\overline{A}$ ,  $\text{conv}(A)$ , and  $\text{cone}(A)$  denote the closure, the convex hull, and the convex cone (containing zero) of  $A \subseteq \mathcal{B}$ . When we write  $C \leq 0$  for some  $C \subseteq \mathbb{R}$ , means  $c \leq 0$  for all  $c \in C$ .

Let  $\hat{x} \in \mathcal{B}$  and let  $\varphi : \mathcal{B} \rightarrow \mathbb{R}$  be a function. The Michel-Penot (M-P, briefly) directional derivative of  $\varphi$  at  $\hat{x}$  in the direction  $v \in \mathcal{B}$  introduced in [4] is given by

$$\varphi^\diamond(\hat{x}; v) := \sup_{w \in \mathcal{B}} \limsup_{a \downarrow 0} \frac{\varphi(\hat{x} + av + aw) - \varphi(\hat{x} + aw)}{a}$$

and the M-P subdifferential of  $\varphi$  at  $\hat{x}$  is given by the set

$$\partial^\diamond \varphi(\hat{x}) := \{\xi \in \mathcal{B}^* \mid \langle \xi, v \rangle \leq \varphi^\diamond(\hat{x}; v) \quad \forall v \in \mathcal{B}\}.$$

The M-P subdifferential is a natural generalization of the Gateaux derivative since it is known (See [4], Proposition 1.3) that when function  $\varphi$  is Gateaux differentiable at  $\hat{x}$ ,

$\partial^\diamond \varphi(\hat{x}) = \{D\varphi(\hat{x})\}$  and  $\varphi^\diamond(\hat{x}; v) = \varphi'(\hat{x}, v)$ , where  $\varphi'(\hat{x}, v)$  denotes the usual directional derivative of  $\varphi$  at  $\hat{x}$  in the direction  $v$ , i.e.,

$$\varphi'(\hat{x}, v) := \lim_{a \downarrow 0} \frac{\varphi(\hat{x} + av) - \varphi(\hat{x})}{a}$$

Moreover when a function  $\varphi$  is convex, the M-P subdifferential coincides with the subdifferential in the sense of convex analysis, denoted by  $\partial$ .

Given a locally Lipschitz function  $\varphi : \mathcal{B} \rightarrow \mathbb{R}$  we say that  $\varphi$  is M-P regular at  $\hat{x}$  if  $\varphi'(\hat{x}; v)$  exists and

$$\varphi^\diamond(\hat{x}; v) = \varphi'(\hat{x}, v)$$

for all  $v \in \mathcal{B}$ . Observe that convex functions are examples for M-P regular functions.

In the following theorem we summarize some important properties of the M-P directional derivative and the M-P subdifferential from [4] and [5] which are widely used in what follows.

**Theorem 1.** Let  $\varphi$  and  $\psi$  be functions from  $\mathcal{B}$  to  $\mathbb{R}$  which are Lipschitz near  $\hat{x}$ . Then, the following assertions hold:

(i) One always has

$$\varphi^\diamond(\hat{x}; v) = \max \{ \langle \xi, v \rangle \mid \xi \in \partial^\diamond \varphi(\hat{x}) \}, \quad (1)$$

$$\partial^\diamond(\varphi + \psi)(\hat{x}) \subseteq \partial^\diamond \varphi(\hat{x}) + \partial^\diamond \psi(\hat{x}). \quad (2)$$

(ii) The function  $v \rightarrow \varphi^\diamond(\hat{x}; v)$  is finite, positively homogeneous and subadditive on  $\mathcal{B}$  and

$$\partial(\varphi^\diamond(\hat{x}; \cdot))(0) = \partial^\diamond \varphi(\hat{x}). \quad (3)$$

(iii)  $\partial^\diamond \varphi(\hat{x})$  is a nonempty, convex and weak star compact of  $\mathcal{B}^*$ .

### 3 Main Results

We consider the following optimization programming problem:

$$(P) \begin{cases} \text{minimize} & f(x) \\ \text{s.t.} & f_t(x) \leq 0, \quad t \in T \end{cases}$$

where  $T$  is an arbitrary set and all emerging functions  $f$  and  $f_t$  for  $t \in T$  are extended real-valued locally Lipschitz from the Banach space  $\mathcal{B}$ .

As a starting, we denote the feasible set of problem  $(P)$  with

$$\Omega := \{x \in \mathcal{B} \mid f_t(x) \leq 0 \quad t \in T\}.$$

The contingent cone, the attainable directions cone and the feasible directions cone of  $\Omega$  at  $\hat{x} \in \Omega$  are respectively defined as:

$$\begin{aligned} K_{\Omega}(\hat{x}) &:= \left\{ z \in \mathcal{B} \mid \exists a_k \downarrow 0, \exists z_k \xrightarrow{\text{in}} z, \text{ such that } \hat{x} + a_k z_k \in \Omega \ \forall k \in \mathbb{N} \right\}, \\ A_{\Omega}(\hat{x}) &:= \left\{ z \in \mathcal{B} \mid \forall a_k \downarrow 0, \exists z_k \xrightarrow{\text{in}} z, \text{ such that } \hat{x} + a_k z_k \in \Omega \ \forall k \in \mathbb{N} \right\}, \\ D_{\Omega}(\hat{x}) &:= \{ z \in \mathcal{B} \mid \exists \varepsilon > 0 \text{ such that } \hat{x} + az \in \Omega \ \forall a \in (0, \varepsilon) \}. \end{aligned}$$

For a given  $\hat{x} \in \Omega$ , let  $T(\hat{x})$  denotes the index set of all active constraints at  $\hat{x}$ , i.e.,

$$T(\hat{x}) := \{ t \in T \mid f_t(\hat{x}) = 0 \}.$$

Based on the above notations and the Michel-Penot sub-differential, we extend the following RCs to non-differentiable infinite problem (P).

**Definition 1.** Let  $\hat{x} \in \Omega$ . We say that

1. The First regularity condition ( $RC_1$ , shortly) holds at  $\hat{x}$  if

$$\left\{ v \in \mathcal{B} \mid \left\langle \bigcup_{t \in T(\hat{x})} \partial^{\diamond} f_t(\hat{x}), v \right\rangle \leq 0 \right\} \subseteq K_{\Omega}(\hat{x}).$$

2. The Second regularity condition ( $RC_2$ , shortly) holds at  $\hat{x}$  if

$$\left\{ v \in \mathcal{B} \mid \left\langle \bigcup_{t \in T(\hat{x})} \partial^{\diamond} f_t(\hat{x}), v \right\rangle \leq 0 \right\} \subseteq \overline{D_{\Omega}(\hat{x})}.$$

3. The Three regularity condition ( $RC_3$ , shortly) holds at  $\hat{x}$  if

$$\left\{ v \in \mathcal{B} \mid \left\langle \bigcup_{t \in T(\hat{x})} \partial^{\diamond} f_t(\hat{x}), v \right\rangle \leq 0 \right\} \subseteq A_{\Omega}(\hat{x}).$$

4. The Fourth regularity condition ( $RC_4$ , shortly) holds at  $\hat{x}$  if

$$\left\{ v \in \mathcal{B} \mid \left\langle \bigcup_{t \in T(\hat{x})} \partial^{\diamond} f_t(\hat{x}), v \right\rangle \leq 0 \right\} \subseteq \overline{\text{conv}}(K_{\Omega}(\hat{x})).$$

It is easy to see that  $\overline{D_{\Omega}(\hat{x})} \subseteq A_{\Omega}(\hat{x}) \subseteq K_{\Omega}(\hat{x}) \subseteq \overline{\text{conv}}(K_{\Omega}(\hat{x}))$  and hence the following relationships among the above constraint qualifications are obvious:

$$RC_2 \Rightarrow RC_3 \Rightarrow RC_1 \Rightarrow RC_4. \quad (4)$$

**Remark 1.** In particular case where  $T$  is a finite set and  $\mathcal{B}$  has finite dimension, the  $RC_1$ ,  $RC_2$ ,  $RC_3$  and  $RC_4$  are similar to the Abadie, Zangwill, Kuhn-Tucker, and Guignard RCs, respectively, which defined for non-differentiable problems (See e.g., [3]). It is worth mentioning that if  $f_t$ s are convex functions, these RCs are equivalent.

The following theorem shows that in M-P regular systems the converse inclusion of  $RC_1$  is always true.

**Theorem 2.** If for each  $t \in T(\hat{x})$ ,  $f_t$  is a regular function at  $\hat{x} \in \Omega$ , then

$$K_{\Omega}(\hat{x}) \subseteq \left\{ v \in \mathcal{B} \left| \left\langle \bigcup_{t \in T(\hat{x})} \partial^{\diamond} f_t(\hat{x}), v \right\rangle \leq 0 \right. \right\}. \quad (5)$$

*Proof.* Suppose that  $z$  is an arbitrary element of  $K_{\Omega}(\hat{x})$ . Then, there exist  $a_k \rightarrow 0^+$  and  $z_k \rightarrow z$  such that  $\hat{x} + a_k z_k \in \Omega$  for all  $k \in \mathbb{N}$ . If  $\ell_t$  denotes the Lipschitzian constant of  $f_t$  near  $\hat{x}$ , then

$$|f_t(\hat{x} + a_k z_k) - f_t(\hat{x} + a_k z)| \leq \ell_t a_k \|z - z_k\| \quad (\text{when } a_k \rightarrow 0).$$

The above relation leads us to

$$\lim_{a_k \rightarrow 0} \left| \frac{f_t(\hat{x} + a_k z_k) - f_t(\hat{x} + a_k z)}{a_k} \right| = 0 \quad \forall t \in T. \quad (6)$$

Let  $\hat{t} \in T(\hat{x})$ . With regard to  $f_{\hat{t}}(\hat{x}) = 0$ , feasibility of  $\hat{x} + a_k z_k$ , regularity of  $f_{\hat{t}}$ s at  $\hat{x}$ , and virtue of (6) we conclude

$$\begin{aligned} f_{\hat{t}}^{\diamond}(\hat{x}; z) &= \lim_{a_k \rightarrow 0} \frac{f_{\hat{t}}(\hat{x} + a_k z) - f_{\hat{t}}(\hat{x})}{a_k} \\ &= \lim_{a_k \rightarrow 0} \frac{f_{\hat{t}}(\hat{x} + a_k z) - f_{\hat{t}}(\hat{x} + a_k z_k)}{a_k} \\ &\quad + \lim_{a_k \rightarrow 0} \frac{f_{\hat{t}}(\hat{x} + a_k z_k) - f_{\hat{t}}(\hat{x})}{a_k} \\ &= \lim_{a_k \rightarrow 0} \frac{f_{\hat{t}}(\hat{x} + a_k z_k) - f_{\hat{t}}(\hat{x})}{a_k} \\ &= \lim_{a_k \rightarrow 0} \frac{f_{\hat{t}}(\hat{x} + a_k z_k)}{a_k} \leq 0. \end{aligned}$$

Since  $\hat{t}$  is an arbitrary index in  $T(\hat{x})$ , the last inequality and (1) imply that

$$\left\langle \bigcup_{t \in T(\hat{x})} \partial^{\diamond} f_t(\hat{x}), z \right\rangle \leq 0,$$

and hence

$$z \in \left\{ v \in \mathcal{B} \left| \left\langle \bigcup_{t \in T(\hat{x})} \partial^{\diamond} f_t(\hat{x}), v \right\rangle \leq 0 \right. \right\}.$$

□

**Remark 2.** If  $\mathcal{B} = \mathbb{R}^n$  then the convexity and closedness of

$$\left\{ v \in \mathcal{B} \mid \left\langle \bigcup_{t \in T(\hat{x})} \partial^\diamond f_t(\hat{x}), v \right\rangle \leq 0 \right\}$$

implies that the inclusion (5) can be rewritten as

$$\overline{\text{conv}}(K_\Omega(\hat{x})) \subseteq \left\{ v \in \mathcal{B} \mid \left\langle \bigcup_{t \in T(\hat{x})} \partial^\diamond f_t(\hat{x}), v \right\rangle \leq 0 \right\}.$$

Therefore, if  $\mathcal{B}$  has finite dimension and  $f_t$ s for  $t \in T(\hat{x})$  are regular at  $\hat{x}$  then the following implications are fulfilled at  $\hat{x}$ :

$$RC_4 \iff \overline{\text{conv}}(K_\Omega(\hat{x})) = \left\{ v \in \mathcal{B} \mid \left\langle \bigcup_{t \in T(\hat{x})} \partial^\diamond f_t(\hat{x}), v \right\rangle \leq 0 \right\}.$$

Before proving the next theorems, we give a lemma, which shall be useful.

**Lemma 1.** Let  $\hat{x}$  be an optimal solution of problem(P), and the mapping  $v \rightarrow f^\diamond(\hat{x}, v)$  be linear. Then,

$$\{v \in \mathcal{B} \mid f^\diamond(\hat{x}; v) < 0\} \cap \overline{\text{conv}}(K_\Omega(\hat{x})) = \emptyset.$$

*Proof.* The proof falls naturally into two parts:

**Part one.** We first establish that

$$\{v \in \mathcal{B} \mid f^\diamond(\hat{x}; v) < 0\} \cap K_\Omega(\hat{x}) = \emptyset. \quad (7)$$

On the contrary, suppose that there exists  $\hat{v} \in K_\Omega(\hat{x})$  such that  $f^\diamond(\hat{x}; \hat{v}) < 0$ . Then

$$\limsup_{a \downarrow 0} \frac{f(\hat{x} + a\hat{v}) - f(\hat{x})}{a} \leq f^\diamond(\hat{x}; \hat{v}) < 0,$$

which implies that there exist  $\delta > 0$  and  $\varepsilon > 0$  such that

$$f(\hat{x} + a\hat{v}) - f(\hat{x}) \leq -\delta a \quad \forall a \in (0, \varepsilon). \quad (8)$$

Since  $f$  is Lipschitzian near  $\hat{x}$ , there exists  $r > 0$  such that

$$f(\hat{x} + a\hat{u}) - f(\hat{x} + a\hat{v}) \leq \ell_f a \|\hat{u} - \hat{v}\|, \quad \forall \hat{u} \in \mathbb{B}(\hat{x}, r), \quad (9)$$

where  $\ell_f$  is the Lipschitz constant of  $f$  near  $\hat{x}$  and  $\mathbb{B}(\hat{x}, r)$  denotes the open ball with center  $\hat{x}$ . By the definition of  $K_\Omega(\hat{x})$ , there exist sequences  $a_k \rightarrow 0^+$  in  $\mathbb{R}$  and  $\hat{v}_k \rightarrow \hat{v}$  in  $\Omega$  such that  $\hat{x} + a_k \hat{v}_k \in \Omega$ .

Therefore for  $k$  large enough, one has

$$\|\hat{v}_k - \hat{v}\| \leq \frac{\delta}{2\ell_f} \quad \text{and} \quad a_k \in (0, \varepsilon).$$

Now from (8), (9) and above inequality we can deduce that

$$\begin{aligned} f(\hat{x} + a\hat{v}_k) - f(\hat{x}) &= f(\hat{x} + a\hat{v}_k) - f(\hat{x} + a\hat{v}) \\ &+ f(\hat{x} + a\hat{v}) - f(\hat{x}) \leq \\ \ell_f a_k \|\hat{v}_k - \hat{v}\| - \delta a_k &< -\frac{\delta}{2} a_k < 0. \end{aligned}$$

But this contradicts the fact that  $\hat{x}$  is an optimal solution of (P) and hence (7) holds.

**Part two.** Let  $v^* \in \text{conv}(K_\Omega(\hat{x}))$ . Then, there exist non-negative scalars  $\beta_1, \beta_2, \dots, \beta_m \in \mathbb{R}$  and vectors  $\hat{v}_1, \hat{v}_2, \dots, \hat{v}_m \in K_\Omega(\hat{x})$ , such that

$$\sum_{l=1}^m \beta_l = 1, \quad v^* = \sum_{l=1}^m \beta_l \hat{v}_l.$$

Using the linearity of  $f^\diamond(\hat{x}; \cdot)$  and part one, we get

$$f^\diamond(\hat{x}; v^*) = f^\diamond\left(\hat{x}; \sum_{l=1}^m \beta_l \hat{v}_l\right) = \sum_{l=1}^m \beta_l f^\diamond(\hat{x}; \hat{v}_l) \geq 0.$$

Taking into consideration the continuity of  $f^\diamond(\hat{x}, \cdot)$  and above inequality, it follows that

$$f^\diamond(\hat{x}; \tilde{v}) \geq 0 \quad \text{for all } \tilde{v} \in \overline{\text{conv}}(K_\Omega(\hat{x})).$$

The proof is complete.  $\square$

**Theorem 3.** Suppose that  $\hat{x}$  is an optimal solution of problem (P) and the  $RC_4$  is satisfied at  $\hat{x}$ . If the mapping  $v \rightarrow f^\diamond(\hat{x}; v)$  is linear, then the following inclusion holds:

$$0 \in \partial^\diamond f(\hat{x}) + \overline{\text{conv}}\left(\bigcup_{t \in T(\hat{x})} \partial^\diamond f_t(\hat{x})\right).$$

*Proof.* Let  $\hat{v}$  is an element of  $\mathcal{B}$  satisfying

$$\left\langle \bigcup_{t \in T(\hat{x})} \partial^\diamond f_t(\hat{x}), \hat{v} \right\rangle \leq 0.$$

Owing to  $RC_4$  and Lemma 1 we conclude

$$f^\diamond(\hat{x}; \hat{v}) \geq 0.$$

Thus, we can obtain

$$f^\diamond(\hat{x}; \hat{v}) \geq 0 \quad \forall \hat{v} \in \left\{ v \in \mathcal{B} \mid \left\langle \overline{\text{conv}}\left(\bigcup_{t \in T(\hat{x})} \partial^\diamond f_t(\hat{x})\right), v \right\rangle \leq 0 \right\},$$

in view of

$$\begin{aligned} &\left\{ v \in \mathcal{B} \mid \left\langle \overline{\text{conv}}\left(\bigcup_{t \in T(\hat{x})} \partial^\diamond f_t(\hat{x})\right), v \right\rangle \leq 0 \right\} \\ &= \left\{ v \in \mathcal{B} \mid \left\langle \bigcup_{t \in T(\hat{x})} \partial^\diamond f_t(\hat{x}), v \right\rangle \leq 0 \right\}. \end{aligned}$$

The above result implies that  $v^* = 0$  is a global minimizer of the convex function

$$v \rightarrow H(v) := f^\diamond(\hat{x}; v) + \Theta(v),$$

where  $\Theta(\cdot)$  denotes the indicator function of set

$$\left\{ v \in \mathcal{B} \left| \left\langle \overline{\text{conv}} \left( \bigcup_{t \in T(\hat{x})} \partial^\diamond f_t(\hat{x}) \right), v \right\rangle \leq 0 \right. \right\};$$

i.e.

$$\Theta(v) = 0 \text{ if } \left\langle \overline{\text{conv}} \left( \bigcup_{t \in T(\hat{x})} \partial^\diamond f_t(\hat{x}) \right), v \right\rangle \leq 0 \text{ and } \Theta(v) = +\infty \text{ otherwise.}$$

Now, by necessary condition for convex optimization problems (See e.g. [2]) and by the sum rule formula (2) (which equality holds there for convex functions), one has

$$0 \in \partial(f^\diamond(\hat{x}; \cdot))(0) + \partial\Theta(0),$$

where  $\partial\varphi$  denotes the subdifferential of convex function  $\varphi$  in the sense of convex analysis. Finally, the virtue of (3) and the fact that  $\Theta(0) = \overline{\text{conv}} \left( \bigcup_{t \in T(\hat{x})} \partial^\diamond f_t(\hat{x}) \right)$ , one can conclude that

$$0 \in \partial^\diamond f(\hat{x}) + \overline{\text{conv}} \left( \bigcup_{t \in T(\hat{x})} \partial^\diamond f_t(\hat{x}) \right).$$

The proof is complete.  $\square$

**Theorem 4.** Suppose that  $\hat{x}$  is an optimal solution of problem (P) and the  $RC_1$  is satisfied at  $\hat{x}$ . Then the following inclusion holds:

$$0 \in \partial^\diamond f(\hat{x}) + \overline{\text{conv}} \left( \bigcup_{t \in T(\hat{x})} \partial^\diamond f_t(\hat{x}) \right).$$

*Proof.* Owing to the equality (7), the proof is similar to Theorem 3.  $\square$

**Theorem 5** (KKT Necessary Condition). Suppose that  $\hat{x}$  is an optimal solution of problem (P) and one of the following conditions holds:

- $RC_4$  at  $\hat{x}$  and linearity of  $f^\diamond(\hat{x}; \cdot)$ .
- $RC_1$  at  $\hat{x}$ .
- $RC_2$  at  $\hat{x}$ .
- $RC_3$  at  $\hat{x}$ .

If  $\text{cone} \left( \bigcup_{t \in T(\hat{x})} \partial^\diamond f_t(\hat{x}) \right)$  is a closed cone, then, there exist  $\lambda_t \geq 0$ ,  $t \in T(\hat{x})$ , where  $\lambda_t \neq 0$  for finitely many  $t \in T(\hat{x})$ , such that

$$0 \in \partial^\diamond f(\hat{x}) + \sum_{t \in T(\hat{x})} \lambda_t \partial^\diamond f_t(\hat{x}).$$



*Proof.* Owing to the Theorems 3 and 4, virtue of (4) and the following fact for convex sets  $A_\gamma$ ,  $\gamma \in \Gamma$  (See e.g., [2]):

$$\text{cone} \left( \bigcup_{\gamma \in \Gamma} A_\gamma \right) = \left\{ \sum_{\gamma \in \Gamma_0} \tau_\gamma a_\gamma \mid \Gamma_0 \text{ is a finite subset of } \Gamma, a_\gamma \in A_\gamma, \tau_\gamma \geq 0 \right\},$$

the result is immediate.  $\square$

Note that  $\text{cone} \left( \bigcup_{t \in T(\hat{x})} \partial^\diamond f_t(\hat{x}) \right)$  is assumed to be closed in Theorem 5. The following example shows that this assumption can not be waived, even when  $\mathcal{B}$  has finite dimension and  $f_t$ s are convex.

**Example 1.** For all  $t \in T := \mathbb{N}$ , take

$$A_t := \{(a_1, a_2) \in \mathbb{R}^2 \mid a_1^2 + a_2^2 - 2ta_2 \leq 0\}.$$

Set  $f(x_1, x_2) := -x_1$  and

$$f_t(x_1, x_2) := \sup_{(a_1, a_2) \in A_t} (a_1 x_1 + a_2 x_2).$$

It is easy to see that  $\Omega = (-\infty, 0] \times (-\infty, 0]$  and  $\hat{x} = (0, 0)$  are respectively the feasible solution set and the optimal solution of the following problem:

$$\inf \{f(x_1, x_2) \mid f_t(x_1, x_2) \leq 0, \quad t \in T\}.$$

We observe that  $T(\hat{x}) = T$ . Since  $f_t$  is support function of  $A_t$ , we obtain  $\partial^\diamond f_t(\hat{x}) = A_t$ , and hence

$$\begin{aligned} \text{cone} \left( \bigcup_{t \in T(\hat{x})} \partial^\diamond f_t(\hat{x}) \right) &= ((0, +\infty) \times (0, +\infty)) \cup \{(0, 0)\}, \\ \{v \in \mathcal{B} \mid \langle \bigcup_{t \in T(\hat{x})} \partial^\diamond f_t(\hat{x}), v \rangle \leq 0\} &= \Omega. \end{aligned}$$

Owing to  $K_\Omega(\hat{x}) = \Omega$  and convexity of  $\Omega$  we conclude that the  $RC_i$  holds at  $\hat{x}$  for  $i = 1, 2, 3, 4$ . Note that  $\text{cone} \left( \bigcup_{t \in T(\hat{x})} \partial^\diamond f_t(\hat{x}) \right)$  is not closed. It is easy to see that there is no sequence of scalars satisfying Theorem 5. Moreover, it can show that

$$0 \in \partial^\diamond f(\hat{x}) + \overline{\text{conv}} \left( \bigcup_{t \in T(\hat{x})} \partial^\diamond f_t(\hat{x}) \right).$$

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#### چکیده

در این مقاله، مسائل بهینه‌سازی شامل تعداد نامتناهی قید نامساوی در یک فضای باناخ مورد بررسی و مطالعه قرار گرفته است. این دسته مسائل به گونه‌ای است که تابع هدف و تمامی توابع فید در نزدیکی نقطه بهینه به طور موضعی لیپ شیتز هستند. هدف، ارائه شرایط لازم بهینگی و بررسی شرایط نظم‌پذیری برای مسائل فوق، توسط زیرمشتق میشل-پینت است.

#### کلمات کلیدی

مسئله برنامه‌ریزی نامتناهی، شرط نظم‌پذیری، شرایط بهینگی، زیرمشتق میشل - پینت.