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Solving Second Kind Volterra-Fredholm Integral Equations by Using Triangular Functions (TF) and Dynamical Systems

A. Soleymanpour Bakefayat *,[1], S. Karamseraji[2]
[1]Department of Mathematics, Farhangian University, Tehran, Iran
[2]Department of Mathematics, Karaj Branch, Islamic Azad University, Alborz, Iran.

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Abstract. The method of triangular functions (TF) could be a generalization form of the functions of block-pulse (Bp). The solution of second kind integral equations by using the concept of TF would lead to a nonlinear equations system. In this article, the obtained nonlinear system has been solved as a dynamical system. The solution of the obtained nonlinear system by the dynamical system through the Newton numerical method has got a particular priority, in that, in this method, the number of the unknowns could be more than the number of equations. Besides, the point of departure of the system could be an infeasible point. It has been proved that the obtained dynamical system is stable, and the response of this system can be achieved by using of the fourth order Runge-Kutta. The results of this method is comparable with the similar numerical methods; in most of the cases, the obtained results by the presented method are more efficient than those obtained by other numerical methods. The efficiency of the new method will be investigated through examples.

Keywords. Second kind Fredholm-Volterra integral equations, Nonlinear systems, Dynamical systems, Triangular functions, Block-pulse functions.

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∗ Corresponding author
asad87000@yahoo.com, rimakool2002@yahoo.com
http://mathco.journals.pnu.ac.ir

1 Introduction

Integral and differential equations comprise a major part of the problems posed in engineering and applied mathematics. Many researchers have worked on the proposed methods in this field to solve various integral equations using analytic and numerical methods. Wavelet method [25], collocation method based on Legendre polynomials [23] and numerous other methods can be named among the numerical methods for solving integral equations. The Bp functions method is the most prominent method for that purpose [11]. These functions are used not only in approximating the product of an integro-differential equation, but also are in control theory for stabilization and control of dynamical systems [11]. Maleknejad et al. [18] solved second kind integral equations by using Galerkin methods with hybrid Legendre and Block-Pulse functions; more over Maleknejad et al. [16] solved linear Fredholm integral equations by using hybrid Taylor and Bp functions. Asadi et al. [2] used Bp and Fourier functions to solve second kind integral equations. In addition, a combination of Bp functions and Chebyshev polynomials were used to solve second kind integral equations [24].

Moreover, Babolian et al. [4] investigated a direct method for solving the first kind Volterra integral equations using operational matrix with Bp functions. They converted the first kind integral equation into a lower linear triangular system and solved it directly by forwarding substitution. In a similar study, Maleknejad et al. [17] made a correction on Bp functions and used them to solve the first kind Volterra integral equations. Bp functions and their combinations with other numerical methods were also used to solve Fredholm-Volterra integral equations in two-dimensional case and integro-differential equations [3, 14]. Additionally, approximation by Bp functions and stochastic operational matrix were used to solve stochastic Volterra integral equations [15]. In this study, the given integral equation was converted into a lower linear triangular system and the linear system was solved directly by forwarding substitution.

In a fairly recent separate set of studies, Mirzaee et al. used Bp functions and their combination with Taylor series and recombination of them with Bernstein polynomials to solve Fredholm-Volterra integral equations, Fuzzy integral equations, three-dimensional integral equations and Fuzzy system of Fredholm integral equations [19, 20, 21, 22]. Similarly, in [6], attempts were made to solve algebric Volterra integral equations. Their study also implemented a lower triangular system. The triangular functions (TF) method is a new method achieved by Bp functions. Due to their approximation ability and orthogonality, TF were used in many fields of sciences and engineering. One of the implications of TF is using them with a structure of operational matrix to analyze dynamical systems [8]. Their study estimated a low bound for mean integral squared error, too. In a similar research, Maleknejad et al. utilized the TF to solve Fredholm-Volterra integral equations. They converted the given integral equation into a nonlinear system and solved it using the Newton numerical method [13].

It is evident in the literature that most of the numerical methods implemented linear and sometimes nonlinear systems to solve integral equations. It should be noted that getting to the precise solution of these systems is of phenomenal importance. There are many known methods for solving linear systems. However, the pervasive method for solving nonlinear systems is the Newton numeric method [12]. This method is utilized in cases in which the product system from a nonlinear integral equation is nonlinear. Nevertheless, Newton numerical method suffers from a number of limitations. The first limitation of the method is to invertible Jacobian matrix which might be singular. The second limitation is the initial point selection. This point must be feasible. meaning that it should satisfy constraints. Finally, it cannot be used if the number of the unknowns is more than the equations. In contrast, dynamical systems are considered as essential tools for solving nonlinear systems [10]. Since appropriate dynamical systems do not suffer from any of the limitation of the Newton method, they can directly be used to solve integral equations with high precision. It is worth mentioning, that dynamical systems are special cases of recurrent neural networks.

In this paper, TFs were used to produce a nonlinear system to solve Fredholm-Volterra integral equations. Then, the obtained system was solved by an appropriate dynamical system. Dynamical system can be solved in several ways such as S-Functions in MATLAB software, orders like ODE45, numerical methods in applied mathematics like Euler method and various order Runge-Kutta methods. In this paper, we have used fourth-order Runge-Kutta method with an appropriate length step to solve dynamical systems. Now, let us introduce the introduction to dynamic systems.

Consider the nonlinear system of differential equations [10]

$$\dot{x} = f(t, x), \quad x(t_0) = x_0, t \geq t_0, \tag{1}$$

where $x \in \mathbb{R}^n$, $t \geq 0$ and f is continuous on a subset of $\mathbb{R}^n$. System (1) is called a nonlinear time-varying dynamical system. We begin by considering the general nonlinear time-invariant or an autonomous dynamical system

$$\dot{x} = f(x(t)), \quad x(0) = x_0, \quad t \geq 0. \tag{2}$$

where $x(t) \in D \subset \mathbb{R}^n, t \in T_{x_0}$, is the system state vector, D is an open set with $0 \in D$, $f : D \longrightarrow \mathbb{R}^n$ is continuous on D and $T_{x_0} = [0, \tau_0), 0 \leq \tau_0 \leq \infty$.

Definition 1. A point $x_e \in D$ is said to be an equilibrium point of (2) at time $t = t_e$ if $f(x_e) = 0$ for all $t \geq t_e$.

It should be noted that if we suppose the right-hand sides of equations (1) and (2) are equal to zero, the equilibrium point of the system can be achieved. In other words, $x = x_e$ may be the equilibrium point of (2) when, by starting from this point $x = x_e$, system states permanently stay in $x = x_e$ and $x = x_e$ is a root of $f(x) = 0$. This solves nonlinear equations or nonlinear system of equations using dynamical systems. Without loss of generality, it can be assumed that $x_e = 0$ is equilibrium point.

Definition 2. Let $\phi(t; x)$ be the solution of (2) that starts at initial state x_0 at time $t = 0$. The region of attraction of the origin, denoted by R_A, is defined by:

$$R_A = \{x \in D | \phi(t; x) \to 0 \ \text{ as } \ t \to \infty\}.$$

Definition 3. (i) The zero solution $x(t) = 0$ to (2) is *Lyapunov stable* if, for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if $\|x(0)\| < \delta$, then $\|x(t)\| < \varepsilon$, $t \geq 0$.
(ii) The zero solution $x(t) = 0$ to (2) is *asymptotically stable* if it is Lyapunov stable and $\lim_{t\to\infty} x(t) = 0$.

The concept of stability is one of the important concepts regarding the equilibrium points. If an equilibrium point be unstable then by starting from an initial point like $x = x_0$, the system states are not convergent to $x = x_e$; therefore, the system is called unstable. For more information on the concept of equilibrium points and different types of stability, refer to [10]. In this paper, Lyapunov and asymptotic stability has been applied. In order to solve the nonlinear system of equations, they were converted in to dynamical systems, and these systems must be stable. Transforming the system of equations into dynamical systems has many advantages. Now, this superiority isshown by two examples.

Example 1. Consider the following nonlinear system of equations

$$F(X) = 0 \triangleq \begin{cases} \sqrt{x} + \sqrt{y} = 5 \\ \sqrt{x} - \sqrt{y} = 1, \end{cases} \tag{3}$$

we solve (3) by using a dynamical system.

Solution. First, we use Newton's method to solve (3). Jacobi matrix is as follows:

$$J(x_n, y_n) = \begin{pmatrix} \frac{1}{2\sqrt{x_n}} & \frac{1}{2\sqrt{y_n}} \\ \frac{1}{2\sqrt{x_n}} & -\frac{1}{2\sqrt{y_n}}. \end{pmatrix} \tag{4}$$

By starting initial point $X_0 = (10, 10)^T$ and 8 iterations, the optimal answer can be obtained as follows:

$$X^* = \begin{pmatrix} 9.0000000000 \\ 4.0000000002 \end{pmatrix}, \quad F(X^*) = \begin{pmatrix} 0 \\ -4.44 \times 10^{-16} \end{pmatrix}$$

Now, consider the following dynamical system:

$$\begin{cases} \dot{x} = -\left(\sqrt{x} + \sqrt{y} - 5\right) \\ \dot{y} = \sqrt{x} - \sqrt{y} - 1 \end{cases} \tag{5}$$

We know that equilibrium points of (5) are the same solutions of (3). It is clear that system (5) is stable; by starting from initial point $X_0 = (0, 0)^t$ and using Runge-Kutta method, solutions of (5) are convergent to the following answer:

$$X^* = \begin{pmatrix} 9.000000000354914 \\ 3.999999995843754 \end{pmatrix}, \quad F(X^*) = \begin{pmatrix} 9.79 \\ 1.09 \end{pmatrix} \times 10^{-10},$$

More over, if we apply the initial point $X_0 = (-1, -1)^t$, then solutions of (5) are convergent to the following answer:

$$X^* = \begin{pmatrix} 9.000000000511225 + 0.000000000000000i \\ 3.999999995868742 + 0.000000000000000i \end{pmatrix}$$

$$\longrightarrow F(X^*) = \begin{pmatrix} 9.47 \times 10^{-10} - 6.2 \times 10^{-17}i \\ 1.11 \times 10^{-9} + 3.03 \times 10^{-17}i \end{pmatrix}$$

In each case, the number of iterations for Runge-Kutta method is about 100. If the number of iterations be high, then the accuracy will increase. Example 1 shows that, in Newton's method, the initial point should be feasible. In other words, the initial point cannot be negative because of the presence of radicals. According to Jacobi matrix, the initial point cannot be zero. However, we observed that these restrictions do not exist for dynamical systems.

Example 2. Consider the following nonlinear system of equations

$$G(x,y,z)=0 \triangleq \begin{cases} (x-1)^2+(y-1)^4+z^2=0 \\ 3|x-1|+\sqrt{y-1}+z^4=0 \end{cases} \tag{6}$$

We want to solve (6) via dynamical systems.

System (6) is a system of two equations with three unknowns. Newton method cannot be applied to solve it. A dynamical system to solve (6) has the following form:

$$\begin{cases} \dot{x}=(x-1)^2+(y-1)^4+z^2 \\ \dot{y}=(x-1)^2+(y-1)^4+z^2 \\ \dot{z}=3|x-1|+\sqrt{y-1}+z^4 \end{cases} \tag{7}$$

System (7), starting from initial point $X_0=(2.5,3,-2)^t$, is convergent to the following solution:

$$X^*=\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1.000000000000000 \\ 1.000000000000339 \\ 0.000000001009657 \end{pmatrix},$$

and

$$G(X^*)=\begin{pmatrix} 1.0194075\times 10^{-18} \\ -5.819089\times 10^{-7} \end{pmatrix}.$$

There are three variables in the system (6). Then, a differential equation is created for each variable, such that equilibrium points of obtained system are the same solutions of (6). The difference of the first equation with the second equation in (7) is that the first shows the variations of x and the second shows the variations of y.

2 Block-Pulse Functions

A set of block-pulse functions (BPF), $\Psi_m(t)$ containing m component functions in the semi-open interval $[0,T)$ is given by [8]

$$\boldsymbol{\Psi_m(t)} \triangleq [\psi_0(t),\psi_1(t),\cdots,\psi_i(t),\cdots,\psi_{m-1}(t)]^t, \tag{8}$$

where $[...]^T$ denotes transpose. The ith component $\psi_i(t)$ of the BPF vector $\boldsymbol{\Psi_m(t)}$ is defined as

$$\psi_i(t)=\begin{cases} 1 & \frac{iT}{m}\le t<\frac{(i+1)T}{m}, \\ 0 & \text{Otherwise,} \end{cases}$$

where $i=0,1,2,\ldots,(m-1)$ and $h=\frac{T}{m}$.
A square-integrable time function $f(t)$ of Lebesgue measure may be expanded into an m-term BPF series in $t\in[0,T)$ as follows:

$$f(t) \approx [c_0, c_1, c_2, \ldots, c_i, \ldots, c_{m-1}]\, \boldsymbol{\Psi_m(t)} \triangleq C^T \boldsymbol{\Psi_m(t)} \tag{9}$$

where constants c_i are given by

$$c_i = \frac{1}{h}\int_{ih}^{(i+1)h} f(t)dt, \tag{10}$$

Other important properties of BPFs such as disjointness, orthogonality, completeness and operational matrices for integration of BPFs are given in [8], which are not required in this paper.

Definition 4. Let $\psi_0(t)$ be the first member of an m-set BPF; hence, we introduce

$$\psi_0(t) = T1_0(t) + T2_0(t),$$

where $T1_0(t)$ and $T2_0(t)$ functions are shown in Figure 1.

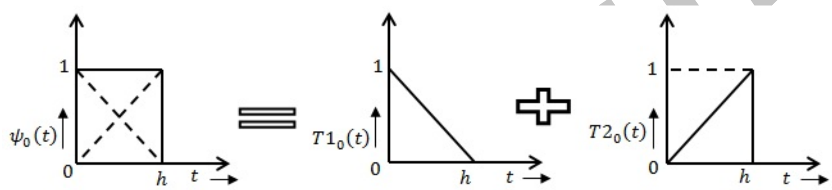

Figure 1: Dissection of BPF into two triangular functions.

For the whole set of BPF, $\boldsymbol{\Psi_m(t)}$, two sets of orthogonal TFs can thus be generated, namely $\boldsymbol{T1_m(t)}$ and $\boldsymbol{T2_m(t)}$ such that

$$\boldsymbol{\Psi_m(t)} = \boldsymbol{T1_m(t)} + \boldsymbol{T2_m(t)}. \tag{11}$$

It is noted that these two sets are complementary to each other as far as block-pulse functions are considered [8]. Now, we can express the m-set triangular function vector as follows:

$$\begin{aligned} \boldsymbol{T1_m(t)} &= [T1_0(t), T1_1(t), \ldots, T1_i(t), \ldots, T1_{m-1}(t)]^T, \\ \boldsymbol{T2_m(t)} &= [T2_0(t), T2_1(t), \ldots, T2_i(t), \ldots, T2_{m-1}(t)]^T, \end{aligned} \tag{12}$$

The ith component of vector $\boldsymbol{T1_m(t)}$ is defined as follows:

$$\boldsymbol{T1_m(t)} = \begin{cases} 1 - \frac{(t-ih)}{h} & ih \le t < (i+1)h, \\ 0 & \text{Otherwise} \end{cases} \tag{13}$$

The ith component of vector $\boldsymbol{T2_m(t)}$ is defined as followes:

$$\boldsymbol{T2_m(t)} = \begin{cases} \frac{(t-ih)}{h} & ih \le t < (i+1)h, \\ 0 & \text{Otherwise} \end{cases} \tag{14}$$

where $i = 0, 1, 2, \ldots, (m-1)$ [8].
Since the elements $T1_i(t)$ and $T2_i(t)$, $i = 0, 1, \ldots, (m-1)$ are mutually disjointed, thereby the condition of orthogonality for TFs is given by

$$\int_0^T T1_i(t)T1_j(t)dt = \delta_{ij}, \qquad \int_0^T T2_i(t)T2_j(t)dt = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. In addition, the other properties of TFs required, such as the product of two TFs vectors or the integration of operational matrices, are shown in [8].
In general, a time function $f(t)$ of Lebesgue measure may be expanded into an m-term TF series in $t \in [0, T)$ as

$$\begin{aligned} f(t) &\simeq [c_0, c_1, \ldots, c_i, \ldots, c_{m-1}]\boldsymbol{T1_m} + [d_0, d_1, \ldots, d_i, \ldots, d_{m-1}]\boldsymbol{T2_m} \\ &= C^T T1_m + D^T T2_m, \end{aligned} \tag{15}$$

where the constant coefficients are the samples of functions such as

$$c_i = f(ih), \quad d_i = f[(i+1)h], \tag{16}$$

where $i = 0, 1, \ldots, m-1$ [8]. Thus, we have $c_{i+1} = d_i$ and for details of approximating properties, TFs refer to [8, 11].

3 Solving Second Kind Fredholm and Volterra Integral Equations via TF Functions and Dynamical Systems

In this section, the numerical solution of the second kind integral equation is considered based on orthogonal triangular functions. Consider the nonlinear second kind Volterra-Fredholm integral equation

$$u(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)F(u(t))dt + \lambda_2 \int_a^b K_2(x,t)G(u(t))dt, \tag{17}$$

where $a \leq x, t \leq b$, $u(t)$ is the unknown function, while $f(x)$, kernels K_1 and K_2 are known in $L^2(\mathbb{R})$. λ_1 and λ_2 are parameters and can be zero. Without loss of generality, suppose that $a = 0$ and $b = 1$. $u(t)$ can be approximated as follows:

$$u(t) \simeq \sum_{k=0}^{m-1} c_k T1_k(t) + \sum_{k=0}^{m-1} c_{k+1} T2_k(t) = C^T \boldsymbol{T1_m(t)} + D^T \boldsymbol{T2_m(t)}, \tag{18}$$

where $\{c_i\}_{k=0}^{m-1}$ and $\{d_k\}_{k=0}^{m-1}$ are defined as in equation (16). Now, the collocation points can be suggested as follows:

$$S_i = ih,$$

where $i = 0, 1, \ldots, m-1$ and $h = \frac{T}{m}$. This is a natural selection for S_i. If we use the orthogonal polynomials in the approximation of function $u(t)$, then roots of orthogonal polynomials could be used instead of S_i, we know that roots may not be equally spaced. Hence, unknown function $u(t)$ has three unknowns by (18) which are as follows:

$$c_0, c_1 = d_0, c_2 = d_1, \ldots, c_m = d_{m-1}. \tag{19}$$

By substituting $u(t)$ into equation (17), an new integral equation is achieved with respect to unknowns in (19). If the integrals are approximated by an appropriate numerical method, then the product relation will be nonlinear. Afterwards, by substituting $S_i = ih$ for x in the product nonlinear equation, a nonlinear system can be achieved which will include $(m+1)$ unknowns and $(m+1)$ equations. In this paper, we approximate integrals by using Gauss-Legendre method on $[0,1]$ (10-points). Thus, we obtain the following nonlinear system:

$$-f(x_i) - \lambda_1 \sum_{j=1}^{10} K_1(x_i,t_j)F(u(t_j))w_j - \lambda_2 \sum_{j=1}^{10} K_2(x_i,t_j)G(u(t_j))w_j + c_i = 0, \tag{20}$$

where $i = 0,1,2,\ldots,m$ and t_j, w_j are knots and weights in the Gauss-Legendre method on $[0,1]$. We transform equation (20) into a dynamical system. The resulting system is as follows:

$$\dot{c}_i = c_i - f(x_i) - \lambda_1 \sum_{j=1}^{10} K_1(x_i,t_j)F(u(t_j))w_j - \lambda_2 \sum_{j=1}^{10} K_2(x_i,t_j)G(u(t_j))w_j, \tag{21}$$

where $i = 0,1,2,\ldots,m$, $x_i = S_i$, $u(x_i) = c_i$ and $u(t_j)$ can be calculated for each j by using equation (15).

Obviously, if we put the right side of equation (21) equal to zero then equilibrium points are obtained. It is also known that the roots of the obtained equations are the solution of equation (20). Therefore, if system (21) is stable, then the states of (21) will converge toward c_i's, which are also the unknowns of the problem. Since the integral equations under investigation have answers, system (20) has a unique answer and the equilibrium point of the system (21) is unique, too. The system (21) includes $c_i, i = 0,1,\ldots,m$ unknowns and fourth order Runge-Kutta method is used to find them. In this method, system (21) is started from an initial point and the states of the system converge into its equilibrium point. As a result, the nonlinear system (20) may be solved. The next section deals with the stability of the dynamical system (21).

4 Stability Analysis

It was shown that the integral equations could be solved by substituting the unknowns by triangular functions and solving the obtained nonlinear system of equations via dynamical systems. This selection is going to show that the obtained dynamical system can be stable. The following theorem is a pre-condition for the stability of a dynamical system.

Theorem 1. (Lyapunov Theorem) Consider the nonlinear dynamical system (2) and assume that there exists a continuously differentiable function $V : D \to \mathbb{R}$ such that

$$V(0) = 0, \tag{22}$$

$$V(x) > 0, \quad x \in D, \quad x \neq 0, \tag{23}$$

$$\dot{V} = V'(x)f(x) \leq 0, \quad x \in D. \tag{24}$$

Then, zero solution $x(t) = 0$ to (2) is Lyapunov stable. In addition, if

$$V'(x)f(x) < 0, \quad x \in D, \quad x \neq 0, \tag{25}$$

then zero solution $x(t) = 0$ to (2) is asymptotically stable.

Proof. See [10]. □

Theorem 1 is an important criterion for assessing the stability of non-linear systems.

Remark 1. In this paper, the following relationships are contracted.

$$V = V(x(t)) \Rightarrow \dot{V} = \frac{dV}{dt}, \quad V' = \frac{dV}{dx}.$$

To solve the nonlinear system of equations:

$$\begin{cases} f_1(x_1, x_2, \ldots, x_n) = 0, \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) = 0. \end{cases}, \tag{26}$$

Consider the following dynamical system:

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, \ldots, x_n), \\ \vdots \\ \dot{x}_n = f_n(x_1, x_2, \ldots, x_n). \end{cases} \tag{27}$$

We know that equilibrium points of (27) are the same solutions of (26). Therefore the solutions of system (26) may easily be achieved if system (27) is stable. The Runge-Kutta numerical method is used for the purpose in which we start from an initial point and considering that the system is stable, then states of system converge into the equilibrium point. This equilibrium point is the same solution to the main problem.

Theorem 2. We consider dynamical system of (27) and define Jacobi matrix J as follows:

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}. \tag{28}$$

If matrix J be a negative definite matrix in a neighborhood of $x = x_e$, then equilibrium point $x = x_e$ will be asymptotically stable.

Proof. Consider the following Lyapunov function:

$$V(x) = \sum_{i=1}^{n} f_i^2(x) = f_1^2(x) + \cdots + f_n^2(x). \tag{29}$$

Derivative of V with respect to t is:

$$
\begin{aligned}
\dot{V} &= 2f_1(x)\left(\dot{x}_1\frac{\partial f_1}{\partial x_1} + \dot{x}_2\frac{\partial f_1}{\partial x_2} + \cdots + \dot{x}_n\frac{\partial f_1}{\partial x_n}\right), \\
&+ 2f_2(x)\left(\dot{x}_1\frac{\partial f_2}{\partial x_1} + \dot{x}_2\frac{\partial f_2}{\partial x_2} + \cdots + \dot{x}_n\frac{\partial f_2}{\partial x_n}\right), \\
&\quad \vdots \\
&+ 2f_n(x)\left(\dot{x}_1\frac{\partial f_n}{\partial x_1} + \dot{x}_2\frac{\partial f_n}{\partial x_2} + \cdots + \dot{x}_n\frac{\partial f_n}{\partial x_n}\right), \\
&= 2f_1(x)\left(f_1\frac{\partial f_1}{\partial x_1} + f_2\frac{\partial f_1}{\partial x_2} + \cdots + f_n\frac{\partial f_1}{\partial x_n}\right), \\
&+ 2f_2(x)\left(f_1\frac{\partial f_2}{\partial x_1} + f_2\frac{\partial f_2}{\partial x_2} + \cdots + f_n\frac{\partial f_2}{\partial x_n}\right), \\
&\quad \vdots \\
&+ 2f_n(x)\left(f_1\frac{\partial f_n}{\partial x_1} + f_2\frac{\partial f_n}{\partial x_2} + \cdots + f_n\frac{\partial f_n}{\partial x_n}\right).
\end{aligned}
$$

After simplification, we have:

$$
\dot{V} = (f_1, f_2, \ldots, f_n)J\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = F^T JF. \tag{30}
$$

It can easily be shown that $V(x_e) = 0$ and $\forall x \in D, V(x) > 0$ so that all the conditions of Lyapunov Theorem are satisfied. Hence, $x = x_e$ is asymptotically stable with Lyapunov function $V(x)$. D is a neighborhood of $x = x_e$. □

If $A = [a_{ij}]_{n\times n}$ be a negative definite matrix, then $\forall i = 1, \ldots, n, a_{ii} < 0$. We note that $\partial f_i/\partial x_i$ are the main diameter arrays of matrix J. By taking each of the measures below, first, the equilibrium points stay unchanged and second, the sign of determinant J is changed.

- Multiplying the right-hand side of each of the equations of (27) by (-1). For example, $\dot{x}_i = -f_i(x)$ instead of $\dot{x}_i = f_i(x)$ is used.
- Changing the right-hand side of (27) by an odd number of equations together.

Since the Runge-Kutta numerical method is used to solve the dynamical system, the sign of $\partial f_i/\partial x_i$ must be checked each time that $f_i(x_i)$ is calculated. In order to meet the assumption of the negative definite for matrix J, if the sign of $f_i(x_i)$ is positive, then the corresponding equation must be multiplied by (-1). Therefore, $\partial f_i/\partial x_i$ can be negative.

Special case) Consider (2) with $x \in \mathbb{R}$. Then dynamical system is as follows:

$$
\dot{x} = -f^2(x), \quad x(0) = x_0 \tag{31}
$$

is stable in finding roots of $f(x) = 0$.

Theorem 3. Dynamical system (31) with equilibrium point $x = x_e$ to find the roots of $f(x) = 0$ is asymptotically stable.

Proof. Consider Lyapunov function as follows

$$V(x) = \int_{x_e}^{x} f^2(y)dy,$$

then, we have

$$\dot{V} = \frac{dV}{dt} = \frac{dV}{dx}.\frac{dx}{dt} = \dot{x} \times f^2(x) = -f^4(x) < 0.$$

□

As a result, if the nonlinear system of equations has only one equation, then we can use (2) or (31) to solve it.

5 Simulation Results

This section is going to provide some examples of linear and nonlinear Fredholm-Volterra integral equations of the second kind and deal with comparing the practicality of this method with other similar methods. In each example, response of dynamical systems is achieved by the fourth-order Runge-Kutta using MATLAB sofware, running on a Pentium 4, 2.4-GHz with 2 GB of RAM. Six examples are presented.

Example 3. Consider the integral equation

$$u(x) + \frac{1}{3}\int_0^1 e^{2x-\frac{5}{3}y} u(y)dy = e^{2x+\frac{1}{3}}, \qquad 0 \le x \le 1. \tag{32}$$

The exact solution is $u(x) = e^{2x}$. Golbabai et al. [9] approximated the solution of this equation by neural networks of radial basis functions (RBF) and achieved the network parameters through creating a cost function using the unconstrained optimization method. By creating the dynamical system mentioned in this article, we have:

$$\dot{c}_i = c_i + \frac{1}{3}\sum_{j=1}^{10} e^{2x_i-\frac{5}{3}t_j} \left(CT1_m(t_j) + DT2_m(t_j)\right) w_j - e^{2x_i+\frac{1}{3}},$$

where $i = 1, 2, \dots, m$ and m is the number of triangular functions mentioned in (12). After solving this system, unknowns are found and therefore, derived the solution to the integral equation. The obtained numerical results are summarized in Table 1. In this example, the number of TF's is 1000 ($m = 1000$), running time is about 15 Min and number of iterations for Runge-Kutta is 80.

Example 4. Consider the nonlinear Fredholm integral equation [5]

$$u(x) + \int_0^1 e^{x-2y} \left(u(y)\right)^3 dy = e^{x+1}, \qquad 0 \le x < 1. \tag{33}$$

Table 1: Comparison of errors of numerical results for Example 3

x_i	Errors in method [9]	Errors (our method)
0	5.40631×10^{-7}	5.7095×10^{-8}
0.1	4.17207×10^{-7}	6.9736×10^{-8}
0.2	1.62255×10^{-7}	8.5176×10^{-8}
0.3	9.97279×10^{-8}	1.0403×10^{-7}
0.4	5.33277×10^{-7}	1.2706×10^{-7}
0.5	5.12821×10^{-7}	1.5520×10^{-7}
0.6	8.86581×10^{-8}	1.8956×10^{-7}
0.7	3.82386×10^{-7}	2.3153×10^{-7}
0.8	6.76977×10^{-7}	2.8279×10^{-7}
0.9	3.36868×10^{-7}	3.4540×10^{-7}
1.0	5.00635×10^{-7}	4.2188×10^{-7}

The exact solution is $u(x) = e^x$. Babolian et al [5]. used the Haar wavelet method to solve this equation. Similar to the previous example, after creating a dynamical system, we solve (33). The relevant errors of numerical results are summarized in Table 2. In this example, $m = 1000$, running time is 10 Min and number of Runge-Kutta iterations is 100.

Example 5. Consider the nonlinear Fredholm integral equation [7]

Table 2: Comparison of errors of numerical results for Example 4

x_i	$\lvert u_{exact} - u_{HaarWavelet} \rvert$ [5]	Errors (our method)
0.1	0.002046	9.9187×10^{-8}
0.2	0.003299	1.0961×10^{-7}
0.3	0.008693	1.2113×10^{-7}
0.4	0.016906	1.3388×10^{-7}
0.5	0.018681	1.4797×10^{-7}
0.6	0.011742	1.6353×10^{-7}
0.7	0.002927	1.8073×10^{-7}
0.8	0.008084	1.9972×10^{-7}
0.9	0.021624	2.2074×10^{-7}

$$u(x) = \sin(\pi x) + \frac{1}{5} \int_0^1 \cos(\pi x) \sin(\pi y) \left(u(y)\right)^3 dy, \qquad 0 \le x \le 1. \tag{34}$$

The exact solution is $u(x) = \sin(\pi x) + \frac{20-\sqrt{391}}{3} cos(\pi x)$. Biazar et al [7]. used Homotopy Perturbation method (HPM) to solve (34). Exact and approximate plots in Figure 2 and errors in Table 3 are shown. In this example we have: $m = 1200$, Runge-Kutta iterations $= 80$ and

Running time is about 15 Min.

Table 3: Absolute errors obtained by [7] and our method for Example 5

x_i	Errors in method [7]	Errors (our method)
0	1.1897×10^{-6}	1.3425×10^{-7}
0.1	1.1315×10^{-6}	1.2768×10^{-7}
0.2	9.6253×10^{-7}	1.0861×10^{-7}
0.3	6.9932×10^{-7}	7.8914×10^{-8}
0.4	3.6765×10^{-7}	4.1487×10^{-8}
0.5	0	5.5511×10^{-16}
0.6	3.6765×10^{-7}	4.1487×10^{-8}
0.7	6.9932×10^{-7}	7.8914×10^{-8}
0.8	9.6253×10^{-7}	1.0861×10^{-7}
0.9	1.1315×10^{-6}	1.2768×10^{-7}
1.0	1.1897×10^{-6}	1.3425×10^{-7}

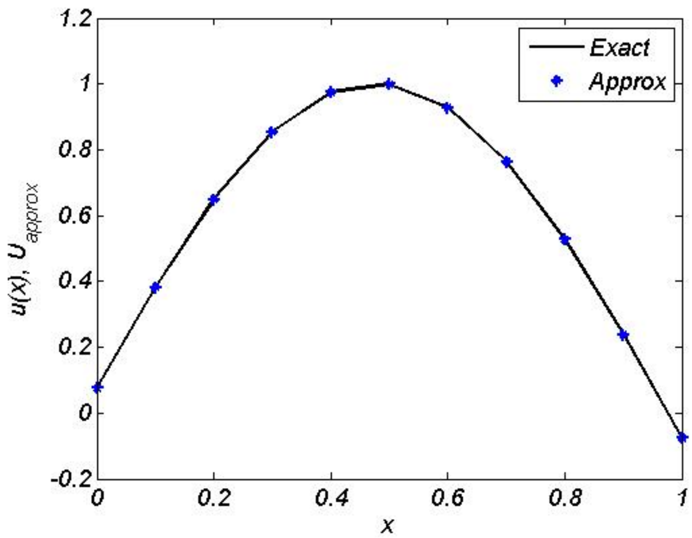

Figure 2: Approximate and exact solutions by our method for Example 5

Example 6. Consider the nonlinear Fredholm-Volterra integral equation [1]

$$u(x) = \frac{1}{6}x + \frac{1}{2}xe^{-x^2} + \int_0^x xte^{-u^2(t)}dt + \int_0^1 xu^2(t)dt, \tag{35}$$

where exact solution is $u(x) = x$ which was dealt with in study of Ahmadi et al. [1]. Their study was concerned with the solvability of non-linear Fredholm-Volterra integral equations;

the existence and uniqueness and linearization methods concerning these equations were investigated. The relevant errors are provided in Table 4. Figure 3 shows the approximate and exact plots; Figure 4 illustrates the numerical coefficients of the triangular functions. We have $m = 4$, number of Runge-Kutta iterations is 100 and running time is about 1 Min.

Table 4: Absolute errors obtained by [1] and our method for Example 6

x_i	Errors in method [1]	Errors (our method)
0	0	0
0.1	3.3830481×10^{-2}	$1.66533453 \times 10^{-16}$
0.2	$6.66696141 \times 10^{-2}$	$3.33066907 \times 10^{-16}$
0.3	$9.64452055 \times 10^{-2}$	$4.44089209 \times 10^{-16}$
0.4	$1.20842155 \times 10^{-1}$	$6.10622663 \times 10^{-16}$
0.5	$1.37304534 \times 10^{-1}$	$8.88178419 \times 10^{-16}$
0.6	$1.43213404 \times 10^{-1}$	$1.11022302 \times 10^{-15}$
0.7	$1.36241284 \times 10^{-1}$	$1.11022302 \times 10^{-15}$
0.8	$1.14769599 \times 10^{-1}$	$1.11022302 \times 10^{-15}$
0.9	$7.80823838 \times 10^{-2}$	$1.11022302 \times 10^{-15}$
1.0	$2.59596401 \times 10^{-2}$	$1.11022302 \times 10^{-15}$

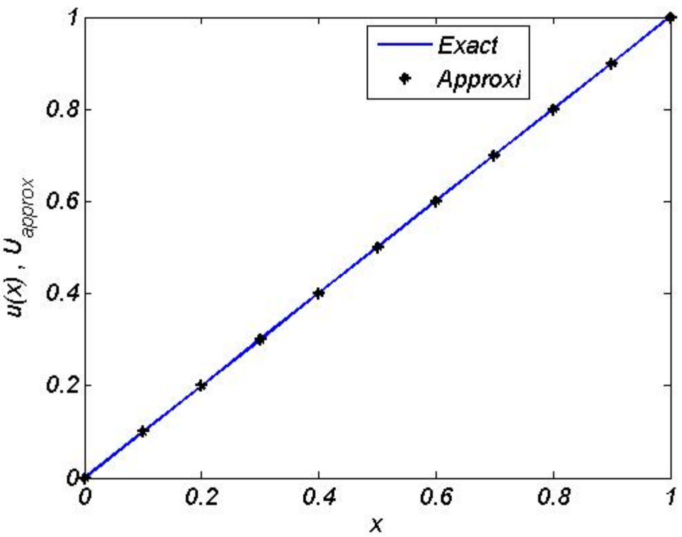

Figure 3: Approximate and exact solutions by our method for Example 6

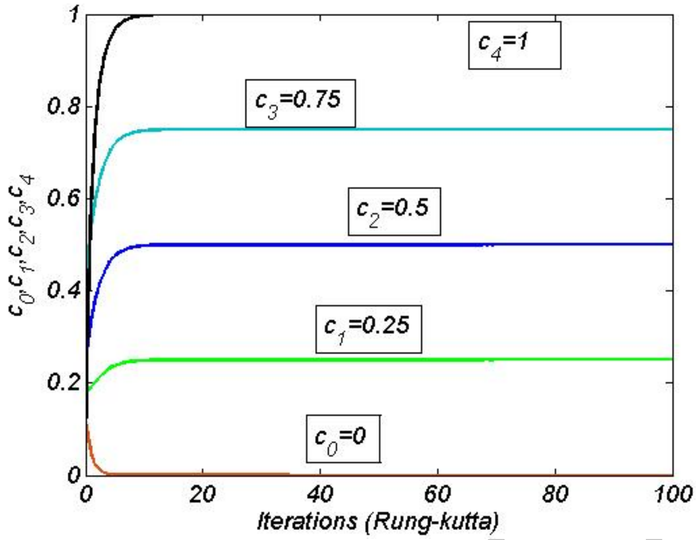

Figure 4: Convergence history of the numerical coefficients of the triangular functions for Example 6

Example 7. Consider the nonlinear Fredholm-Volterra integral equation [19]

$$u(x) = -\frac{t^6}{30} + \frac{t^4}{3} - t^2 + \frac{5}{3}t - \frac{5}{4} + \int_0^t (t-x)\,(u(x))^2\, dx + \int_0^1 (t+x)u(x)dx. \qquad (36)$$

where exact solution is $u(x) = x^2 - 2$. Mirzaee et al. [19] used a combination of Block-Pulse functions and the Taylor methods to solve the equation in which the integration of operational matrix with knots of Newton-Cotes integration method converted the integral equation to an algebaric system. Table 5 summarizes the errors of substituting the triangular functions and creating the corresponding dynamical system. The relevant numerical coefficients of the triangular functions are illustrated in Figure 5. $m = 42$, number of Runge-Kutta iterations is 100 and running time is equal to 1 Min.

Example 8. Consider the nonlinear Fredholm-Volterra integral equation [23]

$$u(x) = e^{-x} - e^{x}(h(x)-1) + \int_0^{h(x)} e^{x+t}u(t)dt - \int_0^1 e^{x+h(t)}u(h(t))dt, \qquad (37)$$

with $h(x) = \ln(x+1)$ and exact solution $u(x) = e^{-x}$. Nemati et al. [23] applied the Legendre polynomials method to solve the equation in which the integral equation was converted to a linear system of equations and was simply solved by an inverse matrix method. The numerical results of applying the triangular functions and creating the dynamical system are summarized in Table 6. Figure 6 illustrates the approximate and exact plots and the numerical coefficients of the triangular functions are provided in Figure 7. Table 7 provides the values of constant coefficients or C_i's. We have $m = 11$, number of Runge-Kutta iterations is 80 and running time is about 1 Min.

Table 5: Absolute errors obtained by [19] and our method for Example 7

x_i	Errors (method [19])	Errors (our method)
0	1.696354×10^{-4}	1.580803×10^{-5}
0.1	1.938751×10^{-4}	1.096763×10^{-4}
0.2	2.015758×10^{-4}	1.519825×10^{-4}
0.3	2.356108×10^{-4}	1.487820×10^{-4}
0.4	2.569098×10^{-4}	9.232611×10^{-5}
0.5	2.727425×10^{-4}	7.799909×10^{-7}
0.6	2.976612×10^{-4}	7.336847×10^{-5}
0.7	3.155343×10^{-4}	8.443672×10^{-5}
0.8	3.434160×10^{-4}	4.097313×10^{-5}
0.9	3.692853×10^{-4}	6.851250×10^{-6}

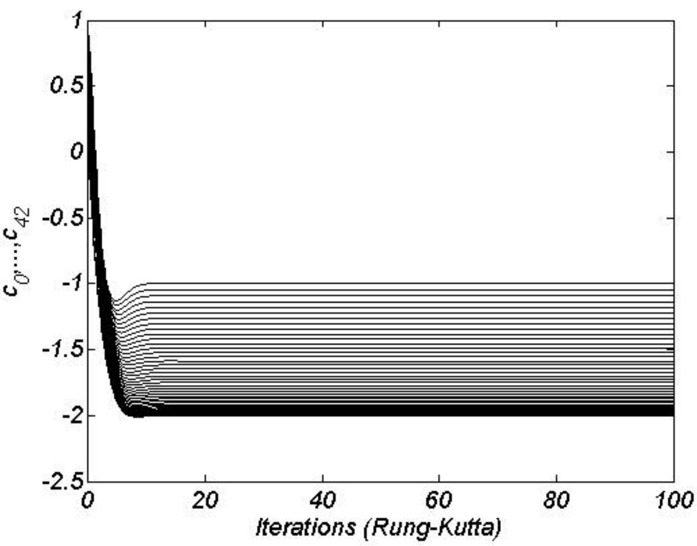

Figure 5: Convergence history of the numerical coefficients of the triangular functions for Example 7

Table 6: Absolute errors obtained by our method for Example 8

x_i	Errors (our method)
0	2.270623×10^{-4}
0.1	7.430852×10^{-5}
0.2	2.981265×10^{-4}
0.3	4.125198×10^{-4}
0.4	4.427352×10^{-4}
0.5	4.364865×10^{-4}
0.6	3.760377×10^{-4}
0.7	3.437967×10^{-4}
0.8	2.095614×10^{-4}
0.9	3.956215×10^{-4}
1.0	2.159911×10^{-4}

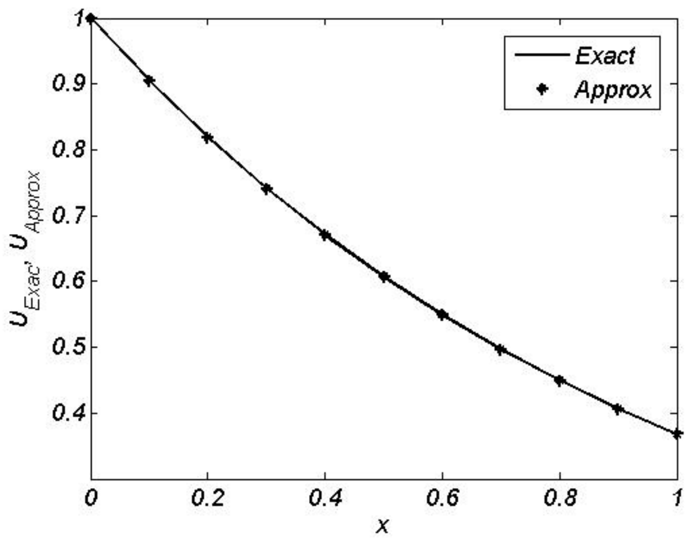

Figure 6: Approximate and exact solutions by our method for Example 8

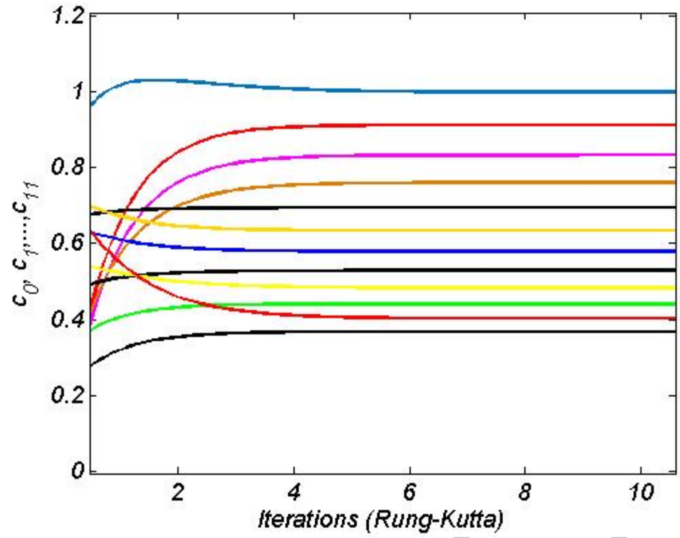

Figure 7: Convergence history of the numerical coefficients, $c_0, \ldots, c_{11}$ for Example 8

Table 7: Coefficients of the triangular functions $c_0, \ldots, c_{11}$ for Example 8

c_0	0.36809543228318
$d_0 = c_1$	0.402289745870384
$d_1 = c_2$	0.441132669386520
$d_2 = c_3$	0.483161950139820
$d_3 = c_4$	0.529052451626724
$d_4 = c_5$	0.579390507148305
$d_5 = c_6$	0.634543785457830
$d_6 = c_7$	0.694908778423649
$d_7 = c_8$	0.761083009964044
$d_8 = c_9$	0.833515347090235
$d_9 = c_{10}$	0.912844657612032
$d_{10} = c_{11}$	0.999727937630924

6 Conclusion

This study utilized a combination of triangular functions and dynamical systems to solve Fredholm-Volterra integral equations. The initial condition for using triangular functions or Block-Pulse functions is to suppose a number of collocation points. These points determine the number of the triangular or block-pulse functions. Using the collocation points to find the coefficients would give a nonlinear system with an equal number of equations and unknowns. However, if other points are used to find coefficients except collocation points, then the number of the unknowns cannot equal that the equations. Thus, the Newton numerical method will be insignificant in finding the solution. Dynamical systems are one of the most accurate methods in such cases as they do not suffer from the constraints of the Newton method. Moreover, the potentials of the dynamical systems can be maximized by implementing the concepts of the control theory.

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حل معادلات انتگرال ولترا-فردهلم نوع دوم با استفاده از توابع مثلثی و سیستم‌های دینامیکی

سلیمانپور باکفایت، ا.
استادیار ریاضی کاربردی-نویسنده مسئول
ایران، تهران، دانشگاه فرهنگیان، گروه ریاضی
asad87000@yahoo.com

کرم سراجی، س.
دانشجوی دکتری ریاضی کاربردی
ایران، البرز، دانشگاه آزاد اسلامی واحد کرج.
rimakool2002@yahoo.com

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چکیده

روش توابع مثلثی می‌تواند تعمیمی از روش بلاک پالس باشد. جواب معادلات انتگرال نوع دوم با استفاده از روش توابع مثلثی، به یک دستگاه معادلات غیرخطی منجر می‌شود. در این مقاله، دستگاه غیرخطی حاصل توسط یک سیستم دینامیکی حل شده است. حل دستگاه غیرخطی حاصل از روش سیستم‌های دینامیکی نسبت به روش عددی نیوتن دارای این مزیت است که در این روش تعداد مجهولات می‌تواند از تعداد معادلات بیشتر باشد. همچنین نقطه شروع سیستم می‌تواند نشدنی باشد. ثابت شده است که سیستم دینامیکی حاصل، پایدار بوده و پاسخ این سیستم از روش عددی رانگ کوتای مرتبه ۴ حاصل می‌شود. نتایج حاصل قابل مقایسه با نتایج روش‌های عددی مشابه است و در اکثر حالات نتایج بدست آمده بهتر از نتایج روش‌های عددی دیگر است. تاثیر روش جدید با ذکر مثال‌هایی بررسی شده است.

کلمات کلیدی

معادلات انتگرال فردهلم-ولترا، سیستم‌های غیرخطی، سیستم‌های دینامیکی، توابع مثلثی، توابع بلاک پالس.