Control and Optimization in Applied Mathematics (COAM)
DOI. 10.30473/coam.2020.50781.1133
Vol. 3, No. 2, Autumn-Winter 2018(1-12), ©2016 Payame Noor University, Iran

Quasi-Gap and Gap Functions for Non-Smooth Multi-Objective Semi-Infinite Optimization Problems

A. Hassani Bafrani[1,*], A. Sadeghieh[2]

[1] Department of Mathematics, Payame Noor University,
P.O. Box, 19395-3697, Tehran, Iran

[2] Department of Mathematics, Yazd branch, Islamic Azad University, Yazd, Iran

Received: January 8, 2020; **Accepted:** April 2, 2020.

Abstract. In this paper, we introduce and study some new single-valued gap functions for non-differentiable semi-infinite multiobjective optimization problems with locally Lipschitz data. Since one of the fundamental properties of gap function for optimization problems is its abilities in characterizing the solutions of the problem in question, then the essential properties of the newly introduced gap functions are established. All results are given in terms of the Clarke subdifferential.

Keywords. Multiobjective optimization, Semi-Infinite Programming, Gap function, Clarke subdifferential

MSC. 90C34; 90C40; 90C26.

* Corresponding author
a.hassani@pnu.ac.ir, alijon.sadeghieh@gmail.com
http://mathco.journals.pnu.ac.ir

1 Introduction

The notion of gap function for mathematical programming problems has been studied in various publications. This concept was first defined by Hearn in [7] for the scalar value convex optimization problems, and was then introduced for variational inequality problem in [1].

For multi-objective optimization problems with smooth data, the gap function has been presented in [4] as a set-valued function. Also, two kinds of set-valued gap functions are introduced for smooth and non-smooth multiobjective optimizations in [14]. Since the initial calculations of set-valued functions are faced with special problems, working with these gap functions is very difficult. Recently, Caristi *et al.* [4] introduced some single-valued gap functions, with complex structures, for multi-objective optimization problems.

All previously mentioned papers considered the (multiobjective) optimization problems with the finite number of constraints. Kanzi and Soleymani-Damaneh [10] studied the concept of gap function for optimization problems with the infinite number of quasi-convex constraints, i.e., quasi-convex semi-infinite problems. Also, the concept of gap function extended to linear semi-infinite multiobjective optimization in [11], and quasi-variational inequality problems in [13].

The purpose of this article is to introduce several scalar-valued gap functions, with simple structures, for semi-infinite multi-objective optimization problems with locally Lipschitz functions. In fact, the purpose of the present paper is to give a generalization of sources listed above. The paper mainly deals with constrained optimization problems formulated as

$$(P) \quad \begin{cases} \text{minimize } f(x) := \big(f_1(x), \dots, f_p(x)\big) \\ \text{subject to } \; g_\alpha(x) \le 0 \; \text{ with } \alpha \in A, \end{cases}$$

where $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ for $i \in \Delta := \{1, \dots, p\}$ and $g_\alpha : \mathbb{R}^n \to \mathbb{R}$ for $\alpha \in A$ are (not necessary differentiable) locally Lipschitz functions, and the index set $A \neq \emptyset$ is arbitrary.

It is worth mentioning that Mastroeni [12] presented a descent method for solving the variational inequalities and optimization problems (under differentiability) based on gap function algorithms. Also, some applications of gap functions in iteration algorithms, proper efficiency, and scalarization of multiobjective optimization can be studied in [4, Section 5].

2 Notations and Preliminaries

In this section, we present definitions and auxiliary results that will be needed in the rest of the paper.

Let $\mathbb{R}^m$ be the $m-$dimensional Euclidean space. Denote by 0_m and $\mathbb{R}^m_+$ the zero vector (i.e., $(\overbrace{0,\dots,0}^{m \text{ times}})$) and the nonnegative orthant of $\mathbb{R}^m$, respectively. Also, the open ball with center $a \in \mathbb{R}^m$ and radius $\varepsilon > 0$ is denoted by $\mathbb{B}_\varepsilon(a)$. The order and weak order in $\mathbb{R}^m$ can respectively be defined by :

$$\left(a^1,...,a^m\right) \leq \left(b^1,...,b^m\right) \iff \begin{cases} a^i \leq b^i, & \forall i = 1,\dots,m, \\ a^l < b^l, & \exists l \in \{1,\dots,m\}, \end{cases}$$

$$\left(a^1,...,a^m\right) < \left(b^1,...,b^m\right) \iff a^i < b^i, \quad \forall i = 1,\dots,m.$$

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The Clarke directional derivative of φ at $\hat{x} \in \mathbb{R}^n$ in the direction $v \in \mathbb{R}^n$, and the Clarke subdifferential of φ at $\hat{x}$ introduced in [8] are respectively given by

$$\varphi^0(\hat{x};v) := \limsup_{y\to\hat{x},\ t\downarrow 0} \frac{\varphi(y+tv)-\varphi(y)}{t},$$

$$\partial_c\varphi(\hat{x}) := \left\{\xi \in \mathbb{R}^n \mid \langle \xi, v\rangle \leq \varphi^0(\hat{x};v) \quad \text{for all } v \in \mathbb{R}^n\right\}.$$

The Clarke subdifferential is a natural generalization of the derivative since it is known that when function φ is continuously differentiable at $\hat{x}$, then $\partial_c\varphi(\hat{x}) = \{\nabla\varphi(\hat{x})\}$.

Theorem 1. *(*Lebourg mean-value [8]*)* Let $x, y \in \mathbb{R}^n$, and suppose that φ is a locally Lipschitz function from $\mathbb{R}^n$ to $\mathbb{R}$. Then, there exists a point u in the open line segment (x, y), such that

$$\varphi(y) - \varphi(x) \in \langle \partial_c\varphi(u), y - x\rangle .$$

Definition 1. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. We say that φ is $c-$quasiconvex (i.e., Clarke quasiconvex) at $\hat{x} \in \mathbb{R}^n$ if for any $x \in \mathbb{R}^n$

$$\varphi(x) \leq \varphi(\hat{x}) \Longrightarrow \left\langle \xi, x - \hat{x}\right\rangle \leq 0 \quad \forall \xi \in \partial_c\varphi(\hat{x}).$$

3 Main Results

As a starting point of this section, we introduce the available set of (P) and the set of active indices a possible point x_0 as follows:

$$S := \{x \in \mathbb{R}^n \mid g_\alpha(x) \leq 0, \quad \forall \alpha \in A\},$$
$$A(x_0) := \{\alpha \in A \mid g_\alpha(x_0) = 0\}.$$

A given point $x_0 \in S$ is said to be an efficient (resp. weakly efficient) solution for (P) if there is no $x \in S$ satisfies $f(x) \leq f(x_0)$ (resp. $f(x) < f(x_0)$). The set of all efficient solutions and weakly efficient solutions of (P) are denoted by E and W, respectively.

For each $x_0 \in S$, let:

$$\widehat{\partial}_c f_i(x_0) := \partial_c f_i(x_0) \setminus \{0_n\}, \qquad \forall i \in \Delta,$$
$$\widehat{\partial}_c f(x_0) := \widehat{\partial}_c f_1(x_0) \times \ldots \times \widehat{\partial}_c f_p(x_0) \subseteq (\mathbb{R}^n)^p,$$
$$\partial_c^\sharp f(x_0) := \partial_c f(x_0) \setminus \{0_{np}\} = \Big(\partial_c f_1(x_0) \times \ldots \times \partial_c f_p(x_0)\Big) \setminus \{0_{np}\}.$$

It is easy to see that

$$\widehat{\partial}_c f(x_0) = \Big\{(\xi_1, \ldots, \xi_p) \in \partial_c f(x_0) \mid \xi_i \neq 0_n \text{ for all } i \in \Delta\Big\},$$
$$\partial_c^\sharp f(x_0) = \Big\{(\xi_1, \ldots, \xi_p) \in \partial_c f(x_0) \mid \xi_i \neq 0_n \text{ for some } i \in \Delta\Big\},$$
$$\widehat{\partial}_c f(x_0) \subseteq \partial_c^\sharp f(x_0) \subseteq \partial_c f(x_0).$$

First, we introduce a quasi-gap function for (P).

Definition 2. For each $(x, y, z) \in S \times S \times \mathbb{R}^n$ and $\xi := (\xi_1, \ldots, \xi_p) \in \partial_c f(z)$, the quasi-gap function $\varphi_y(x, z, \xi)$ is defined as:

$$\varphi_y(x, z, \xi) := \sum_{i=1}^{p} \big\langle \xi_i, x - y \big\rangle.$$

Theorem 2. let f_i be $c-$quasiconvex function at $x_0 \in S$ for $i \in \Delta$.

(I) If for each $y \in S$ there exists some $\xi^{(y)} \in \widehat{\partial}_c f(x_0)$ with $\varphi_y(x_0, x_0, \xi^{(y)}) \leq 0$, then $x_0 \in E$.

(II) If for each $y \in S$ there exists some $\xi^{(y)} \in \partial_c^\sharp f(x_0)$ with $\varphi_y(x_0, x_0, \xi^{(y)}) \leq 0$, then $x_0 \in W$.

Proof. **(I)** Suppose that $x_0 \notin E$. Then, we can find some $x^* \in S$ and $k \in \Delta$, satisfying

$$f_i(x^*) - f_i(x_0) \leq 0, \ \ \forall i \in \Delta, \quad \text{and} \quad f_k(x^*) - f_k(x_0) < 0. \tag{1}$$

The above inequalities and the $c-$quasiconvexy of f_i functions at x_0 imply that

$$\big\langle \xi_i, x^* - x_0 \big\rangle \leq 0, \qquad \forall i \in \Delta, \ \forall \xi_i \in \partial_c f_i(x_0). \tag{2}$$

At the other hand, the assumptions of theorem yield that there exists an $\xi^{(x^*)} \in \widehat{\partial}_c f(x_0)$ such that

$$\varphi_{x^*}(x_0, x_0, \xi^{(x^*)}) \leq 0. \tag{3}$$

It is sufficient to prove that

$$\left\langle \xi_k^{(x^*)}, x^* - x_0 \right\rangle < 0, \tag{4}$$

since (2) and (4) imply $\varphi_{x^*}(x_0, x_0, \xi^{(x^*)}) = \sum_{i=1}^{p} \left\langle \xi_i^{(x^*)}, x_0 - x^* \right\rangle > 0$, which contradicts (3).

If (4) does not hold, in view of (2) we obtain $\left\langle \xi_k^{(x^*)}, x^* - x_0 \right\rangle = 0$. By latter and $\xi_k^{(x^*)} \neq 0_n$ we can find some sequence $\{w_t\} \to x^* - x_0$ such that $\left\langle \xi_k^{(x^*)}, w_t \right\rangle > 0$ for all $t \in \mathbb{N}$. Since $w_t = (w_t + x_0) - x_0$, the latter inequality and $c-$quasiconvexity of f_k lead us to

$$\left\langle \xi_k^{(x^*)}, (w_t + x_0) - x_0 \right\rangle > 0 \implies f_k(w_t + x_0) - f_k(x_0) > 0, \quad \forall t \in \mathbb{N}.$$

Hence, the continuity of f_k concludes that:

$$\lim_{t \to \infty} \big(f_k(w_t + x_0) - f_k(x_0)\big) \geq 0 \implies f_k(x^*) - f_k(x_0) \geq 0,$$

which contradicts (1). Thus (4) holds.

(II) If $x_0 \notin W$, then there exists an $x^* \in S$ such that $f_i(x^*) - f_i(x_0) < 0$, for all $i \in \Delta$. By definition of $\partial_c^{\sharp} f(x)$, there exists a $k \in \Delta$, such that $\xi_k^{(x^*)} \neq 0_n$. Similar to the proof of (I), it can be seen that $\left\langle \xi_k^{(x^*)}, x^* - x_0 \right\rangle < 0$. The remainder of proof is similar to (I) and is hence omitted.

□

The following example shows that the converse of the above theorem does not valid.

Example 1. : Consider the following problem:

$$\begin{cases} \min \Big(|x_1| + x_1 \ , \ |x_2| + x_2 \Big) \\ \\ \quad \text{subject to} \ \ x_1 + x_2 \leq 0. \end{cases}$$

In fact, $f_1(x_1, x_2) = |x_1| + x_1$, $f_2(x_1, x_2) = |x_2| + x_2$, and $g(x_1, x_2) = x_1 + x_2$. Considering $x_0 = (0, 0)$, we have $x_0 \in E$, and

$$\partial_c f_1(x_0) = [0, 2] \times \{0\},$$
$$\partial_c f_2(x_0) = \{0\} \times [0, 2].$$

Taking $\hat{y} = (\hat{y}_1, \hat{y}_2) = (-1, -1) \in S$, for each $\xi_1^{(\hat{y})} \in \widehat{\partial}_c f_1(x_0)$ and $\xi_2^{(\hat{y})} \in \widehat{\partial}_c f_2(x_0)$, we have $\xi_1^{(\hat{y})} = (a_1, 0)$ and $\xi_2^{(\hat{y})} = (0, a_2)$ for some $a_1, a_2 \in (0, 2]$. Thus,

$$\varphi_{\hat{y}}(x_0, x_0, \xi^{(\hat{y})}) = \big\langle\, (a_1, 0)\,, (-\hat{y}_1, -\hat{y}_2)\,\big\rangle + \big\langle\, (0, a_2)\,, (-\hat{y}_1, -\hat{y}_2)\,\big\rangle = a_1 + a_2 > 0.$$

□

Theorem 3. If $x_0 \in E$, then for each $y \in S$ and $m \in \mathbb{N}$, there exists $z^{(m)} \in \mathbb{B}_{1/m}(x_0)$ and $\xi^{(m)} := (\xi_1^{(m)}, \dots, \xi_p^{(m)}) \in \partial_c f(z^{(m)})$, such that

$$\big\langle \xi_i^{(m)}, y - x_0 \big\rangle \geq 0, \qquad \forall i \in \Delta, \tag{5}$$

or

$$\big\langle \xi_k^{(m)}, y - x_0 \big\rangle > 0, \qquad \exists k \in \Delta.$$

Proof. Since the proof is the same as [4, Theorem 4.2], it is omitted, An only different point of these proves is that in [4, Theorem 4.2] the feasible set is convex, and here it is not necessarily convex. □

Remark 1. The result of Theorem 3 can be written as

$$\begin{aligned} x_0 \in E \implies & \forall y \in S,\ \ \forall m \in \mathbb{N},\ \ \exists z^{(m)} \in \mathbb{B}_{1/m}(x_0),\ \ \exists (\xi_1^{(m)}, \dots, \xi_p^{(m)}) \in \partial_c f(z^{(m)}), \\ & \Big(\langle \xi_1^{(m)}, y - x_0 \rangle, \langle \xi_2^{(m)}, y - x_0 \rangle, \dots, \langle \xi_p^{(m)}, y - x_0 \rangle \Big) \nleq 0_p. \end{aligned}$$

The similar proof of Theorem 3 shows that:

$$\begin{aligned} x_0 \in W \implies & \forall y \in S,\ \ \forall m \in \mathbb{N},\ \ \exists z^{(m)} \in \mathbb{B}_{1/m}(x_0),\ \ \exists (\xi_1^{(m)}, \dots, \xi_p^{(m)}) \in \partial_c f(z^{(m)}), \\ & \Big(\langle \xi_1^{(m)}, y - x_0 \rangle, \langle \xi_2^{(m)}, y - x_0 \rangle, \dots, \langle \xi_p^{(m)}, y - x_0 \rangle \Big) \nless 0_p. \end{aligned}$$

Definition 3. Suppose that x_0 is an efficient solution to (P). The point $y \in S$ is said to be compatible with x_0 if the number of natural numbers m, which is satisfied in (5) is infinite. The set of all compatible points with x_0 is denoted by $S(x_0)$.

The following corollary of Theorem 3, is stated as the approximation converse of Theorem 2.

Theorem 4. Suppose that $x_0 \in E$ and $y \in S(x_0)$. Then there exists a sequence $\{z^{(m)}\}_{m=1}^{\infty}$ converging to x_0, and $\{\xi^{(m)}\}_{m=1}^{\infty}$ with $\xi^{(m)} \in \partial_c f\left(z^{(m)}\right)$, such that:

$$\varphi_y\left(x_0, z^{(m)}, \xi^{(m)}\right) \leq 0, \qquad \forall m \in \mathbb{N}.$$

Now, we introduce a new gap function for the problem (P).

Definition 4. For each $(x,z) \in S \times \mathbb{R}^n$ and $\xi := (\xi_1, \ldots, \xi_p) \in \partial_c f(z)$, the gap function $\varphi(x,z,\xi)$ is defined as:

$$\varphi(x,z,\xi) := \sup_{y \in S} \Big\{ \sum_{i=1}^{p} \langle \xi_i, x-y \rangle \Big\}.$$

It is easy to see that

$$\varphi(x,z,\xi) = \sup_{y \in S} \varphi_y(x,z,\xi).$$

Notice that the above gap function is more suitable than the gap function, which is defined in [4], because of $z = x$ in that gap function, so our gap function is its extension. Moreover, the gap function presented in [4] is more complicated in calculus, since its style is infimum of superior.

Lemma 1. For each $x \in S,\ z \in \mathbb{R}^n$, and $\xi \in \partial_c f(z)$, we have:

$$\varphi(x,z,\xi) \geq 0.$$

Proof. By taking $y = x$ in definition of $\varphi(x,z,\xi)$, the result is clear. □

Now, we can state the following famous theorem.

Theorem 5. Suppose that f_i is a $c-$quasiconvex function at $x_0 \in S$ for each $i \in \{1, \ldots, p\}$.

(I) If $\varphi(x_0, x_0, \hat{\xi}) = 0$ for some $\hat{\xi} \in \widehat{\partial}_c f(x_0)$, then $x_0 \in E$.

(II) If $\varphi(x_0, x_0, \xi^\sharp) = 0$ for some $\xi^\sharp \in \partial_c^\sharp f(x_0)$, then $x_0 \in W$.

Proof. **(I)** $\varphi(x_0, x_0, \hat{\xi}) = 0$ implies that for each $y \in S$ we have $\varphi_y(x_0, x_0, \hat{\xi}) \leq 0$. Theorem 2 justifies the result.

(II) Applying the proof of part (I), the result holds.

□

Remark 2. In the best of our knowledge, the inverse of Theorem 5 is not valid, even by convexity and differentiability of involving functions. However, in [4] shows that the inverse of Theorem 5 holds for set-valued gap function at a proper, efficient solution under some suitable assumptions. However, the characterization of situations for the satisfactory of the inverse of Theorem 5 is an important open problem.

Now, we introduce another gap function for the problem (P), in which satisfies in the converse of Theorem 5.

Definition 5. For each $x \in S$, $\xi := (\xi_1, \ldots, \xi_p) \in \partial_c f(x)$, and $\lambda := (\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^p_+$ with $\sum_{i=1}^p \lambda_i = 1$, we define:

$$\varphi^*(x, \xi, \lambda) := \sup_{y \in S} \sum_{i=1}^{p} \lambda_i \langle \xi_i, x - y \rangle.$$

It is trivial that by using the proof of Theorem 5, if f_i for each $i = 1, \ldots, p$ is $c-$quasiconvex at $x_0 \in S$, and if $\varphi^*(x_0, \hat{\xi}, \lambda) = 0$ for some $\hat{\xi} \in \widehat{\partial}_c f(x_0)$ and $\lambda > 0_p$, then $x_0 \in E$. The proof of the converse of this result needs the following definition.

Definition 6. $\hat{x} \in S$ is said a Karush-Kuhn-Tucker point for problem (P) if there exist $\lambda := (\lambda_1, \ldots, \lambda_p) \geq 0_p$ with $\sum_{i=1}^p \lambda_i = 1$, and $\mu_\alpha \geq 0$ for $\alpha \in A(\hat{x})$, a finite number of them are nonzero, such that:

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_c f_i(\hat{x}) + \sum_{\alpha \in A(\hat{x})} \mu_\alpha \partial_c g_\alpha(\hat{x}).$$

$\hat{x} \in S$ is said to be strong Karush-Kuhn-Tucker point for problem (P) if the above inclusion holds for some $\lambda := (\lambda_1, \ldots, \lambda_p) > 0_p$. The set of all Karush-Kuhn-Tucker points (resp. strong Karush-Kuhn-Tucker points) of (P) is denoted by $\mathcal{K}$ (resp. $\mathcal{SK}$).

Many authors have studied necessary conditions for optimality of multiobjective semi-infinite programming; see, for example, [2, 5, 8, 9]. We can formulate these necessary conditions as follows:

$$x_0 \in W \Longrightarrow x_0 \in \mathcal{K},$$
$$x_0 \in E \Longrightarrow x_0 \in \mathcal{SK}.$$

The above mentioned necessary optimality conditions hold under some assumptions (same as closedness of *cone* $\left(\bigcup_{\alpha \in A(x_0)} \partial_c g_\alpha(x_0)\right)$ and\or compactness of index set A) and suitable constraint qualifications (same as Abadie, or Mangasarian-Fromovitz). These special conditions differ from paper to paper, and none of them play a role in proving converse of the Theorem 5, so, naturally, we use $x_0 \in \mathcal{K}$ and $x_0 \in \mathcal{SK}$ in place of $x_0 \in E$ and $x_0 \in W$.

Theorem 6. Let $x_0 \in \mathcal{K}$. If g_α functions are c-quasiconvex at x_0 for $\alpha \in A(x_0)$, then there exist $\xi \in \partial_c f(x_0)$ and $\lambda \in \mathbb{R}^n_+$ such that $\varphi^*(x_0, \xi, \lambda) = 0$.

Proof. By definition of $\mathcal{K}$, there exist some $\lambda := (\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^p_+$ with $\sum_{i=1}^p \lambda_i = 1$, and nonnegative $\mu_{\alpha_1}, \ldots, \mu_{\alpha_q}$ with $\{\alpha_1, \ldots, \alpha_q\} \subseteq A(x_0)$,and $\xi_i \in \partial_c f_i(x_0)$ for $i = 1, \ldots, p$, and $\zeta_{\alpha_m} \in \partial_c g_{\alpha_m}(x_0)$ for $m = 1, \ldots, q$, such that:

$$\sum_{i=1}^{p} \lambda_i \xi_i + \sum_{m=1}^{q} \mu_{\alpha_m} \zeta_{\alpha_m} = 0. \tag{6}$$

Let $y \in S$. Then,

$$g_{\alpha_m}(y) \leq 0 = g_{\alpha_m}(x_0), \qquad \forall m = 1, \ldots, q.$$

Thus, according to c-quasiconvexity of g_{α_m} functions

$$\langle \zeta_{\alpha_m}, y - x_0 \rangle \leq 0, \qquad \forall m = 1, \ldots, q.$$

The last inequality and (6) imply that:

$$\sum_{i=1}^{p} \lambda_i \langle \xi_i, y - x_0 \rangle = - \sum_{m=1}^{q} \mu_{\alpha_m} \langle \zeta_{\alpha_m}, y - x_0 \rangle \geq 0.$$

Therefore,

$$\sum_{i=1}^{p} \lambda_i \langle \xi_i, x_0 - y \rangle \leq 0.$$

From this and $\sum_{i=1}^{p} \langle \xi_i, x_0 - x_0 \rangle = 0$, the result is proved. □

As mentioned in Remark 2, the converse of Theorem 5 is not valid in general. The following example shows this invalidity.

Example 2. Considering the problem that is considered in Example 1. we saw that $x_0 = (0, 0) \in E$ and

$$\varphi_y(x, z, \widehat{\xi}) = -a_1 y_1 - a_2 y_2,$$

for each $y = (y_1, y_2) \in S$ and $\hat{\xi}_1 = (a_1, 0)$ and $\hat{\xi}_2 = (0, a_2)$ with $a_1, a_2 \in (0, 2]$. Hence,

$$\varphi\Big(x_0, x_0, (\hat{\xi}_1, \hat{\xi}_2)\Big) = \sup\big\{ - a_1 y_1 - a_2 y_2 \mid y_1 + y_2 \leq 0 \big\}.$$

Since $a_1, a_2 > 0$, taking $y_1 < 0$ and $y_2 < 0$, implies that:

$$\varphi\Big(x_0, x_0, (\hat{\xi}_1, \hat{\xi}_2)\Big) > 0.$$

In a similar way it can be shown that for each $(\xi_1^\sharp, \xi_2^\sharp) \in \partial_c^\sharp f(x_0)$ we have

$$\varphi\Big(x_0, x_0, (\xi_1^\sharp, \xi_2^\sharp)\Big) > 0.$$

□

The following example summarizes our results.

Example 3. Consider the following problem:

$$\begin{cases} \min \left(\begin{cases} x^{\frac{1}{2}} & \text{if} \quad x \in (0,1) \\ x^{\frac{3}{2}} & \text{if} \quad x \in [1, +\infty) \\ 0 & \text{if} \quad x \in (-\infty, 0] \end{cases} , \begin{cases} x - 1 & \text{if} \quad x \in [2, +\infty), \\ 3 - x & \text{if} \quad x \in (-\infty, 2) \end{cases} \right) \\ \\ \text{subject to} \quad |x - \frac{1}{2}| - \frac{1}{2} \leq 0. \end{cases}$$

In fact, $f_1(x) = \begin{cases} x^{\frac{1}{2}} & \text{if} \quad x \in (0,1) \\ x^{\frac{3}{2}} & \text{if} \quad x \in [1,+\infty) \\ 0 & \text{if} \quad x \in (-\infty,0] \end{cases}$, $f_2(x) = \begin{cases} x-1 & \text{if} \quad x \in [2,+\infty), \\ 3-x & \text{if} \quad x \in (-\infty,2) \end{cases}$, and $g_1(x) = |x-\frac{1}{2}| - \frac{1}{2}$. It is easy to check that $\partial_c f_1(1) = \left[\frac{1}{2}, \frac{3}{2}\right]$, $\partial_c f_2(1) = \{-1\}$, $\partial_c g_1(1) = \{0\}$, and $A(1) = \{1\}$. Thus, taking $\hat{\xi} := (1,-1) \in \widehat{\partial}_c f(1)$, we conclude that $\varphi(1,1,\hat{\xi}) = 0$, and so $1 \in E$ by Theorem 5.
On the other hand, since

$$0 \in \partial_c f_1(1) + \partial_c f_2(1) + \partial_c g_1(1),$$

then $1 \in \mathcal{SK} \subseteq \mathcal{K}$ by setting $\lambda_1 = \lambda_2 = \mu_1 = 1$. This fact and Theorem 6 deduce that $\varphi^*(1,\hat{\xi},\widehat{\lambda}) = 0$ for $\widehat{\lambda} := (1,1)$. □

4 Conclusion

In conclusion, for each $x, y \in S$, $z \in \mathbb{R}^n$, $\xi_i \in \partial_c f_i(z)$, and $\lambda_i \geq 0$ with $\sum_{i=1}^{p} \lambda_i = 1$, let

$$\widehat{\varphi}_y(x,z,\xi,\lambda) := \sum_{i=1}^{p} \lambda_i \langle \xi_i, y-x \rangle,$$
$$\widehat{\varphi}(x,z,\xi,\lambda) := \sup_{y \in S} \widehat{\varphi}_y(x,z,\xi,\lambda).$$

$\widehat{\varphi}$, as a generalization of φ and φ^*, is a new general form of gap function for (P). In similar way to Theorems 3, 5, and 6 (apart from some small differences), the following theorems can be proved:

Theorem 7. Suppose that the f_i (for $i = 1,\ldots,p$) and g_α (for $\alpha \in A(x_0)$) are $c-$quasiconvex functions at x_0. Then, the following assertions hold:

(I) $\exists \hat{\xi} \in \widehat{\partial}_c f(x_0),\ \exists \lambda > 0_p,\ \widehat{\varphi}(x_0,x_0,\hat{\xi},\lambda) = 0 \implies x_0 \in E.$

(II) $x_0 \in E \overset{suitable\ conditions}{\Longrightarrow} x_0 \in \mathcal{SK} \Longrightarrow \exists \xi \in \partial_c f(x_0),\ \exists \lambda > 0_p,\ \widehat{\varphi}(x_0,x_0,\xi,\lambda) = 0.$

Theorem 8. Suppose that the f_i (for $i = 1,\ldots,p$) and g_α (for $\alpha \in A(x_0)$) are $c-$quasiconvex functions at x_0. Then, the following assertions hold:

(I) $\exists \xi^\sharp \in \partial_c^\sharp f(x_0),\ \exists \lambda > 0_p,\ \widehat{\varphi}(x_0,x_0,\xi^\sharp,\lambda) = 0 \implies x_0 \in W.$

(II) $x_0 \in W \overset{suitable\ conditions}{\Longrightarrow} x_0 \in \mathcal{K} \Longrightarrow \exists \xi \in \partial_c f(x_0),\ \exists \lambda \geq 0_p,\ \widehat{\varphi}(x_0,x_0,\xi,\lambda) = 0.$

Theorem 9. Suppose that each f_i (for $i = 1,\ldots,p$) is a $c-$quasiconvex function at x_0. Then, the following assertions hold:

(I) $\forall y \in S,\ \exists \xi_{(y)} \in \partial_c f(x_0),\ \exists \lambda > 0_p,\ \widehat{\varphi}_y(x_0, x_0, \xi_{(y)}, \lambda) \leq 0 \implies x_0 \in E.$

(II) $x_0 \in E \Longrightarrow \forall y \in S(x_0),\ \exists \{z^{(m)}\} \to x_0,\ \exists \xi^{(m)} \in \partial_c f(z^{(m)}),\ \forall \lambda \geq 0_p,\ \widehat{\varphi}_y(x_0, z^{(m)}, \xi^{(m)}, \lambda) \leq 0\ \ \forall m \in \mathbb{N}.$

Remark 3. It is easy to show that the condition $\exists \lambda > 0_p$ in Theorem 8(I) can be replaced by the weaker condition $\exists \lambda \geq 0_p$, if

$$\xi_k^\sharp \neq 0_n \implies \lambda_k \neq 0.$$

References

[1] L. Altangerel, R.I. Bot, G. Wanka, Conjugate duality in vector optimization and some applications to the vector variational inequality, Journal of Mathematical Analysis and Applications 329 (2007) 1010-1035.

[2] G. Caristi, N. Kanzi, Karush-Kuhn-Tuker Type Conditions for Optimality of Non-Smooth Multiobjective Semi-Infinite Programming, International Journal of Mathematical Analysis 9 (2015) 1929-1938.

[3] G. Caristi, N. Kanzi, M., Soleymani-Damaneh, On gap functions for nonsmooth multiobjective optimization problems, Optim. Lett. 12 (2018) 273-286.

[4] C.Y. Chen, C.J. Goh, X.Q. Yang, The gap function of a convex multicriteria optimization problem, European Journal of Operational Research 111 (1998) 142-151.

[5] T.D. Chuong, D.S. Kim, Nonsmooth semi-infinite multiobjective optimization problems, Journal of Optimization; Theory and Applications 160 (2014) 748-762 .

[6] F.H. Clarke, Optimization and nonsmooth analysis. Wiley, Interscience (1983).

[7] D.W. Hearn, The gap function of a convex program, Operations Research Letters 1 (1982) 67-71.

[8] N. Kanzi, Necessary and Sufficient Conditions for (Weakly) Efficient of Non-Differentiable Multi-Objective Semi-Infinite Programming Problems, Iran J. Sci. Technol. Trans. Sci. 42 (2018) 1537-1544.

[9] N. Kanzi, Karush-Kuhn-Tucker Types Optimality Conditions for Non-Smooth Semi-Infinite Vector Optimization Problems, J. Mathematical Extension 9 (2015) 45-56.

[10] N. Kanzi, M. Soleyman-damaneh, Slater CQ, optimality and duality for quasiconvex semi-infinite optimization problems, J. Math.Anal.Appl. 434 (2016) 638-651.

[11] N. Kanzi, J. Shaker Ardekani, G. Caristi, Optimality, scalarization and duality in linear vector semi-infinite programming, Optimization. 67 (2018) 523-536.

[12] G. Mastroeni, Gap functions for equilibrium problems, J. Glob. Optim. 27 (2003) 411–426.

[13] H. Mirzaee, M. Soleimani-damaneh, Optimality, duality and gap function for quasi variational inequality problems, ESAIM Control Optim. Calc. Var. 23 (2017) 297–308.

[14] M. Soleimani-damaneh, The gap function for optimization problems in Banach spaces, Nonlinear Analysis 69 (2008) 716-723.

توابع شبه‌شکاف و شکاف برای مسائل نیمه-نامتناهی چندهدفه غیرهموار

حسنی بافرانی، ع.
مربی، گروه ریاضی کاربردی-نویسنده مسئول،
ایران، تهران، دانشگاه پیام‌نور، گروه ریاضی، صندوق پستی ۳۶۹۷-۱۹۳۹۵
a.hassani@pnu.ac.ir

صادقیه، ع.
استادیار، گروه ریاضی کاربردی،
ایران، یزد، دانشگاه آزاد اسلامی-واحد یزد، گروه ریاضی،
alijon.sadeghieh@gmail.com

تاریخ دریافت: ۱۸ دی ۱۳۹۸ تاریخ پذیرش: ۱۴ فروردین ۱۳۹۹

چکیده

در این مقاله ما به معرفی و مطالعه چند تابع شکاف تک‌مقداری جدید برای مسائل بهینه‌سازی چندهدفه نیمه نامتناهی غیرمشتق‌پذیر با داده‌های موضعا لیپ‌شیتز پرداخته‌ایم. از آنجا که یکی از خواص اصلی هر تابع شکافی برای یک مسئله بهینه‌سازی، توانایی آن در مشخص‌سازی جواب‌های آن مسئله است، این خاصیت توابع شکاف جدید معرفی شده نیز ارائه شده است. تمامی احکام بر حسب زیرمشتق کلارک بیان شده‌اند.

کلمات کلیدی

بهینه‌سازی چندهدفه، برنامه‌ریزی نیمه-نامتناهی، تابع شکاف، زیرمشتق کلارک.