

## Charge gradient effects on modulated dust lattice wave packets in dusty plasma crystals

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### Abstract

Nonlinear Dust lattice modes are studied in a hexagonal two-dimensional dusty plasma lattice, in presence of charge gradient of dust particles. In this lattice, such gradients affect nonlinear behavior of dust lattice waves. The amplitude modulation of off-plane transverse dust lattice wave packets is investigated considering the anisotropy of interactions, caused by the height-dependent charge variations. A nonlinear Schrodinger equation described time evolution of modulated off-plane transverse dust lattice wave packet. Calculations show that the charge gradient changes the stability condition of the solution of the nonlinear Schrodinger equation.

**Keywords:** Dusty plasma crystal, Dust lattice mode, Nonlinear wave, Charge fluctuation, Modulated wave packets

### 1. Introduction

Dust lattice waves are produced by the oscillations of regularly spaced charged micro particles suspended in a plasma crystal, which form as a result of strong mutual coulomb interaction [1, 2]. Dust lattices support a variety of linear [3-7] and nonlinear modes [8-15]. Recently, Yaroshenko et al, studied the vertical vibrations of a one-dimensional string of magnetized particles, taking into account the magnetic force associated with gradients of an external magnetic field, and they found a new low-frequency oscillatory mode [18]. The influence of an inhomogeneous magnetic field, ion focusing effect and equilibrium charge gradient on the propagation of the modified dust lattice modes in a one-dimensional paramagnetic lattice is considered in Ref [19]. Dust lattice waves in hexagonal dusty plasma crystal have been studied in various investigations [8-22]. The occurrence of 2D modulated transverse dust-lattice wave (TDLW) packets moving at a negative group velocity, i.e., the wave is backward propagating, has been established by recent numerical (molecular dynamics) [13] and experimental [14] investigations. This “*bending mode*” was theoretically investigated in linear and nonlinear region [23]. Mode coupling instability in hexagonal dusty plasma crystals has been studied recently [24].

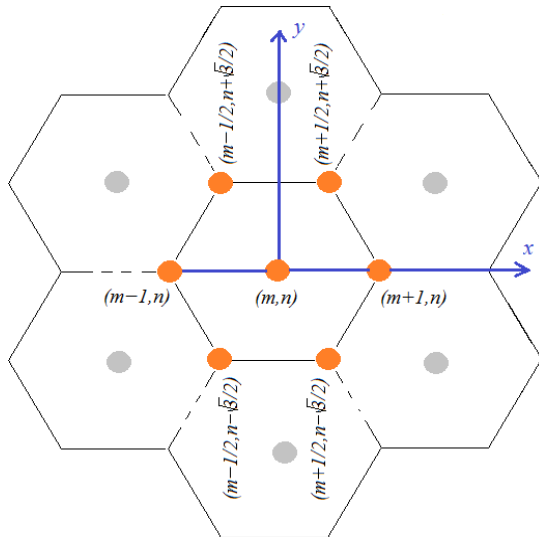
In this paper, we consider a two dimensional monolayer of micro particles forming the hexagonal-type crystal, in the presence of an external electric field.

Propagation of the dust lattice waves is studied theoretically. The aim of this work is to study the effects of anisotropy of interactions, caused by the height-dependent charge variations (HDCV) on the amplitude modulation of off-plane transverse dust lattice wave (TDLW) packets. The modulational instability is investigated. We must point out for rigor that the modulation theory employed here is a *mildly* nonlinear theory, which claims to model weak vertical displacements. The latter point justifies our choice in neglecting the coupling to in-plane dust grain motion, since we are only interested in the effect of height-dependent charge variations on off-plane motion.

The outline of the manuscript is as follows. The equation of motion is derived in Sec. 2 and simplified by adopting a continuum approximation. The derivation of an evolution equation for the modulated wave amplitude is presented in Sec. 3 by assuming transverse wave propagation either along a principal axis of the hexagonal structure or perpendicular to it. The modulational stability is investigated in Sec. 4, and the results are then summarized in Sec. 5.

### 2. Vibrational modes in a hexagonal lattice

In order to describe the vertical modes in dusty plasma, we consider a hexagonal lattice, where the spherical dust grains have the same charge  $Q$  and mass  $M$ , separated by average distance  $d$  (figure 1). The



**Figure 1.** Hexagonal structure of dusty plasma crystal.

electrostatic potential of each particle is assumed to be the screened Coulomb potential. The electrostatic potential energy between neighboring grains of hexagonal lattice can be written as

$$U_{o,i} = \frac{Q_o Q_i}{4\pi\epsilon_o |r_{o,i}|} \exp\left(-\frac{|r_{o,i}|}{\lambda_D}\right), \quad (1)$$

where  $\lambda_D$  is the screening length of the plasma,  $|r_{o,i}|$  is the distance between the central particle and other neighbors, and  $i \equiv (m \pm 1, n), (m \pm 1/2, n \pm \sqrt{3}/2)$ . The electrostatic force acting on the central particle can be obtained as  $\vec{F}_{o,i} = -\partial U_{o,i} / \partial \vec{r}_o$ . Then the  $z$ -component of equation of motion for central particle in the lattice is

$$M\ddot{z}_{m,n} + \nu M\dot{z}_{m,n} = F_E - Mg + \sum_{i=1}^6 F_{o,i}, \quad (2)$$

where  $F_E$  is the electric force acting on the central grain due to an external electric field. We shall assume a smooth, continuous variation of the intensity  $\mathbf{E}$ , as well as the grain charge  $Q$  near the equilibrium position  $z = 0$ . Thus, we may develop

$$E(z) = E_0 + E'_0 z + \frac{1}{2!} E''_0 z^2 + \frac{1}{3!} E'''_0 z^3 + \dots,$$

$$Q(z) = Q_0 + Q'_0 z + \frac{1}{2!} Q''_0 z^2 + \frac{1}{3!} Q'''_0 z^3 + \dots,$$

where the prime and the subscript “0” denote differentiation with respect to  $z$  and evaluation at  $z = 0$ , respectively.

Using eqs.(1) and (2), the equation of motion become

$$\ddot{z}_{m,n} + \nu \dot{z}_{m,n} = -\omega_g^2 z_{m,n} - K_1 z_{m,n}^2 - K_2 z_{m,n}^3 + K_3 \sum_{i=1}^6 z_i + K_4 z_{m,n} \sum_{i=1}^6 z_i + K_5 z_{m,n}^2 \sum_{i=1}^6 z_i + K_6 \sum_{i=1}^6 z_i^2$$

$$\begin{aligned} &+ K_7 z_{m,n} \sum_{i=1}^6 z_i^2 + K_8 \sum_{i=1}^6 z_i^3 + K_9 \sum_{i=1}^6 (z_i - z_{m,n}) \\ &+ K_{10} z_{m,n} \sum_{i=1}^6 (z_i - z_{m,n}) + K_{11} z_{m,n}^2 \sum_{i=1}^6 (z_i - z_{m,n}) \\ &+ K_{12} \sum_{i=1}^6 (z_i - z_{m,n})^2 + K_{13} z_{m,n} \sum_{i=1}^6 (z_i - z_{m,n})^2 \\ &+ K_{14} \sum_{i=1}^6 z_i (z_i - z_{m,n})^2 + K_{15} \sum_{i=1}^6 z_i (z_i - z_{m,n}) \\ &+ K_{16} z_{m,n} \sum_{i=1}^6 z_i (z_i - z_{m,n}) + K_{17} \sum_{i=1}^6 z_i^2 (z_i - z_{m,n}) \\ &+ K_{18} \sum_{i=1}^6 (z_i - z_{m,n})^3, \end{aligned} \quad (3)$$

where  $z_{m,n}$ ,  $z_i$  refers to vertical displacement of the central particle and six neighbors from their equilibrium positions, respectively. Coefficients  $K_i$  ( $i = 1, \dots, 18$ ) and  $\omega_g$  are defined in the Appendix. Here gravitation is compensated by two terms of force.

$$Mg = Q_o E_o - 6 \frac{Q_o Q'_o}{4\pi\epsilon_o d}. \quad (4)$$

## 2.1. Linear dispersion relation

Waves can propagate along an arbitrary direction, which is here denoted by an angle  $\theta$ , representing the angle between the wave vector  $k$  and a primitive translation vector (along the  $x$  axis), i.e.,  $k_x = k \cos \theta$  and  $k_y = k \sin \theta$ . Retaining only linear contribution in the form of “phonons” of the type

$$z_{m,n} = z_0 \exp[-i\omega t + ikd(m \cos \theta + n \sin \theta)] + c.c., \quad (5)$$

we obtain an inverse-optic-mode-like dispersion relation from eq. (3),

$$\begin{aligned} \omega(\omega + i\nu) = &\omega_g^2 - 6K_3 + 4(K_3 + K_9) \\ &\times \left[ \sin^2(kd \cos \theta / 2) \right. \\ &+ \sin^2[kd \cos(\theta + \pi/3) / 2] \\ &\left. + \sin^2[kd \cos(\theta - \pi/3) / 2] \right]. \end{aligned} \quad (6)$$

The dispersion relation obtained here provides the frequency-wave number dependence for TDLW propagation at any direction inside the  $x$ - $y$  plane. This expression is identical to the expression obtained by Vladimirov *et al.* [23].

## 2.2. Continuum approximation

If the characteristic length scale of the wave form, say,  $L$ , is much larger than the interparticle spacing  $a$ , then the continuum approximation can be invoked in order to

convert the difference [eq. (3)] into a differential equation for  $z_{m,n}$ , now expressed as continuous function  $z(x,t)$ . We expand  $z_{m\pm 1,n}$  and  $z_{m\pm 1/2,n\pm\sqrt{3}/2}$  around  $z_{m,n}$  in powers of  $d/L$  and retain terms of the order of  $(d/L)^4$  (higher powers of  $d/L$  lead only to some corrections and do not include any new physical concepts) to obtain

$$z_{m\pm 1,n} = z \pm d \frac{\partial z}{\partial x} + \frac{d^2}{2} \frac{\partial^2 z}{\partial x^2} \pm \frac{d^3}{6} \frac{\partial^3 z}{\partial x^3} + \frac{d^4}{24} \frac{\partial^4 z}{\partial x^4} + \dots \quad (7)$$

$$z_{m\pm 1/2,n\pm\sqrt{3}/2} = z \pm \frac{d}{2} \frac{\partial z}{\partial x} \pm \frac{\sqrt{3}d}{2} \frac{\partial z}{\partial y} + \frac{d^2}{8} \frac{\partial^2 z}{\partial x^2} + \frac{3d^2}{8} \frac{\partial^2 z}{\partial y^2} \pm \frac{\sqrt{3}d^2}{4} \frac{\partial^2 z}{\partial x \partial y} + \dots \quad (8)$$

Substituting (7), (8) into eq. (3), and retaining terms of order in  $d^4$ , the classical Newton's laws take the form of the differential equation for the particle displacements,

$$\begin{aligned} \ddot{z} + \nu \dot{z} = & -\omega_g^2 z - K_1 z^2 - K_2 z^3 \\ & + H_1 [z_{xx} + z_{yy} + \frac{d^2}{16} (z_{xxxx} + z_{yyyy} + 2z_{xxyy})] \\ & + H_2 [z_x^2 + z_y^2 + \frac{3d^2}{16} (z_{xx}^2 + z_{yy}^2 + 4z_{xy}^2)] \\ & + H_3 z [z_{xx} + z_{yy} + \frac{d^2}{16} (z_{xxxx} + 2z_{xxyy} + z_{yyyy})] \\ & + H_4 [z_x z_{xxx} + z_y z_{yyy} + z_x z_{xyy} + z_y z_{xxy} \\ & + \frac{1}{2} z_{xx} z_{yy}] + H_5 z [z_x^2 + z_y^2 + \frac{3d^2}{16} \\ & (z_{xx}^2 + z_{yy}^2 + 4z_{xy}^2)] + H_6 z^2 [z_{xx} + z_{yy} \\ & + \frac{d^2}{16} (u_{xxxx} + u_{yyyy} + 2u_{xxyy})] \\ & + H_7 z [z_x z_{xxx} + z_x z_{xyy} + z_y z_{xxy} \\ & + z_y z_{yyy} + \frac{1}{2} z_{xx} z_{yy}] \\ & + H_8 [3z_x^2 z_{xx} + 3z_y^2 z_{yy} + z_x^2 z_{yy} + z_y^2 z_{xx} \\ & + 4z_x z_y z_{xy}], \end{aligned} \quad (9)$$

where coefficients are defined in the appendix. In the following, we shall assume a very small damping rate and will therefore neglect damping by setting  $\nu = 0$  in the nonlinear analysis to follow. This is expected to incur a relative error of the order of  $(\nu/\omega_g)^2$ , which is reportedly small in experiments. Our results will later be extended by incorporating dissipation effects omitted here.

### 3. Amplitude modulation

Assuming, the transversal wave propagate in the

$\hat{x} \cos \theta + \hat{y} \sin \theta$  direction. We shall employ the standard lattice version of the reductive perturbation technique [25, 26] in the quasi-continuum limit. Allowing for a slight departure from the small amplitude (linear) assumption, one may consider

$$z = \varepsilon z_1 + \varepsilon^2 z_2 + \dots, \quad (10)$$

where  $(\varepsilon \ll 1)$  is a small (real) parameter characterizing the strength of the nonlinearity. The function  $u_j$  at each order is assumed to be a sum of  $l$  th order harmonics, viz.

$$z_j = z_{j0} + \sum_{l=1}^{\infty} [z_{jl} \exp[il(mkd \cos \theta + nkd \sin \theta - \omega t)]] + c.c. \quad (11)$$

The amplitudes  $z_{jl}$  are assumed to be slowly varying function of time and space via the set of independent stretched variables  $\xi = (x_1 - v_g t_1)$ ,  $\eta = y_1$  and  $\tau = t_2 = \varepsilon t_1$ . We shall now substitute these expansions into the equation of motion (9) and collect the contributions appearing in each power in  $\varepsilon$ .

At first order, we obtain the dispersion relation in the form

$$\omega^2 = \omega_g^2 + H_1 k^2 \left( 1 - \frac{k^2 d^2}{16} \right), \quad (12)$$

and

$$z_{10} = 0. \quad (13)$$

In the second order, considering the annihilation of secular terms, we obtain the following expression for the propagation velocity,  $v_g$ ,

$$v_g = \frac{H_1 k}{\omega} \left( 1 - \frac{k^2 d^2}{8} \right), \quad (14)$$

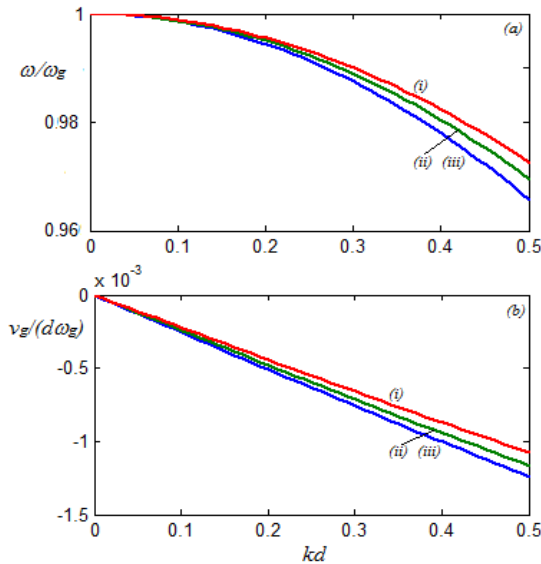
and also the expressions for harmonic amplitudes can be obtained

$$z_{20} = \frac{2}{\omega_g^2} [-K_1 + H_2 k^2 (1 + 3k^2 d^2 / 16) - H_3 k^2 (1 - \frac{k^2 d^2}{16}) - H_4 k^4] |z_{11}|^2 = n_1 |z_{11}|^2, \quad (15)$$

$$\begin{aligned} z_{22} = & \frac{-K_1 - H_2 k^2 (1 - 3k^2 d^2 / 16) - H_3 k^2 (1 - \frac{k^2 d^2}{16}) + H_4 k^4}{-4\omega^2 + \omega_g^2 + 4k^2 H_1 (1 - k^2 d^2 / 4)} \\ & \times z_{11}^2 = n_2 z_{11}^2. \end{aligned} \quad (16)$$

It is easy to verify that  $v_g = d\omega/dk$ . In the third order, we obtain the following NLS equation

$$i \frac{\partial Z}{\partial \tau} + P \frac{\partial^2 Z}{\partial \xi^2} + R \frac{\partial^2 Z}{\partial \eta^2} + QZ(Z)^2 = 0 \quad (17)$$



**Figure 2.** (a) The dispersion relation and (b) the behavior of normalized group velocity versus  $kd$  for  $s_1 = s_2 = s_3 = s_4 = 0$  (line (i)),  $s_1 = 0.05$ ,  $s_2 = 0.005s_1$ ,  $s_3 = 0.0005s_2$ ,  $s_4 = 0.00005s_3$  (line (ii)), and  $s_1 = 0.1$ ,  $s_2 = 0.01s_1$ ,  $s_3 = 0.001s_2$ ,  $s_4 = 0.0001s_3$  (line (iii)).

which describes the evolution of the fundamental (carrier) harmonic amplitude  $Z = z_{11}(\xi, \eta, t)$ . The dispersion coefficient  $P$  is given by

$$P = -\frac{v_g^2}{2\omega} + \frac{H_1}{2\omega} \left(1 - \frac{3k^2 d^2}{8}\right). \quad (18)$$

$$R = \frac{H_1}{2\omega} \left(1 - \frac{k^2 d^2}{8}\right). \quad (19)$$

$$Q = \frac{-2n_1 K_1}{\omega} + \frac{-3K_2}{2\omega} + \frac{n_2}{2\omega} \times \left[ -2K_1 + 4H_2 k^2 \left(1 + \frac{3k^2 d^2}{8}\right) - H_3 k^2 \left(5 - \frac{17k^2 d^2}{16}\right) - 10H_4 k^4 \right] + \frac{k^2 H_5}{2\omega} \left(2 + \frac{9k^2 d^2}{16}\right) - \frac{3k^2 H_6}{2\omega} \times \left(1 - \frac{k^2 d^2}{16}\right) - \frac{k^4 H_7}{2\omega} - \frac{3k^4 H_8}{2\omega}. \quad (20)$$

where coefficients are defined in the appendix. In sum, the dynamics of the wave fundamental harmonic amplitude  $z_{11} = Z(\xi, \eta, \tau)$ , is governed by the NLS [eq. (17)] and is thus by the interplay among the dispersion and nonlinearity coefficients  $P$  and  $Q$ . Their analytical behavior on relevant parameters will be investigated below once their role in dynamics is briefly summarized in what follows.

#### 4. Modulation instability

The amplitude dynamics of a TDWL packet was shown

to be governed by the NLS [eq. (17)] above. Two physical phenomena that are generally modeled via this formulation are wave collapse via modulational instability and the formation of envelope excitations. Without reproducing the whole existing theory, which may be found, for e.g., in Ref. 27, we shall provide the basic information needed to understand our findings in what follows.

The detailed analysis of the NLS [Eq. (17)] reveals that a modulated wave packet whose amplitude obeys the NLS equation [Eq. (17)] is modulationally stable for (a)  $P > 0$ ,  $R > 0$  and  $Q < 0$  or if (b)  $P < 0$ ,  $R < 0$  and  $Q > 0$  [28]. In a special case, suppose that the amplitude perturbation occurs only in the  $\xi$  direction. Assuming a perturbation of amplitude  $\Psi_0$ , and characteristic wave number  $\tilde{k}$ , the maximum growth rate  $\sigma = Q|\Psi_0|^2$  is attained for a perturbation wave number

$$\tilde{k} = \sqrt{\frac{2Q}{P}} |\Psi_0| \quad (21)$$

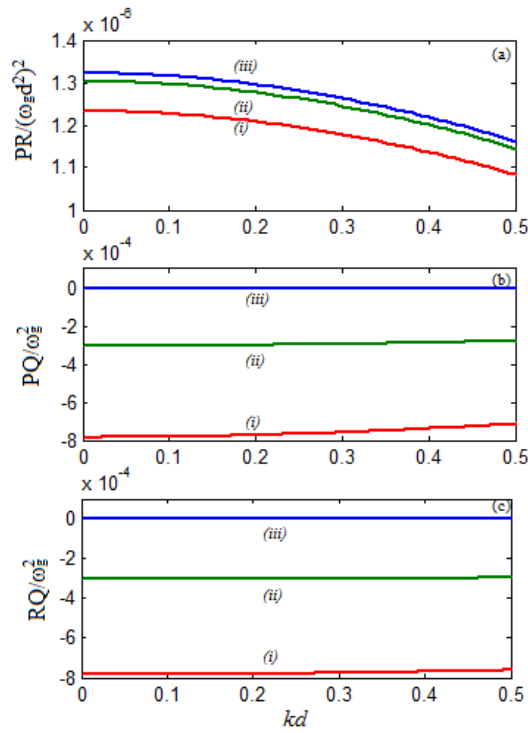
The coefficients  $P$  and  $Q$  therefore determine the occurrence and first stage evolution of the instability.

Only the first evolution stage of the instability outlined above can be described analytically. The further evolution of the instability can only be modeled numerically. In the case  $PQ > 0$ , *bright-type solitons* are formed: these model localized envelope pulses, which confine the fast carrier wave and move at or near the group velocity, and are formally equivalent to bright pulses in nonlinear fiber optics. On the other hand, for  $PQ < 0$ , modulated wave packets may propagate in the form of *dark/gray envelope solitons*, modeling localized voids amidst constant values everywhere else.

The products  $PR$ ,  $PQ$  and  $RQ$  are depicted in figure 3. Recalling that the sign of the products  $PQ$  and  $RQ$  determines the stability profile of the wave. Since  $PR$  is positive and both  $PQ$  and  $RQ$  are also negative, stable dark-type envelope structures should therefore be sustained in the system. It must be added for rigor that the sign of  $P$  (and presumably  $Q$ ) may change by taking into account HDCV, thus affecting the stability profile of modulated wave packets, and the type of envelope solitons susceptible to occur.

#### 5. Conclusion

The propagation of off-plane transverse dust lattice wave in 2D hexagonal dusty plasma crystals including the height-dependent charge variations was investigated. We need to point out for rigor that the approach (the continuum approximation) employed here is only valid for small values of  $kd$ . The linear dispersion characteristics of transverse dust lattice waves were studied, including the dispersion relation, and group velocity (figure 2), and an evolution equation for the modulated amplitude of the first harmonic was derived.



**Figure 3.** (a) The variation of  $PR$ , (b)  $PQ$  and (c)  $RQ$  versus  $kd$  for  $s_1 = s_2 = s_3 = s_4 = 0$  (line (i)),  $s_1 = 0.05$ ,  $s_2 = 0.005s_1$ ,  $s_3 = 0.0005s_2$ ,  $s_4 = 0.00005s_3$  (line (ii)) and  $s_1 = 0.1$ ,  $s_2 = 0.01s_1$ ,  $s_3 = 0.001s_2$ ,  $s_4 = 0.0001s_3$  (line (iii)).

The dispersion relation shows a negative group velocity of the wave. These results are in excellent agreement with earlier numerical,[13] experimental,[14] and theoretical [15,23] results.

Using multiple scale theory, we have shown that transverse wave packets will in principle be stable in the long wavelength region. Furthermore, in an attempt to determine the region of validity of our study, in as much rigor as possible, we have investigated the role of the height-dependent charge variations. We have shown that this effect can lead to modulational instability (figure 3). Furthermore, in the presence of HDCV we predict the formation of both bright and dark-type envelope solitons in regions similar to the bright envelope structures observed in laboratory experiments[14]. Modulational instability may also be the first stage of the generic (i.e., for any symmetric potential) structural instability suggested in Ref. 29.

Our work is of relevance in dusty plasma crystal experiments in the laboratory, where our predictions for the type and stability of modulated wave packets can be tested and will hopefully be confirmed. Beyond dusty (complex) plasma physics, we view this work as a fundamental investigation of nonlinear transverse motion in hexagonal crystals of potential relevance (either currently or in the future) in other physical contexts, where electrostatic-interaction sustained crystalline structures occur (such as ultracold plasmas or one-component plasmas), or in lattice theory and in discrete

dynamical systems where pulse formation and wave packet localization occur.

### Appendix

$$\omega_{g0}^2 = -(QE)'_0 / M,$$

$$\omega_o^2 = Q_o^2 \exp(-\kappa) / 4\pi\epsilon_o M d^3,$$

$$\kappa = d / \lambda_D,$$

$$\omega_g^2 = \omega_{g0}^2 + 6Q_o^n d^2 \omega_o^2 / Q_o$$

$$s_1 = Q_o' d / Q_o,$$

$$s_2 = Q_o'' d^2 / Q_o,$$

$$s_3 = Q_o''' d^3 / Q_o,$$

$$s_4 = Q_o^{(4)} d^4 / Q_o,$$

$$K_1 = -(QE)_0'' / 2!M + 3\omega_o^2 Q_o^{(3)} d^2 / Q_o,$$

$$K_2 = -(QE)_0''' / 3!M + \omega_o^2 Q_o^{(4)} d^2 / Q_o,$$

$$K_3 = -s_1^2 \omega_o^2,$$

$$K_4 = -s_1 s_2 d^{-1} \omega_o^2,$$

$$K_5 = -s_1 s_3 d^{-2} \omega_o^2 / 2,$$

$$K_6 = -s_1 s_2 d^{-1} \omega_o^2 / 2,$$

$$K_7 = -s_2^2 d^{-2} \omega_o^2 / 2,$$

$$K_8 = -s_1 s_3 d^{-2} \omega_o^2 / 6,$$

$$K_9 = -\omega_o^2 (1 + \kappa),$$

$$K_{10} = -s_1 d^{-1} \omega_o^2 (1 + \kappa),$$

$$K_{11} = -s_2 d^{-2} \omega_o^2 (1 + \kappa),$$

$$K_{12} = s_1 d^{-1} \omega_o^2 (1 + \kappa) / 2,$$

$$K_{13} = s_2 d^{-2} \omega_o^2 (1 + \kappa) / 2,$$

$$K_{14} = s_1^2 d^{-2} \omega_o^2 (1 + \kappa) / 2,$$

$$K_{15} = -s_1 d^{-1} \omega_o^2 (1 + \kappa),$$

$$K_{16} = -s_2^2 d^{-2} \omega_o^2 (1 + \kappa),$$

$$K_{17} = -s_2 d^{-2} \omega_o^2 (1 + \kappa) / 2,$$

$$K_{18} = d^{-2} \omega_o^2 (3 + 3\kappa + \kappa^2) / 2,$$

$$H_1 = -\frac{3}{2} \omega_o^2 d^2 [s_1^2 + (1 + \kappa)],$$

$$H_2 = -\frac{3}{2} \omega_o^2 d [s_1 s_2 + s_1 (1 + \kappa)],$$

$$H_3 = -\frac{3}{2} \omega_o^2 d [s_1 s_2 + 2s_1 (1 + \kappa)],$$

$$H_4 = -\frac{3}{8} \omega_o^2 d^3 [s_1 s_2 + s_1 (1 + \kappa)],$$

$$H_5 = -\frac{3}{2} \omega_o^2 [s_2^2 + s_1^2 (1 + \kappa) - 2s_2 (1 + \kappa)],$$

$$H_6 = -\frac{3}{2} \omega_o^2 [s_2^2 + s_1^2 (1 + \kappa) + \frac{3}{2} s_2 (1 + \kappa) + \frac{3}{2} s_1 s_3],$$

$$H_7 = -\frac{3}{8}\omega_0^2 d^2 [s_2^2 - 2s_2(1+\kappa) + s_1^2(1+\kappa)],$$

$$H_8 = \frac{3}{8}\omega_0^2 d^2 [3(3+3\kappa+\kappa^2) - 3s_1s_3 + \frac{9}{8}s_1^2(1+\kappa) - \frac{3}{2}s_2(1+\kappa)],$$

where the frequency  $\omega_{g0}$  is typically of order of

$\omega_{g0} / 2\pi = 20 \text{ Hz}$ , and also

$$(QE)_0'' / 2!M = -7.896 \times 10^3 (\text{Hz}^2 / \text{meter}),$$

$$(QE)_0''' / 3!M = 1.11 \times 10^3 (\text{Hz}^2 / \text{meter}^2) [\text{Ref. 15}].$$

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