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GENERALIZED NONLINEAR 3D EULER-BERNOULLI BEAM THEORY*

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Abstract– The issue of the new elastic terms discovered in the nonlinear dynamic model of an enhanced nonlinear 3D Euler-Bernoulli beam is discussed. While the elastic orientation is negligible, the nonlinear dynamic model governing tension-compression, torsion and two spatial bendings is presented. Considering this model, some new elastic terms can be identified in the variation of elastic potential energy in each bending motion equation, and in each transverse shear force. Due to the new terms, each term of a bending equation and a transverse shear force, finds a counterpart in the other bending equation and transverse shear force, but the equations remain asymmetric. The new terms have arisen, since variation of strains and variation of elastic potential energy are derived from exact strains and exact deformations regarding considerable elastic orientation, then the elastic orientation is neglected. The new terms perish in the nonlinear 3D Euler-Bernoulli beam theory, since elastic orientation is neglected first, then variation of strains and variation of elastic potential energy are derived from the approximated strains.

Keywords– 3D Euler-Bernoulli beam theory

1. INTRODUCTION

Deriving a set of partial differential equations governing the motions of a spatial beam is a prerequisite stage in many engineering fields. This stage becomes more complicated as the beam flexibility increases due to the demands for saving more material and for producing lighter structures.

Zohoor and Khorsandijou [1] have derived the boundary conditions and the ten coupled nonlinear partial differential motion equations of an enhanced nonlinear 3D Euler-Bernoulli beam with flying support. They [2] have exposed the dynamic model of a flying manipulator with two highly flexible links within which the flexibility has been modeled similar to that of [1]. The dynamic model in [2] includes sixteen coupled nonlinear partial differential motion equations along with the boundary conditions. They [3] have derived the nonlinear dynamic model of a mobile robot with flexible links experiencing considerable and negligible elastic orientation in their cross-sectional frames. Elastic orientation is considerable in long links and is negligible in short links. They [3] have derived the variation of elastic potential energy of long links of a spatial mobile flexible robot that might be reduced to the fully-enhanced variation of elastic potential energy and to the enhanced variation of elastic potential energy, both of which belong to short links.

In [1-3], when the elastic orientation of a cross-section is negligible, the variation of the strain field and the variation of elastic potential energy are derived from the exact strains and deformations regarding

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considerable elastic orientation, and then the elastic orientation is neglected. As a result, some new elastic terms have arisen in the variation of elastic potential energy, in motion equations and in transverse shear forces. The new elastic terms would perish in the nonlinear 3D Euler-Bernoulli beam theory [4], since the variation of strains as well as the variation of elastic potential energy are derived from the approximated strains corresponding to negligible elastic orientation. Hiller [5] has considered only three elastic degrees of freedom for each link as an Euler-Bernoulli beam and inner constraints are assumed for the other three elastic coordinates including axial deformation. Shi *et al*. [6] have found that the traditional deformation field used for Euler–Bernoulli beams fails to produce an elastic rotation matrix that is complete to second-order in the deformation variables. They have proposed a complete second-order deformation field along with the equations needed to incorporate the beam model into a graph-theoretic formulation for flexible multibody dynamics. They have presented two examples to demonstrate the effects of the proposed second-order deformation field on the response of a flexible multibody system.

Novozhilov [7] has studied the deformation of thin prismatic rods of an arbitrary cross-section. He has considered 1^{st} and 2^{nd} order approximations for displacement components of an arbitrary point of cross-section, using Taylor-series expansions in terms of the two components of the position vector of the arbitrary point apparent in the cross-sectional frame. Strain components are derived from these approximated displacements.

An analysis of shear correction factors has been carried out by Okumus [8] for a homogeneous polyethylene thermoplastic cantilever beam which is reinforced by steel fibers with a rectangular cross-section. It has been shown that an applied shear force produces extension-shear coupling. A linear and non-linear bubble finite strip method of analysis has been used by Azhari [9] to study the local, distortional and lateral buckling monosymmetric I-beams under pure bending.

The present paper highlights a verification of [1] regarding the nonlinear dynamic model of an enhanced nonlinear 3D Euler-Bernoulli beam. Geometric nonlinearity of an enhanced nonlinear 3D Euler-Bernoulli beam with circular cross-section is taken into consideration, and material nonlinearity is neglected. The nonlinear dynamic model of the beam is composed of the boundary conditions and four coupled nonlinear partial differential equations over tension-compression, torsion and two spatial bendings of the beam, whose elastic orientation remains negligible. Although the elastic orientation is negligible, two new elastic terms can be identified in the variation of elastic potential energy, in each bending motion equation and in each transverse shear force.

A justification is made for the necessity of the existence of the new elastic terms in the two coupled nonlinear partial differential bending equations and in the transverse shear forces of an enhanced nonlinear 3D Euler-Bernoulli beam. Due to the new terms, each term in a bending equation and a transverse shear force has found a counterpart in the other bending equation and transverse shear force, but the equations are still asymmetric. The bending motion equations and transverse shear forces derived in accordance with the nonlinear 3D Euler-Bernoulli beam theory [4] are asymmetric, and some terms in a bending equation and transverse shear force do not have a counterpart in the other bending equation and transverse shear force.

The new terms have arisen, since the variation of strains and the variation of elastic potential energy are derived from exact strains and exact deformation components regarding considerable elastic orientation, then the elastic orientation is neglected. They are lost in the nonlinear 3D Euler-Bernoulli beam theory [4], since elastic orientation is neglected first, then the variation of strains and the variation of elastic potential energy are derived from the approximated strains.

Some other new elastic terms that are not discussed in this paper can be revealed if the motion equations are derived from the fully-enhanced variation of elastic potential energy, or from the exact variation of elastic potential energy before substitution of the rotational elastic coordinates with zero. In the present paper the motion equations are derived from the enhanced variation of elastic potential energy that is an inexact variation of the elastic potential energy within which the rotational elastic coordinates have been substituted by zero.

2. NONLINEAR 3-D EULER-BERNOULLI BEAM

In this paper a slender beam undergoing tension, compression, torsion and spatial bendings, is modeled according to the nonlinear 3-D Euler-Bernoulli beam theory. So the beam cross-sections are assumed to remain plane and perpendicular to its center line before and after spatial elastic bending deflection. The assumption of a plane cross-section during bending implies that the out-of-plane warping of the cross-section has been neglected. Since the beam experiences elastic torsion beyond bendings and axial deformations, the out-of-plane warping of cross-section arises according to Saint Venant's theory of torsion, unless the cross section is circular. Therefore, to comply with the Bernoulli's hypothesis, only circular cross-sections are considered in this paper. It should be noted that spatial bending induces torsion, even in isotropic beams.

In-plane warping of the cross-section is neglected in this paper, so the Poisson's ratio is substituted with zero in the formulations. In order to prepare a necessary and sufficient condition for neglecting in-plane and out-of-plane warpings of the cross-section, the components of p, shown in Fig. 1, are assumed to be constant.

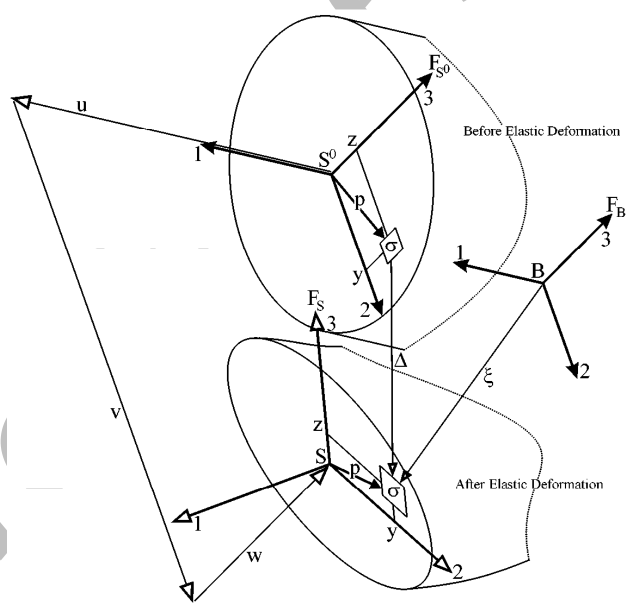

Fig. 1. Displacement field [1]; circular cross-section

This paper emphasizes the new terms of the beam dynamic model due to the geometric nonlinearity, while neglecting the material nonlinearity. The beam is assumed to be made from a linearly elastic isotropic material with uniform density and cross-sectional area. It is straight before elastic deformation and its cross-sectional frame undergoes large elastic orientation. As shown in Figs. 1-3, F_S is the principal frame of the cross-section within which the two moments of cross-sectional area about the 2nd and 3rd axes are equal. The coordinate reference frames are right-handed orthogonal in Figs. 1-3, and their axes are marked by numbers to indicate 1st, 2nd and 3rd axes respectively.

Spatial elastic deformation of a cross-section is shown by six coordinates along with two holonomic constraints in Figs. 2-3. In Fig. 3, the length of the beam element is Δs before, and $(1+e)\Delta s$ after elastic deformation. Therefore one can write Eq. (1) and find the axial strain of the centerline as Eq. (2).

In the left-hand-side illustration of Fig. 3 the origins of F_{S^0} and F_S coincide via translation to show that the pose of the cross-section facing S in the deformed element relative to the cross-section facing S^0 in the undeformed element is determined by Δu, Δv, Δw, α, β and γ, whereas the pose of F_S relative to F_{S^0} is determined by u, v, w, α, β and γ. Pose is a short term instead of the position and orientation of a frame.

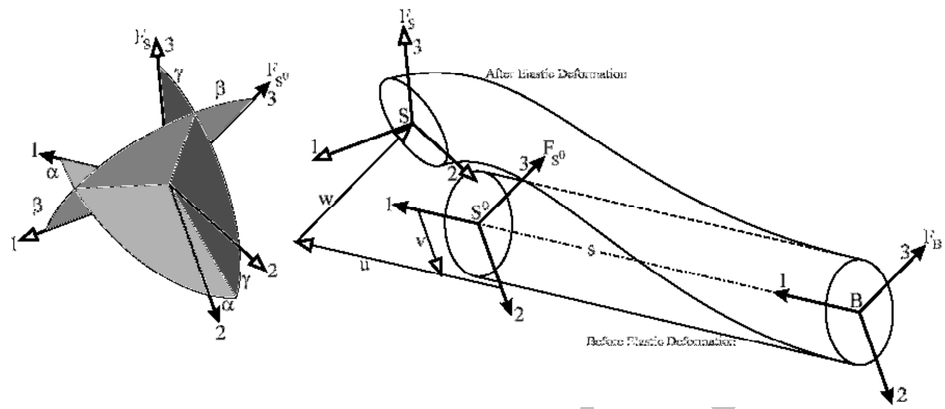

Fig. 2. Six dependent spatial elastic coordinates [1-3]; circular cross-section

$$R\left[(\Delta s+\Delta u)\quad \Delta v\quad \Delta w\right]^T=\left[(1+e)\Delta s\quad 0\quad 0\right]^T \tag{1}$$

$$e=\sqrt{\left(1+u'\right)^2+v'^2+w'^2}-1 \tag{2}$$

Considering the two triangles in the left-hand-side illustration of Fig. 3, the beam structural holonomic constraints, Eqs. (3, 4), are derived to eliminate two superfluous coordinates, namely α and β. Therefore, the nonlinear 3D Euler-Bernoulli beam is a holonomic system with u, v, w, and γ as its independent elastic degrees of freedom.

$$\alpha=\lim_{\Delta s\to 0}\left[\tan^{-1}\frac{\Delta v}{\Delta s+\Delta u}\right]=\tan^{-1}\frac{v'}{1+u'} \tag{3}$$

$$\beta=\lim_{\Delta s\to 0}\left[\tan^{-1}\frac{-\Delta w}{\sqrt{(\Delta s+\Delta u)^2+\Delta v^2}}\right]=-\tan^{-1}\frac{w'}{r} \tag{4}$$

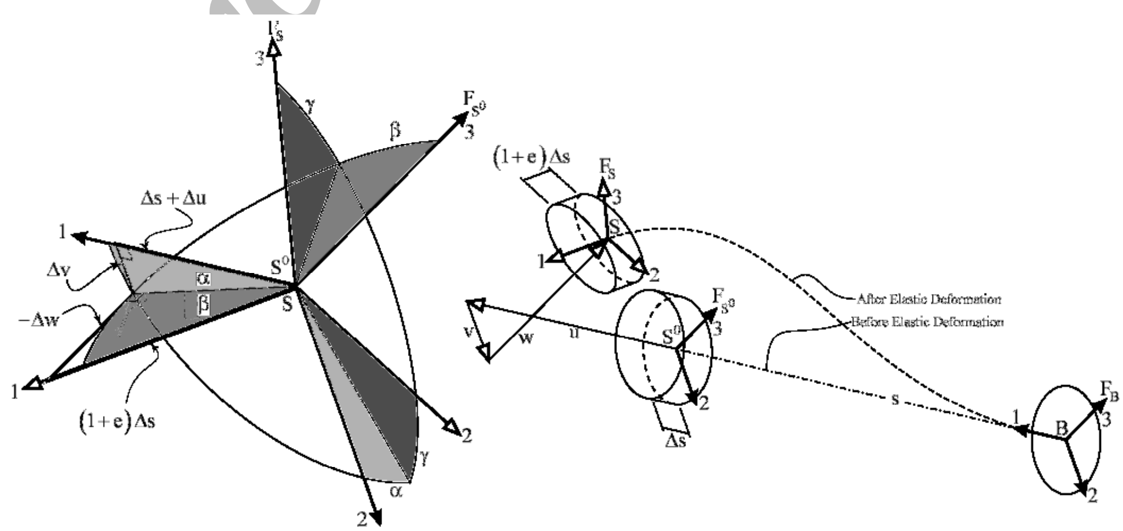

Fig. 3. Two holonomic constraints among elastic coordinates [1-3]; circular cross-section

Exact elastic orientation is determined by the Euler angles α, β, and γ, which obviously have an unchangeable sequence. This unchangeable sequence has been mathematically interpreted by the fact that the matrix multiplication is not commutative. Exact elastic orientation of the cross-section is described nonlinearly by the elastic degrees of freedom in Eq. (5), which is a rotation transformation matrix projecting a vector from F_B onto F_S. Due to the nonlinearity of Eq. (5), R_{zz}, R_{zy}, R_{zx}, R_{yz}, R_{yy}, R_{yx}, R_{xz}, R_{xy} and R_{xx} are not converted into R_{yy}, $-R_{yz}$, R_{yx}, $-R_{zy}$, R_{zz}, $-R_{zx}$, R_{xy}, $-R_{xz}$ and R_{xx} respectively by replacing v with $-w$ and w with v. This is referred to as the formulation asymmetry of Eq. (5) apart from its matrix asymmetry.

$$R=\begin{bmatrix} (u'+1)/(e+1) & v'/(e+1) & w'/(e+1) \\ -v'\cos\gamma/r - w'(u'+1)\sin\gamma/r(e+1) & (u'+1)\cos\gamma/r - w'v'\sin\gamma/r(e+1) & r\sin\gamma/(e+1) \\ v'\sin\gamma/r - w'(u'+1)\cos\gamma/r(e+1) & -(u'+1)\sin\gamma/r - w'v'\cos\gamma/r(e+1) & r\cos\gamma/(e+1) \end{bmatrix} \tag{5}$$

Elastic angular velocity and normalized curvature are given by Eqs. (6, 7) according to the Kirchhoff's kinetic analogy [1-3]. Elastic angular acceleration is given by Eq. (8). The Eqs.(6-8) are asymmetric since Ω_z, Ω_y, Ω_x, κ_z, κ_y, κ_x, $\dot{\Omega}_z$, $\dot{\Omega}_y$ and $\dot{\Omega}_x$ are not converted into Ω_y, $-\Omega_z$, Ω_x, κ_y, $-\kappa_z$, κ_x, $\dot{\Omega}_y$, $-\dot{\Omega}_z$ and $\dot{\Omega}_x$ respectively by replacing v with $-w$ and w with v.

$$\Omega=\begin{bmatrix}\Omega_x & \Omega_y & \Omega_z\end{bmatrix}^T=\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T\dot{\gamma}+C\begin{bmatrix}\dot{u}' & \dot{v}' & \dot{w}'\end{bmatrix}^T \tag{6}$$

$$\kappa=\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T\gamma'+C\begin{bmatrix}u'' & v'' & w''\end{bmatrix}^T \tag{7}$$

$$\dot{\Omega}=\begin{bmatrix}\dot{\Omega}_x & \dot{\Omega}_y & \dot{\Omega}_z\end{bmatrix}^T=\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T\ddot{\gamma}+C\begin{bmatrix}\ddot{u}' & \ddot{v}' & \ddot{w}'\end{bmatrix}^T+\dot{C}\begin{bmatrix}\dot{u}' & \dot{v}' & \dot{w}'\end{bmatrix}^T \tag{8}$$

where

$$C=\frac{1}{r(e+1)^2}\begin{bmatrix} -(e+1)w'v'/r & (u'+1)(e+1)w'/r & 0 \\ -v'(e+1)\sin\gamma + w'(u'+1)\cos\gamma & (u'+1)(e+1)\sin\gamma + v'w'\cos\gamma & -r^2\cos\gamma \\ -v'(e+1)\cos\gamma - w'(u'+1)\sin\gamma & (u'+1)(e+1)\cos\gamma - v'w'\sin\gamma & r^2\sin\gamma \end{bmatrix} \tag{9}$$

3. EXACT STRAIN FIELD

Elastic displacement of a general point of the beam's medium is illustrated in Fig. 1, within which the cross-section of the beam is assumed not to experience in-plane and out-of-plane warpings. The exact displacement field of the beam is given by Eq. (10), which is an asymmetric formulation, since Δ_z, Δ_y and Δ_x are not converted into Δ_y, $-\Delta_z$ and Δ_x respectively, by replacing v with $-w$ and w with v.

$$\Delta(s,t)=\begin{bmatrix}u\\v\\w\end{bmatrix}+R^T\begin{bmatrix}0\\y\\z\end{bmatrix}-\begin{bmatrix}0\\y\\z\end{bmatrix}=\begin{bmatrix} u-\frac{y}{r}\left[v'\cos\gamma+\left(\frac{u'+1}{e+1}\right)w'\sin\gamma\right]+\frac{z}{r}\left[v'\sin\gamma-\left(\frac{u'+1}{e+1}\right)w'\cos\gamma\right] \\ v+\frac{y}{r}\left[(u'+1)\cos\gamma-\frac{v'w'}{e+1}\sin\gamma-r\right]-\frac{z}{r}\left[(u'+1)\sin\gamma+\frac{v'w'}{e+1}\cos\gamma\right] \\ w+y\frac{r}{e+1}\sin\gamma+z\left(\frac{r}{e+1}\cos\gamma-1\right) \end{bmatrix} \tag{10}$$

The linear part of the Green-Lagrange geometric strain tensor is taken into consideration. The exact components of this strain are given by Eqs. (11), which are asymmetric due to the nonlinearity and formulation asymmetry of Eq. (5). As shown by Eqs. (11), the strains' formulation is asymmetric, whereas the strain tensor is a symmetric matrix. The asymmetry of the strains' formulation in Eqs. (11) can be verified by the fact that ε_{xx}, ε_{xy} and ε_{xz} are not converted into ε_{xx}, $-\varepsilon_{xz}$ and ε_{xy} respectively by replacing v with $-w$ and w with v.

$$
\begin{aligned}
\varepsilon_{xx} &= u' + y\left[\frac{v'}{r}\kappa_x \sin\gamma - \frac{1+u'}{1+e}\left(\kappa_z + \frac{w'}{r}\kappa_x \cos\gamma\right)\right] + z\left[\frac{v'}{r}\kappa_x \cos\gamma + \frac{1+u'}{1+e}\left(\kappa_y + \frac{w'}{r}\kappa_x \sin\gamma\right)\right] \\
\varepsilon_{xy} &= \frac{v'}{2} - \frac{y}{2}\left[\frac{1+u'}{r}\kappa_x \sin\gamma + \frac{v'}{1+e}\left(\kappa_z + \frac{w'}{r}\kappa_x \cos\gamma\right)\right] - \frac{z}{2}\left[\frac{1+u'}{r}\kappa_x \cos\gamma - \frac{v'}{1+e}\left(\kappa_y + \frac{w'}{r}\kappa_x \sin\gamma\right)\right] \\
\varepsilon_{xz} &= \frac{w'}{2} + \frac{y}{2}\left[\frac{-w'\kappa_z + r\kappa_x \cos\gamma}{1+e}\right] + \frac{z}{2}\left[\frac{w'\kappa_y - r\kappa_x \sin\gamma}{1+e}\right] \\
\varepsilon_{yy} &= \varepsilon_{zz} = \varepsilon_{yz} = 0
\end{aligned}
\tag{11}
$$

The motion equations of this paper and [1] are considered to be valid for negligible elastic orientation, but the variation of strains used in the variation of elastic potential energy must be derived from the exact strains of Eqs. (11). Then the elastic rotational degrees of freedom are substituted with zero in the formulations, some of which are given by Eqs. (12) [1-3]. Some of the parameters become symmetric, as the elastic rotational degrees of freedom; i.e. α, β and γ are substituted with zero. By replacing v with $-w$ and w with v in Eqs. (12), e, R_{zz}, R_{zy}, R_{zx}, R_{yz}, R_{yy}, R_{yx}, R_{xz}, R_{xy}, R_{xx}, κ_z, κ_y, κ_x, $\delta\kappa_z$ and $\delta\kappa_y$ are converted into e, R_{yy}, $-R_{yz}$, R_{yx}, $-R_{zy}$, R_{zz}, $-R_{zx}$, R_{xy}, $-R_{xz}$, R_{xx}, κ_y, $-\kappa_z$, κ_x, $\delta\kappa_y$ and $-\delta\kappa_z$ respectively, But $\delta\kappa_x$ is not converted into $\delta\kappa_x$. As a result, the strains' variation and the variation of elastic potential energy and consequently the equations of motion, two coupled nonlinear partial differential bending equations; i.e. Eqs. (26, 27) and equations of transverse shear forces; i.e. Eqs. (30, 31) will become asymmetric.

As the elastic rotational degrees of freedom are substituted with zero, Ω_z, Ω_y, Ω_x, $\dot{\Omega}_z$, $\dot{\Omega}_y$, $\dot{\Omega}_x$, Δ_z, Δ_y, Δ_x, ε_{xx}, ε_{xy} and ε_{xz} are also converted into Ω_y, $-\Omega_z$, Ω_x, $\dot{\Omega}_y$, $-\dot{\Omega}_z$, $\dot{\Omega}_x$, Δ_y, $-\Delta_z$, Δ_x, ε_{xx}, $-\varepsilon_{xz}$ and ε_{xy} respectively, by replacing v with $-w$ and w with v.

$$
\begin{aligned}
& v' \approx \alpha \approx 0 \;,\quad w' \approx \beta \approx 0 \;,\quad \gamma \approx 0 \;,\quad r \approx 1+u' \;,\quad e \approx u' \;,\quad \delta e \approx \delta u' \;,\quad R^T \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \;, \\
& \kappa_x \approx \gamma' \;,\quad \kappa_y \approx \frac{-w''}{1+u'} \;,\quad \kappa_z \approx \frac{v''}{1+u'} \;,\quad \delta\kappa_x \approx \frac{v''}{\left(1+u'\right)^2}\delta w' + \delta\gamma' \;, \\
& \delta\kappa_y \approx \frac{1}{1+u'}\left[v''\delta\gamma - \delta w'' + \frac{1}{1+u'}\left(w''\delta u' + u''\delta w'\right)\right] \;,\quad \delta\kappa_z \approx \frac{1}{1+u'}\left[w''\delta\gamma + \delta v'' - \frac{1}{1+u'}\left(v''\delta u' + u''\delta v'\right)\right]
\end{aligned}
\tag{12}
$$

4. VARIATION OF ELASTIC POTENTIAL ENERGY

When the elastic rotational degrees of freedom are substituted with zero, the variation of elastic potential energy is given by Eq. (13), within which the stresses are derived using Hooke's law for a linearly elastic isotropic uniform beam with a circular cross-section [1-3]. Poisson's ratio has been substituted with zero, because in-plane warping does not exist.

$$\delta U^e = \int_0^L \int_A \left\{ \tau_{xx}\delta\varepsilon_{xx} + 2\tau_{xy}\delta\varepsilon_{xy} + 2\tau_{xz}\delta\varepsilon_{xz} \right\} dAds =$$
$$= \int_0^L \left\{ EAe\delta e + EI\kappa_z\delta\kappa_z + EI\kappa_y\delta\kappa_y + G2I\kappa_x\delta\kappa_x + (E-G)I\frac{\kappa_x\kappa_y}{r}\delta v' + (E-G)I\frac{\kappa_x\kappa_z}{r}\delta w' \right\} ds \tag{13}$$

The terms (14) that are the last two terms in the integrand of Eq. (13) would be lost according to the nonlinear 3D Euler-Bernoulli beam theory, since strains' variation therein are derived from the approximated strain field of Eqs. (15) concerning negligible elastic orientation. The inaccurate strains' variation of Eqs. (16) produces Eqs. (17) for the inaccurate variation of elastic potential energy in the theory. The terms (14) improve the nonlinear 3D Euler-Bernoulli beam theory.

$$+(E-G)I\frac{\kappa_x\kappa_y}{r}\delta v' \quad , \quad +(E-G)I\frac{\kappa_x\kappa_z}{r}\delta w' \tag{14}$$

$$\varepsilon_{xx} = e - y\kappa_z + z\kappa_y \quad , \quad \varepsilon_{xy} = -\frac{1}{2}\kappa_x z \quad , \quad \varepsilon_{xz} = \frac{1}{2}\kappa_x y \quad , \quad \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{yz} = 0 \tag{15}$$

$$\delta\varepsilon_{xx} = \delta e - y\delta\kappa_z + z\delta\kappa_y \quad , \quad \delta\varepsilon_{xy} = -\frac{1}{2}z\delta\kappa_x \quad , \quad \delta\varepsilon_{xz} = \frac{1}{2}y\delta\kappa_x \quad , \quad \delta\varepsilon_{yy} = \delta\varepsilon_{zz} = \delta\varepsilon_{yz} = 0 \tag{16}$$

$$\delta U^e = \int_0^L \left\{ EAe\delta e + EI\kappa_z\delta\kappa_z + EI\kappa_y\delta\kappa_y + G\,2I\kappa_x\delta\kappa_x \right\} ds \tag{17}$$

5. MOTION EQUATIONS

Four coupled nonlinear partial differential equations governing tension-compression, torsion and two lateral bendings of a beam with negligible elastic orientation are derived using Hamilton's principle. They are more accurate than the motion equations of a nonlinear 3D Euler-Bernoulli beam. Non-conservative forces and moments that produce damping and exciting terms in the motion equations are neglected.

The variation of the elastic potential energy of Eq. (13) is not exact, since therein the rotational elastic coordinates have been substituted by zero. Equation (13) is substituted in Eq. (21) which is the Hamilton's principle, and is differentiated by the partial differential operations of Eqs. (18). Therefore the motion equations of this paper have lost some other elastic terms that are not discussed here. They would appear if the exact variation of elastic potential energy were used instead of Eq. (13).

Equation (22) will result, if the time integration of the variation of kinetic energy, i.e. Eq. (19) and the variation of the gravitational potential energy, i.e. Eq. (20), are substituted in Eq. (21) [1-3].

$$\int F\delta u' ds = F\delta u - \int F'\delta u\, ds \quad , \quad \int F\delta u'' ds = F\delta u' - F'\delta u + \int F''\delta u\, ds \tag{18}$$

$$\int_0^t \delta T dt = \int_0^t \int_0^L \int_A -\rho\ddot{\xi}^T\delta\xi dAdsdt =$$
$$-\rho\int_0^t \left\{ \int_0^L \left\{ \left(\dot{\Omega}^T[I] - \Omega^T[I]\tilde{\Omega}\right)\begin{bmatrix}1\\0\\0\end{bmatrix}\delta\gamma + \left[A\ddot{d}^T - \left\langle\left(\dot{\Omega}^T[I] - \Omega^T[I]\tilde{\Omega}\right)C\right\rangle'\right]\begin{bmatrix}\delta u\\\delta v\\\delta w\end{bmatrix}\right\} ds + \left(\dot{\Omega}^T[I] - \Omega^T[I]\tilde{\Omega}\right)C\begin{bmatrix}\delta u\\\delta v\\\delta w\end{bmatrix}\Bigg|_{s=0}^{s=L} \right\} dt \tag{19}$$

$$\delta U^g = g\rho A\begin{bmatrix}0 & 0 & 1\end{bmatrix}\int_0^L (\delta d)\, ds \tag{20}$$

$$\int_0^t \left(\delta T - \delta U^g - \delta U^e\right) dt = 0 \tag{21}$$

$$\int_0^t \left\{ \int_0^L \left(A_1\delta\gamma + A_2\delta u + A_3\delta v + A_4\delta w \right) ds + \left(B_1\delta\gamma + B_2\delta u + B_3\delta v + B_4\delta w + \bar{B}_2\delta u' + \bar{B}_3\delta v' + \bar{B}_4\delta w' \right) \Big|_{s=0}^{s=L} \right\} dt = 0 \quad (22)$$

The agent variables of Eq. (22) are introduced by Eqs. (24-34). Four coupled nonlinear partial differential Eqs. (24-27) are the motion equations of an enhanced nonlinear 3D short Euler-Bernoulli beam with a fixed support. They govern the tension-compression, torsion and two lateral bendings of a beam experiencing negligible elastic orientation. Eqs. (24-27) should be solved under the boundary conditions of Eqs. (23) at $s = 0$ and $s = L$.

$$\begin{aligned} &\left(B_1 = \gamma' = 0 \ \text{ or } \ \gamma = 0\right) \text{ and } \left(B_2 = 0 \ \text{ or } \ u = 0\right) \text{ and } \left(B_3 = 0 \ \text{ or } \ v = 0\right) \text{ and } \left(B_4 = 0 \ \text{ or } \ w = 0\right) \\ &\text{and } \left(\bar{B}_2 = 0 \ \text{ or } \ u' = 0\right) \text{ and } \left(\bar{B}_3 = v'' = 0 \ \text{ or } \ v' = 0\right) \text{ and } \left(\bar{B}_4 = w'' = 0 \ \text{ or } \ w' = 0\right) \end{aligned} \quad (23)$$

The coupling among spatial bending, torsion and axial elastic deformation of an isotropic beam is determined by the nonlinear partial differential Eq. (24-27). Therefore spatial bending induces torsion and extension in an isotropic beam. It should be noted that bending-induced torsion or axial deformation are nonlinear phenomena and cannot be sensed in linear modeling.

$$A_1 = 2GI\gamma'' - \rho\left(\dot{\Omega}^T[I] - \Omega^T[I]\tilde{\Omega}\right)[1 \ \ 0 \ \ 0]^T = 0 \quad (24)$$

where

$$\dot{\Omega}^T[I] - \Omega^T[I]\tilde{\Omega} = I\left[\left(2\dot{\Omega}_x\right) \ \ \left(\dot{\Omega}_y + \Omega_x\Omega_z\right) \ \ \left(\dot{\Omega}_z - \Omega_x\Omega_y\right)\right]$$

$$A_2 = -\rho A\ddot{u} + \rho\left\langle\left(\dot{\Omega}^T[I] - \Omega^T[I]\tilde{\Omega}\right)C\right\rangle'\begin{bmatrix}1\\0\\0\end{bmatrix} + EAu'' - EI\left\langle\frac{v''^2 + w''^2}{\left(1+u'\right)^3}\right\rangle' = 0 \quad (25)$$

$$A_3 = -\rho A\ddot{v} + \rho\left\langle\left(\dot{\Omega}^T[I] - \Omega^T[I]\tilde{\Omega}\right)C\right\rangle'\begin{bmatrix}0\\1\\0\end{bmatrix} - EI\left\langle\frac{v''}{\left(1+u'\right)^2}\right\rangle'' - EI\left\langle\frac{u''v''}{\left(1+u'\right)^3} + \frac{\gamma w''}{\left(1+u'\right)^2}\right\rangle'' + GI\left\langle\frac{\gamma w''}{\left(1+u'\right)^2}\right\rangle'' = 0 \quad (26)$$

$$A_4 = -\rho A\ddot{w} + \rho\left\langle\left(\dot{\Omega}^T[I] - \Omega^T[I]\tilde{\Omega}\right)C\right\rangle'\begin{bmatrix}0\\0\\1\end{bmatrix} - EI\left\langle\frac{w''}{\left(1+u'\right)^2}\right\rangle'' - EI\left\langle\frac{u''w''}{\left(1+u'\right)^3} - \frac{\gamma v''}{\left(1+u'\right)^2}\right\rangle'' + GI\left\langle\frac{\gamma v''}{\left(1+u'\right)^2}\right\rangle'' - g\rho A = 0 \quad (27)$$

Equation (13) has two more elastic terms than Eq. (17), eventually the additional elastic term in Eq. (26) is the term (35), in Eq. (27) is the term (36), in Eq. (30) is the term (37) and in Eq. (31) is the term (38). These new terms have improved the formulations of the nonlinear 3D Euler-Bernoulli beam theory. The new terms have arisen, since the strains' variation and the variation of elastic potential energy are derived from exact strains and exact deformation components regarding considerable elastic orientation, then the elastic orientation is neglected. They are lost in the nonlinear 3D Euler-Bernoulli beam theory [4], since the elastic orientation is neglected first, then the strains' variation and the variation of elastic potential energy are derived from the approximated strains.

Equations (24, 25) are respectively the motion equations corresponding to torsional and axial deformation of the beam. Equations (26, 27) are the motion equations corresponding to spatial bending deformation of the beam. Equations (28, 29) give the twisting moment and the axial force of the beam respectively. In the Euler-Bernoulli beam theory, the twisting moment and the axial force are respectively $G2I\gamma'$ and EAu'. Equations (30, 31) give the transverse shear forces of the beam along the 2nd and the 3rd axes of F_{S^0} respectively. In the planar Euler-Bernoulli beam theory the transverse shear force is either EIv'' or EIw''.

$$B_1 = -2GI\gamma' \tag{28}$$

$$B_2 = -\rho\left(\dot{\Omega}^T[I] - \Omega^T[I]\tilde{\Omega}\right)C\begin{bmatrix}1\\0\\0\end{bmatrix} - EAu' + EI\frac{v''^2 + w''^2}{\left(1+u'\right)^3} \tag{29}$$

$$B_3 = -\rho\left(\dot{\Omega}^T[I] - \Omega^T[I]\tilde{\Omega}\right)C\begin{bmatrix}0\\1\\0\end{bmatrix} + EI\left\langle \frac{v''}{\left(1+u'\right)^2} \right\rangle' + EI\frac{u''v''}{\left(1+u'\right)^3} + EI\frac{\gamma' w''}{\left(1+u'\right)^2} - GI\frac{\gamma w''}{\left(1+u'\right)^2}' \tag{30}$$

$$B_4 = -\rho\left(\dot{\Omega}^T[I] - \Omega^T[I]\tilde{\Omega}\right)C\begin{bmatrix}0\\0\\1\end{bmatrix} + EI\left\langle \frac{w''}{\left(1+u'\right)^2} \right\rangle' + EI\frac{u''w''}{\left(1+u'\right)^3} - EI\frac{\gamma' v''}{\left(1+u'\right)^2} - GI\frac{\gamma v''}{\left(1+u'\right)^2}' \tag{31}$$

$$\bar{B}_2 = 0 \tag{32}$$

$$\bar{B}_3 = -EI\frac{v''}{\left(1+u'\right)^2} \tag{33}$$

$$\bar{B}_4 = -EI\frac{w''}{\left(1+u'\right)^2} \tag{34}$$

Equation (32) is always zero for circular cross-sections and small elastic orientation. Equations (33, 34) give, respectively, the bending moment of the beam about the 3rd and the 2nd axes of F_{S^0}. In the planar Euler-Bernoulli beam theory the bending moment is either EIv'' or EIw''.

$$(G-E)I\left\langle \frac{\gamma' w''}{\left(1+u'\right)^2} \right\rangle' \tag{35}$$

$$(E-G)I\left\langle \frac{\gamma' v''}{\left(1+u'\right)^2} \right\rangle' \tag{36}$$

$$(E-G)I\frac{\gamma' w''}{\left(1+u'\right)^2} \tag{37}$$

$$(G-E)I\frac{\gamma' v''}{\left(1+u'\right)^2} \tag{38}$$

6. CONCLUSION

- The set of four nonlinear partial differential equations of motion of a spatial beam is exposed. Twisting, tension, compression and spatial bendings of an isotropic beam are nonlinearly coupled even if the elastic orientation is small.

- New terms have arisen in the formulations, since the strains' variation and the variation of elastic potential energy are derived from exact strains and exact deformation field regarding considerable elastic orientation, then the elastic orientation is neglected. They are lost in the nonlinear 3D Euler-Bernoulli beam theory [4], since elastic orientation is neglected first, then the variation of strains and the variation of elastic potential energy are derived from the approximated strains regarding small elastic orientation. The additional elastic term in Eq. (26) is the term (35), in Eq. (27) is the term (36), in Eq. (30) is the term (37) and in Eq. (31) is the term (38). The new elastic terms (14, 35-38) account for the enhancement of the nonlinear 3D Euler-Bernoulli beam theory. As far as the accuracy of formulation is important, the nonlinear terms, including the new elastic terms (35-38), should be considered, even if the elastic orientation is negligible.
- Due to the existence of the new elastic terms in the two coupled nonlinear partial differential bending equations and in the equations of transverse shear forces, each term in a bending Eq. (26 or 27) and in a transverse shear force Eq. (30 or 31) has found a counterpart in the other bending Eq. (27 or 26) and in the other transverse shear force Eq. (31 or 30). Some of the formulations become symmetric as the elastic rotational degrees of freedom; i.e. α, β and γ are substituted with zero, but the motion equations and transverse shear forces remain asymmetric. This is due to the fact that $\delta\kappa_x$ is asymmetric and causes the strains' variation and the variation of elastic potential energy to remain asymmetric. As a result, the coupled nonlinear partial differential bending equations and equations of transverse shear forces, i.e. Equations (26, 27, 30, 31) will become asymmetric. By replacing v with $-w$ and w with v in these equations, it can be verified that, A_4, A_3, B_4 and B_3 are not converted into A_3, $-A_4$, B_3 and $-B_4$ respectively.
- The bending equations and transverse shear forces derived in accordance with the nonlinear 3D Euler-Bernoulli beam theory [4], are asymmetric and some terms in a bending equation and transverse shear force do not have a counterpart in the other bending equation and transverse shear force. It can be verified by excluding the new terms (35-38) from the Eqs. (26, 27, 30, 31)
- Some other new elastic terms that are not discussed in this paper can be revealed if Eq. (21) is derived from the fully-enhanced variation of elastic potential energy, or from the exact variation of elastic potential energy before the substitution of the rotational elastic coordinates with zero. In this paper Eq. (21) and the motion equations are derived from the enhanced variation of elastic potential energy, namely Eq. (13), which is not the exact variation of elastic potential energy, since therein the rotational elastic coordinates have been substituted by zero. As a result, the mentioned terms have vanished.
- If u' is assumed to be small and the nonlinear terms of the Eqs. (24-27) are eliminated, then $\left(\dot{\Omega}^T[I]-\Omega^T[I]\tilde{\Omega}\right)$ will be approximated by $[2I\ddot{\gamma} \quad 0 \quad 0]$ and the well-known Eqs. (39) appear. Equations (39) governs the torsional, longitudinal and two lateral vibrations of a uniform Euler-Bernoulli beam. Each individual equation of Eqs. (39) is valid for small elastic rotation angle, but the set of Eqs. (39) cannot model the three dimensional flexibility of a beam, even if the elastic orientation is negligible.

$$A_1 = 2GI\gamma'' - \rho 2I\ddot{\gamma} = 0, \quad A_2 = -\rho A\ddot{u} + EAu'' = 0, \quad A_3 = -\rho A\ddot{v} - EIv'''' = 0, \quad A_4 = -g\rho A - \rho A\ddot{w} - EIw'''' = 0 \tag{39}$$

Equations (28-31, 33-34) are reduced to Eqs. (40) that gives the twisting moment, axial force, transverse shear forces and bending moments of the beam.

$$B_1 = -G2I\gamma' \quad , \quad B_2 = -EAu' \quad , \quad B_3 = +EIv''' , \quad B_4 = +EIw''' , \quad \overline{B}_3 = -EIv'' , \quad \overline{B}_4 = -EIw'' \tag{40}$$

NOMENCLATURE

A	cross-sectional area
d	elastic displacement vector of S from B that is projected onto F_B; It is equal to: $\left[u(s,t)+s \quad v(s,t) \quad w(s,t)\right]^T$
e	axial strain of centerline
F_B	inertial reference frame having a 3rd axis in the opposite direction of gravity.
F_S	cross-sectional frame after elastic deformation; This is a curvilinear coordinate frame having a 1st axis tangent to the curve created by cross-sectional area centers.
F_{S^0}	cross-sectional frame before elastic deformation
I, $2I$	second moment and polar moment of a circular cross-sectional area
$[I]$	area tensor with respect to F_S; It is equal to: $\begin{bmatrix} 2I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$
L	length of the beam before elastic deformation
p	position vector of σ from S projected onto F_S; It is equal to: $\left[0 \quad y \quad z\right]^T$
$\tilde{p}$	$\begin{bmatrix} 0 & -z & y \\ z & 0 & 0 \\ -y & 0 & 0 \end{bmatrix}$
R	elastic rotation transformation matrix projecting a vector from F_B onto F_S; It is equal to:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma(s,t) & \sin\gamma(s,t) \\ 0 & -\sin\gamma(s,t) & \cos\gamma(s,t) \end{bmatrix} \begin{bmatrix} \cos\beta(s,t) & 0 & -\sin\beta(s,t) \\ 0 & 1 & 0 \\ \sin\beta(s,t) & 0 & \cos\beta(s,t) \end{bmatrix} \begin{bmatrix} \cos\alpha(s,t) & \sin\alpha(s,t) & 0 \\ -\sin\alpha(s,t) & \cos\alpha(s,t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

r	agent variable; It is equal to: $\sqrt{\left(1+u'\right)^2 + v'^2}$
S	center of cross-sectional area after elastic deformation
S^0	center of cross-sectional area before elastic deformation
s	Lagrangian coordinate denoting distance of S from B before deformation
u	elastic axial deflection at S along the 1st axis of F_B
v	elastic bending deflection at S along the 2nd axis of F_B
w	elastic bending deflection at S along the 3rd axis of F_B
y, z	two components of position vector of a point apparent in the cross-sectional frame
α	so-called elastic bending rotation angle at S about the 3rd axis of F_B (1st Euler angle)
β	so-called elastic bending rotation angle at S about the 2nd axis of the updated F_B by α (2nd Euler angle)
γ	so-called elastic twisting angle at S about the 1st axis of F_S (3rd Euler angle)
δd	$\left[\delta u \quad \delta v \quad \delta w\right]^T$
ρ	density
σ	general point of the beam cross-section in Fig. 1

$\tilde{\Omega}$ $\begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}$ with a dual vector $\Omega = \begin{bmatrix} \Omega_x & \Omega_y & \Omega_z \end{bmatrix}^T$

$\dot{[\ ]}$ $\frac{\partial}{\partial t}[\ ]$

$[\ ]'$ $\frac{\partial}{\partial s}[\ ]$

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