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ANALYSIS OF CRACKED MEMBERS THE GOVERNING EQUATIONS AND EXACT SOLUTIONS*

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Abstract– In this work a considerable number of papers regarding the cracked beam like structures in the literature are cited. A new and general governing differential equation for eigenvalue analysis of cracked members is derived. Buckling analysis of cracked columns, lateral free vibration of cracked beams, axial free vibration of cracked bars, torsion free vibration of cracked shafts etc. may be considered as special cases. The proposed standard ordinary differential equation is solved and the exact analytical solutions for eigenvalues and mode shapes of these members are determined. Through customizing the general solutions for special conditions the predefined solutions are obtained and the accuracy and robustness of the present study is verified.

Keywords– Cracked members, exact solutions, Laplace transform, buckling analysis, free vibration

1. INTRODUCTION

In many engineering problems involving beam like structures, continuity of the physical and geometrical properties can be interrupted by singularities due to the presentation of concentrated cracks. Cracks, in a structural element in the form of initial defects within the material or caused by fatigue or stress concentration, can reduce the natural frequencies and change vibration mode shapes due to local flexibility introduced by the crack. The effects of concentrated cracks have been extensively studied in the literature. A crack is modeled by describing the variation of the stiffness of the member in the vicinity of a crack. The presence of a crack in a structural member introduces a local compliance that affects its response to different loads. The change in characteristics can be measured and lead to identification of structural changes, which eventually might lead to the detection of a structural flaw. A wealth of analytical, experimental and numerical investigations now exists. In deriving the governing equations and their solution for different loadings such as axial, flexure, shear, torsion, etc., quite different styles have been used in the literature. There is no unified and general style in the form of the governing equations and in the solution of cracked members. Hundreds of papers considering the effect of cracks and other defects on the behavior of beam like structures have been published in the last twenty years. From these, some typical work in buckling and bending vibration of beams and columns, axial vibration of bars and buildings, shear vibration of beams and shear buildings and vibration of shafts are cited here.

Vibration-based inspection is an area of active research. This task is performed by estimating the effects of structural damage on the eigen-parameters of structures. The problem of detecting, locating, and quantifying the extent of damage was under study for several decades [1, 2]. In order to investigate the prevailing effects of damage present in the structure under examination, a mathematical model of the damage must be introduced into the model of the structure at the location of the fault. While focusing on

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transverse vibrations, a simple stiffness reduction of the damaged region was used in Yuen [3]. Dimarogonas [4] introduced a local flexibility model, a rotational massless spring, for analysis of cracked beams. This model has been the milestone of most papers since its introduction. Several investigators have considered the analysis of vibration of cracked beams. The authors of references [5-7] investigated the influence of small cracks on the natural frequencies of slender structures by perturbation method as well as by Transfer Matrix Method (TMM). Kikidis and Papadopoulos [8] analyzed the influence of the slenderness ratio of a non-rotating cracked shaft on the dynamic characteristics of the structure. Zheng and Fan [9] analyzed the free vibration of a non-uniform beam with multiple cracks by using a kind of Fourier series. Shifrin and Ruotolo [10] proposed a technique for calculating natural frequencies of a vibrating beam with an arbitrary finite number of transverse open cracks. They used a rotational massless spring for the crack model. Their paper has a good mathematical foundation. Chondros et al. [11] used a continuous cracked beam theory for the prediction of a simply supported beam with a breathing crack. The equation of motion and the boundary conditions of the cracked beam, considered as a one-dimensional continuum, were used. They tested their method for evaluation of the lowest natural frequency of the lateral vibration of beams with a single breathing crack. Khiem and Lien [12] and Lin et al. [13] used the TMM for the natural frequency analysis of beams with an arbitrary number of cracks. Li [14] presented an approach for free vibration analysis of a non-uniform beam with an arbitrary number of cracks and concentrated masses, using the fundamental solutions to obtain the mode shape function of vibration of a non-uniform beam. The main advantage of their proposed method is that the eigenvalue equation of the beam can be conveniently determined from a second order determinant. Chondros et al. [15] proposed a continuous cracked beam vibration theory for the lateral vibration of cracked Euler-Bernoulli beams with single or double-edge open cracks. They used the Hu-Washizu-Bar variational formulation to develop the differential equation and the boundary conditions of the cracked beam as a one dimensional continuum. The displacement field about the crack was used to modify the stress and displacement field throughout the bar. Behzad, et al. [16], based on the Hamilton principle, developed the equation of motion and corresponding boundary conditions for bending vibration of a beam with an open edge crack. The natural frequencies of a uniform Euler-Bernoulli beam have been calculated using the new developed model in conjunction with the Galerkin projection method. Patil and Maiti [17] experimentally verified a method for the prediction of the location and size of multiple cracks based on the measurement of natural frequencies of slender multi-cracked cantilever beams. Vibration of beams with multiple open cracks subjected to axial force is studied by Binici [18]. He proposed a method to obtain the eigen-frequencies and eigen-modes of beams containing multiple cracks and subjected to axial force. The method uses one set of boundary conditions to determine mode shapes. Another set of boundary conditions yields second-order determinants that needs to be solved for eigenvalues. He considered both the vibration and the buckling of the structure. Kisa and Gurel [19] developed a numerical model for the modal analysis of multi cracked beams with a circular cross section. Their model divides a beam into a number of parts from the crack sections and couples them by flexibility matrices considering the interaction forces that are derived from the fracture mechanic theory. The main feature of their work was the possibility of analysis of beams with any kind of boundary conditions. Ruotolo and Surace [20] proposed a method for the calculation of natural frequencies of a vibrating isotropic bar with an arbitrary finite number of symmetric transverse open cracks. The TMM, the finite element method and the smooth function method are considered for analysis. A bar with three cracks was analyzed and the results of the three methods were compared. Schoefs et al. [21] proposed a new cracked beam finite element with a view to introducing the effect of large through-cracks in the structural analysis for framed structures like jacket offshore platforms. This cracked beam element, based on strain energy, involves four parameters (two eccentricities and two stiffnesses) which represent the loss of stiffness. These parameters have been identified on the basis of several joint configurations. The cracked beam element model has been applied

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on a T-tubular joint. The results were in good agreement with the strength of the material and with the 3D finite elements. A simply supported Euler-Bernoulli beam with an open crack is considered in Pakrashi et al [22]. A new wavelet-kurtosis based approach has been proposed to identify and calibrate the crack. For obtaining the mode shape, crack models of different levels of complexities, i.e. lumped, continuous and smeared, have been used. The problem of the integration of the static governing equations of the uniform Euler-Bernoulli beam with discontinuities is studied by Biondi and Caddemi [23, 24]. Closed form solutions of the governing differential equation, requiring the knowledge of boundary conditions only, are proposed. The proposed solution for the case of slope discontinuity is compared with the solution of a beam having an internal hinge with a rotational spring reproducing the slope discontinuity. Availability of explicit expression of the response functions for beams with discontinuities allows the introduction of such beams as frame elements in finite element codes. Wang and Qiao [25] presented a general solution of vibration of an Euler-Bernoulli beam with arbitrary type and location of discontinuity. Unlike the commonly used approach in the literature, the modal displacement of the whole beam is expressed by a single function using the Heaviside's unit step function to account for discontinuities. The general modal displacement function is then solved by using Laplace transform. The presented solution reduced the complexity of the vibration of beams with arbitrary discontinuities to the same order of the case without discontinuity. The problem of buckling analysis of cracked shells is introduced and its significance is clarified by Vafai and Estekamch [26]. The latest work on analysis of cracked members is reported by Caddemi and Calio [27]. They used the Dirac's distribution in the buckling analysis of a Euler-Bernoulli column. Their method of analysis is more advanced than the previous work, but their derived governing equation is not in a standard form. As a result, special methods were used for their numerical analysis. This paper was very helpful in the development of the present work.

The derivation styles, the governing equation obtained and the method of solutions used are diverse. The aim of the present paper is to derive a unified formulation which can be used equally for analysis of cracked members under bending moment, axial load, shear force, torque and their combination. The proposed governing equation simplifies the analysis of cracked members to the same level as that of the intact one. That is, the number of boundary conditions for both cases is the same. The compatibility conditions at cracked points are considered in the derivation and no longer need to be considered. Moreover, the proposed governing equation paves the way for simple finite element analysis of cracked members. This last advantage is not considered here. The paper is organized as follows. The literature review and introduction are presented in section 1. The governing equations for eigenvalue analysis of cracked members are developed in section 2. In section 3 the exact solution for the developed ordinary differential equations is derived. A special verification is included in section 4. The conclusions obtained from this study are presented in section 5.

2. THE GOVERNING EQUATION

The governing equation for a typical eigenvalue analysis (buckling, free vibration, etc.) of beam-like members is defined as follows:

$$\Gamma^{(n)}\left(k\,\Gamma^{(n)}(V)\right) \pm Q(V+B) = 0,\ \Gamma^{(n)} = \frac{d^n}{dx^n},\ B = B_1 + B_2 x,\ x \in [0 - L] \tag{1}$$

where k is stiffness, Q is force, B_1, and, B_2, are unknown coefficients, x is axial coordinate, V is displacement, L is length of the member and n is an integer. The parameter n may be 1 or 2. Integrate equation 1, n times, to obtain the following equation:

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$$k\,\Gamma^{(2n)}(W) \pm QW = 0, W = \int^{n}(V+B)dx + B_3, W^{(n)} = (V+B) \tag{2}$$

in which $\int^{n}(V+B)dx$ denotes n times integration of the kernel (V+B). The change of variable from V to W is performed in order to pave the way for the innovative formulation of cracked members. The boundary conditions for W are determined from that of V using the right side of Eq. (2).

Cracks introduce stress concentration in the members. This stress concentration reduces the eigenvalues of the intact eigen problem. The aim of this paper is to investigate the effect of cracks on the eigenvalues and eigen vectors (or mode shapes) of the solution of Eq. (2).

In fracture mechanics, a crack is modeled by a spring with a specified flexibility. That is, a spring is inserted in the member in the position of the crack. The flexibility of the spring is explicitly specified in terms of crack depth, member's cross section height and mechanical properties of the member's material. The introduction of the spring produces a jump in a function of the member's displacement. The jump in this function, at point $x = x_j$, corresponding to Eq. (2) is defined as follows:

$$\Delta\Gamma^{(n-1)}(V)_{crack} = C_j\Gamma^{(n)}(V_j) \rightarrow \Gamma^{(n-1)}(V)_{crack} = C_j\Gamma^{(n)}(V_j)H(x-x_j) \tag{3}$$

Where Δ denotes the jump, H is the Heaviside unit step function, x_j is the crack position coordinate, and C_j is the equivalent spring flexibility. Taking derivative from both sides of Eq. (3) (note that $\Gamma^{(n)}(V_j)$ is constant) leads to the following equation:

$$\Gamma^{(n)}(V)_{crack} = C_j\delta_j\Gamma^{(n)}(V_j) \rightarrow k\,\Gamma^{(2n)}(W)_{crack} = C_j\delta_j k\,\Gamma^{(2n)}(W_j), \delta_j = \delta(x-x_j) \tag{4}$$

In which δ_j is the Dirac's delta distribution [28].

Determine $k\,\Gamma^{(2n)}(W)$ from Eq. (2). Substitute it into Eq. (4) to obtain the crack contribution as follows:

$$k\,\Gamma^{(2n)}(W)_{crack} = \mp C_j\delta_j Q(W+B) \tag{5}$$

The differential operator in the governing differential equation of a cracked member is composed of two parts, intact and cracked, as follows:

$$k\,\Gamma^{(2n)}(W) = k\,\Gamma^{(2n)}(W)_{intact} + k\,\Gamma^{(2n)}(W)_{crack} \tag{6}$$

Compute the intact contribution from Eq. (6) and substitute it into Eq. (2) to obtain:

$$k\,\Gamma^{(2n)}(W) - k\,\Gamma^{(2n)}(W)_{crack} \pm Q(W+B) = 0 \tag{7}$$

Insert the crack contribution, from Eq. (5), into the Eq. (7) and simplify to obtain the following equation:

$$k\,\Gamma^{(2n)}(W) \pm (1 + C_j\delta_j)Q(W+B) = 0 \tag{8}$$

Equation (8) may be extended for multi cracked members as follows:

$$k\,\Gamma^{(2n)}(W) \pm (1 + C_j\delta_j)Q(W+B) = 0,\ j = 1, n_c \tag{9}$$

In Eq. (9) the Einstein's summation convention is assumed. In this convention, repeated index (j here) denotes summation over the index range (n_c here). Equation (9) is the governing equation of cracked members. Equation (9) is a standard differential equation. It can be precisely solved by known methods of

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solution. This elegant specific property of the proposed equation is not available in the related previous work in the literature.

3. EXACT SOLUTIONS

In engineering problems the orders 2 and 4 (n=1 and 2) of the governing equation are the most prevalent cases. For these two cases the exact solutions are derived.

a) Buckling analysis of multi-cracked columns

The governing equation for buckling analysis of multi-cracked columns is defined as follows:

$$W'' + \lambda^2 \left(1 + C_j \delta_j\right)\left(W + B_1 + B_2 x\right) = 0,\ j = 1, n_c,\ EI\lambda^2 = P, W = V \tag{10}$$

Take the Laplace transform (LT) [29] from Eq. (10) to obtain the following equation:

$$s^2 L_W - sW_0 - W_0' + \lambda^2 \left(L_W + \frac{B_1}{s} + \frac{B_2}{s^2} \right) + \lambda^2 C_j \left(W_j + B_1 + B_2 x_j\right) e^{-x_j s} = 0 \tag{11}$$

The LT of the displacement function is obtained as follows:

$$L_W = \frac{sW_0}{\left(s^2 + \lambda^2\right)} + \frac{W_0'}{\left(s^2 + \lambda^2\right)} - \frac{B_1 \lambda^2}{s\left(s^2 + \lambda^2\right)} - \frac{B_2 \lambda^2}{s^2\left(s^2 + \lambda^2\right)} - \frac{\lambda^2 C_j \left(W_j + B_1 + B_2 x_j\right) e^{-x_j s}}{\left(s^2 + \lambda^2\right)} \tag{12}$$

By taking the inverse of LT the general solution for this case is defined as follows:

$$W = W_0 \cos \lambda x + \frac{W_0'}{\lambda} \sin \lambda x - B_1 \left(1 - \cos \lambda x\right) - B_2 \left(x - \frac{\sin \lambda x}{\lambda} \right)$$

$$- \lambda C_j \left(W_j + B_1 + B_2 x_j\right) \sin \lambda \left(x - x_j\right) H\left(x - x_j\right) \tag{13}$$

Where W_j is the value of W at the cracked point j, defined as follows:

$$W_j = W_0 \cos \lambda x_j + \frac{W_0'}{\lambda} \sin \lambda x_j - B_1 \left(1 - \cos \lambda x_j\right) - B_2 \left(x_j - \frac{\sin \lambda x_j}{\lambda} \right) \tag{14}$$

b) Axial vibration of multi-cracked bars

For axial free vibration the governing equation is defined, a special case of Eq. (9), as follows:

$$W'' + \lambda^2 \left(1 + C_j \delta_j\right) W = 0,\ j = 1, n_c,\ EA\lambda^2 = m\omega^2, W' = V \tag{15}$$

Set B_1 and B_2 in Eqs. (13) equal to zero to obtain the solution for this case as follows:

$$W = W_0 \cos \lambda x + \frac{W_0'}{\lambda} \sin \lambda x - \lambda C_j W_j \sin \lambda \left(x - x_j\right) H\left(x - x_j\right) \tag{16}$$

Where

$$W_j = W_0 \cos \lambda x_j + \frac{W_0'}{\lambda} \sin \lambda x_j \tag{17}$$

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The equations of this section are equally valid for the shear vibration of beams and the torsional vibration of shafts.

c) Lateral vibration of multi-cracked beams

The governing equation for lateral vibration of multi-cracked beams is given by the following equation:

$$W^{IV} - \lambda^4\left(1 + C_j\delta_j\right)W = 0,\ j = 1, n_c,\ W = \iint V dx + B,\ W'' = V,\ EI\lambda^4 = m\omega^2 \tag{18}$$

The LT of this equation is written as follows:

$$s^4 L_W - s^3 W_0 - s^2 W_0' - sW_0'' - W_0''' - \lambda^4 L_W - \lambda^4 C_j W_j e^{-x_j s} = 0 \tag{19}$$

And the LT of the displacement function is obtained as follows:

$$L_W = \frac{s^3}{s^4 - \lambda^4}W_0 + \frac{s^2}{s^4 - \lambda^4}W_0' + \frac{s}{s^4 - \lambda^4}W_0'' + \frac{1}{s^4 - \lambda^4}W_0''' + \lambda^4 C_j W_j \frac{e^{-x_j s}}{s^4 - \lambda^4} \tag{20}$$

Or

$$L_W = \frac{W_0}{2}\left[\frac{s}{s^2 - \lambda^2} + \frac{s}{s^2 + \lambda^2}\right] + \frac{W_0'}{2}\left[\frac{1}{s^2 - \lambda^2} + \frac{1}{s^2 + \lambda^2}\right]$$
$$+ \frac{s}{s^4 - \lambda^4}W_0'' + \frac{1}{s^4 - \lambda^4}W_0''' + \lambda^4 C_j W_j \frac{e^{-x_j s}}{s^4 - \lambda^4} \tag{21}$$

The function W is obtained by inverse LT of equation 21. The resulted W and its derivatives are defined by the following equation:

$$W^{(n)}(x) = F_{k(i,n)} W_0^{(i)} \lambda^{-i+n} + C_j W_j \lambda^{i+n+1} F_{k(3,n)}\left(x - x_j\right) H\left(x - x_j\right),\ i = 0,3,\ j = 1, n_c \tag{22}$$

In this equation summation convention is assumed on i and j. The (n) and (i) denote derivatives with respect to x, the superscript without brace denotes power and the subscript k is the element of a matrix. The matrix k and the parameter W_j are defined as follows:

$$[k(i,n)] = \begin{bmatrix} 0 & 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 & 1 \\ 2 & 1 & 0 & 3 & 2 \\ 3 & 2 & 1 & 0 & 3 \end{bmatrix},\ W_j = F_{k(i,n)}\left(b_j\right) W_0^{(i)} \lambda^{-i+n},\ i = 0,3,\ j = 1, n_c \tag{23}$$

The functions $F_k,\ k = 0,3$ are defined as follows:

$$F_0 = \left(\cosh \lambda x + \cos \lambda x\right),\ F_1 = \left(\sinh \lambda x + \sin \lambda x\right) \tag{24}$$

And

$$F_2 = \left(\cosh \lambda x - \cos \lambda x\right),\ F_3 = \left(\sinh \lambda x - \sin \lambda x\right) \tag{25}$$

Equation (14) of Wang and Qiao [25] is included as follows:

$$L_V = \frac{s^3}{s^4 - \lambda^4}V_0 + \frac{s^2}{s^4 - \lambda^4}V_0' + \frac{s}{s^4 - \lambda^4}V_0'' + \frac{1}{s^4 - \lambda^4}V_0''' + \Delta V_j' s^2 \frac{e^{-x_j s}}{s^4 - \lambda^4} \tag{26}$$

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The first four terms of Eq. (26) are the same as that of Eq. (20) except in the name of parameters. The parameter V is the lateral displacement while W is an integral of V as defined in Eq. (2). The fifth term in the first, $\left(\Delta V_j' s^2\right)$, is different from that in the second, $\left(\lambda^4 C_j W_j\right)$. As a result, the solution based on equation (26) needs the discontinuity in the slopes to be known. Otherwise, it needs the relation $\Delta V_j' = C_j V''$ to be inserted into the equation. The final solution based on equation (20) is relatively simpler than that of Wang and Qiao.

4. VERIFICATION

In order to verify the formulation and its implementation, three examples are included in this section.

Example 1: Buckling of simply supported multi-cracked columns

The boundary conditions for this case are defined as follows:

$$W(0) = W(L) = 0,\ B_1 = B_2 = 0 \tag{27}$$

For simply supported case the general displacement is defined as follows:

$$W = \frac{W_0'}{\lambda}\sin\lambda x - \lambda C_j W_j \sin\lambda\left(x - x_j\right)H\left(x - x_j\right) \tag{28}$$

And

$$W_j = \frac{W_0'}{\lambda}\sin\lambda x_j \equiv \sin\lambda x_j \tag{29}$$

Substitute the $W(L) = 0$ boundary condition to obtain:

$$W(L) = \sin\lambda L - \lambda C_j W_j \sin\lambda\left(L - x_j\right) = 0 \tag{30}$$

From this equation the eigenvalues may be determined.

The mode shapes corresponding to this case are defined as follows:

$$W = \sin\lambda x - \lambda C_j W_j \sin\lambda\left(x - x_j\right)H\left(x - x_j\right) \tag{31}$$

Note that for a member with a single crack, the eigenvalue equation is defined as follows:

$$\sin\lambda L - \lambda C_1 \sin\lambda x_1 \sin\lambda\left(L - x_1\right) = 0 \tag{32}$$

This is a well known equation which may be found in the literature [18]. Equation (32) is solved by the Newton-Raphson method [29]. The effect of crack depth and crack position on the buckling capacity ratio $R_P = P_{cr}/P_u$, where P_{cr} and P_u are cracked and intact buckling loads, respectively, is shown in Fig. 1. For a crack at the middle of the column the effect of slenderness ratio (h/L) on the buckling load for different crack depth ratios is shown in Fig. 2. In these figures, $x_i = \xi = a/h$, $beta = x_j/L$ and a are crack depth ratio, crack position ratio and crack depth respectively. The coefficient C_1 is defined as follows [30].

$$\begin{aligned} C_1 = 5.346(h)(1.862\xi^2 - 3.95\xi^3 + 16.375\xi^4 \\ - 37.226\xi^5 + 76.81\xi^6 - 126.9\xi^7 + 172\xi^8 \\ - 143.97\xi^9 - 66.56\xi^{10}) \end{aligned} \tag{33}$$

For the cases of axial, shear and torsion free vibration the governing equation is the same. A typical axial problem is presented for completeness.

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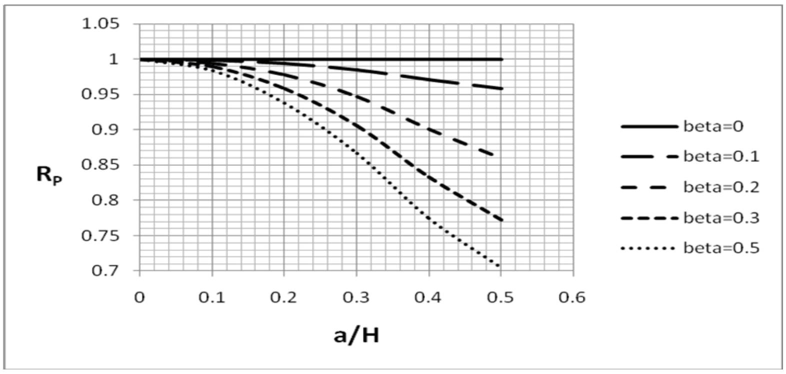

Fig. 1. The effect of crack depth and position on the buckling load

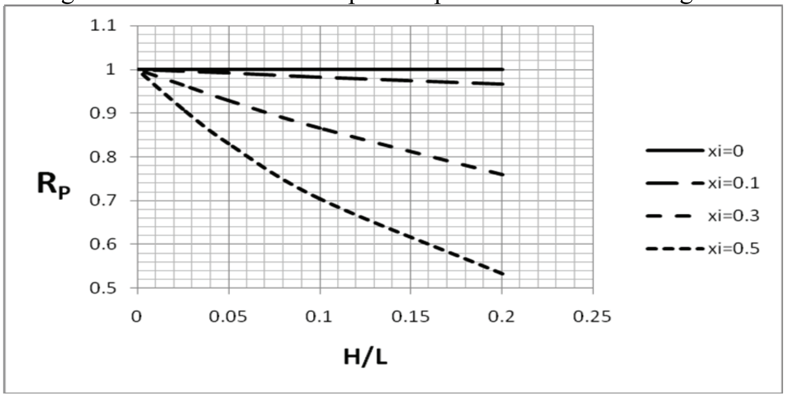

Fig. 2. The effect of slenderness ratio on the buckling load

Example 2: Free vibration of a fixed-free multi-cracked bar

Determine the dynamic characteristics of a fixed-free bar with two cracks at x_1 and x_2. Denote the corresponding springs' flexibilities as C_1 and C_2 respectively.

Solution:

The boundary conditions are defined as follows:

$$V(0) = W_0'(0) = 0, V_0'(L) = W''(L) = 0 \tag{34}$$

Substitution of the first boundary condition leads to the solution as follows:

$$W = \cos \lambda x - \lambda C_j W_j \sin \lambda (x - x_j) H(x - x_j), j = 1,2 \tag{35}$$

And substitution of the second boundary condition leads to the following equation for computation of λ.

$$W''(L) = 0 \rightarrow \cos \lambda L - \lambda C_j W_j \sin \lambda (L - x_j) = 0, j = 1,2 \tag{36}$$

Where

$$W_j = \cos \lambda x_j, j = 1,2 \tag{37}$$

Equations 35 and 34 are used for determination of λ and *W* respectively. The mode shape and its derivative are determined as follows:

$$V = \sin \lambda x + \lambda C_j W_j \cos \lambda (x - x_j) H(x - x_j) \tag{38}$$

And

$$V' = \lambda [\cos \lambda x - \lambda C_j W_j \sin \lambda (x - x_j) H(x - x_j)] \tag{39}$$

Variation of λ versus $C_2 \, (0 \prec C_2 \prec 5), C_1 = 1, x_1 = 0.2L, x_2 = 0.5L, L = 10$ for the first mode is shown in Fig. 3. The first mode shape for $C_1 = 1$ and $C_2 = 2$ is shown in Fig. 4.

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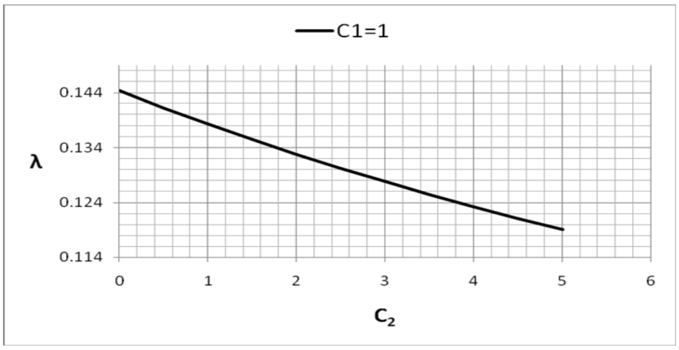

Fig. 3. Variation of λ versus C_2 for $C_1 = 1$

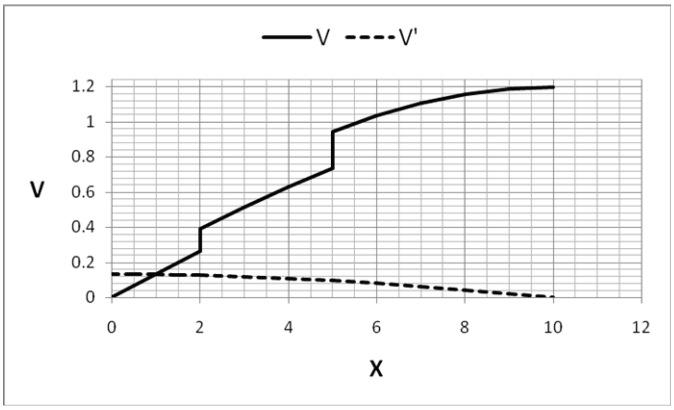

Fig. 4. Variation of *V* versus *x*

The variation of V' is also shown in this figure. From this figure the relation $\Delta V_{crack} = CV'$ is accurately checked at two cracked points. This observation verifies the accuracy of the proposed method, its derivation and its implementation. Note that the other modes may be computed in the same way.

Example 3: Free vibration of a fixed-free multi-cracked beam

Determine the dynamic characteristics of a fixed-free beam of $L = 0.8m$ and square cross-section $b = h = 0.02m$. The material properties are assumed as Young's modulus $E = 2.1\times10^{11}\ N/m^2$, Poisson's ratio $\nu = 0.35$ and mass density $\rho = 7800\,kg/m^3$. The beam has two cracks at $b_1 = 0.12m\ with\ a_1 = 2mm$ and $b_2 = 0.4m\ with\ a_2 = 3mm$ respectively.

Solution:

The boundary conditions are defined as follows:

$$V(0)\rightarrow W''(0)=0,\ V''(0)\rightarrow W(L)=0,\ V''(L)\rightarrow W(0)=0,\ V'''(L)\rightarrow W'(0)=0 \qquad (40)$$

The general solution from Eq. (22) is written as follows:

$$2W = \left(W_0 + W_0''\lambda^{-2}\right)\cosh\lambda x + \left(W_0 - W_0''\lambda^{-2}\right)\cos\lambda x + \left(W_0'\lambda^{-1} + W_0'''\lambda^{-3}\right)\sinh\lambda x$$
$$+ \left(W_0'\lambda^{-1} - W_0'''\lambda^{-3}\right)\sin\lambda x + \lambda C_j W_j \left(\sinh\lambda(x-b_j) - \sin\lambda(x-b_j)\right)H(x-b_j) \qquad (41)$$

Substitution of boundary conditions into the above equation leads to the following characteristics equation.

$$T_{11}T_{22} - T_{21}T_{12} = 0 \qquad (42)$$

Where

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$$
\begin{aligned}
T_{11} &= \left[2(\cosh\lambda L+\cos\lambda L)+\lambda C_j(\cosh\lambda b_j+\cos\lambda b_j)(\sinh\lambda(L-b_j)-\sin\lambda(L-b_j))\right]\\
T_{12} &= \left[2(\sinh\lambda L+\sin\lambda L)+\lambda C_j(\sinh\lambda b_j+\sin\lambda b_j)(\sinh\lambda(L-b_j)-\sin\lambda(L-b_j))\right]\\
T_{21} &= \left[2(\sinh\lambda L-\sin\lambda L)+\lambda C_j(\cosh\lambda b_j+\cos\lambda b_j)(\cosh\lambda(L-b_j)-\cos\lambda(L-b_j))\right]\\
T_{22} &= \left[2(\cosh\lambda L+\cos\lambda L)+\lambda C_j(\sinh\lambda b_j+\sin\lambda b_j)(\cosh\lambda(L-b_j)-\cos\lambda(L-b_j))\right]
\end{aligned} \tag{43}
$$

The natural frequencies (Hz) determined from Eq. (42) (ANAL) are compared with that of Shifrin and Ruotolo in Table 1. The results are in excellent agreement.

Table 1. Comparison of natural frequencies for a beam with two cracks

	Frequency							
	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
ANAL	26.095	163.320	459.601	895.132	1486.443	2247.501	3101.997	3828.458
Ref.[10]	26.095	163.322	459.601	-	-	-	-	-
Diff. %	0.000	0.001	0.000	-	-	-	-	-

By customizing the general solution for special conditions (e.g. for single crack and specified boundary conditions), the well known solutions are obtained. This special agreement of the results may well be considered as a verification of the derived governing equations and the proposed exact solutions.

5. CONCLUSION

From this research the following conclusions may be obtained.

a. A general governing differential equation for eigenvalue analysis of multi-cracked beam like structures is derived.
b. The derived governing equation is a standard ordinary differential equation.
c. With the help of Laplace Transform a general exact solution is proposed.
d. Through customizing the general solution for special cases with well known solutions, the accuracy and robustness of the proposed solutions are verified.

To the best of the author's knowledge, there is no such simple and general formulation in the literature to date. This work is a satisfactory ending to the ongoing eigenvalue analysis of multi-cracked beam like structures.

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