BICRITERIA RESOURCE ALLOCATION PROBLEM IN PERT NETWORKS

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Abstract We develop a bicriteria model for the resource allocation problem in PERT networks, in which the total direct costs of the project as the first objective, and the mean of project completion time as the second objective are minimized. The activity durations are assumed to be independent random variables with either exponential or Erlang distributions, in which the mean of each activity duration is a non-increasing function of the amount of resource allocated to it. The direct cost of each activity is assumed to be a non-decreasing function of the amount of resource allocated to it. Finally, we use the goal attainment method to solve the related bicriteria optimal control problem numerically, by converting this problem to a related bicriteria nonlinear programming, and obtain the optimal values of the resources allocated to the activities.

Key Words Project Management, Multiple Objective Programming, Control

چکیده در این مقاله مدل دو هدفی برای تخصیص منابع در شبکه های PERT توسعه داده شده است که در آن هزینه های مستقیم پروژه به عنوان اولین هدف و میانگین زمان ختم پروژه به عنوان هدف دوم کمینه می‌شوند. زمان فعالیتها متغیر های تصادفی مستقل با توزیع نمایی یا ارلنگ در نظر گرفته شده است. میانگین زمان هر فعالیت تابعی غیر صعودی از میزان منابعی است که به آن تخصیص می‌یابد. هزینه مستقیم هر فعالیت نیز تابعی غیر نزولی از میزان منابعی که به آن اختصاص می یابد در نظر گرفته شده است. در نهایت روش دستیابی هدف را برای حل عددی مساله کنترل بهینه دو هدفی به کار گرفته ایم که با تبدیل مساله به یک برنامه ریزی غیر خطی دو هدفی، مقادیر بهینه منابع تخصیصی فعالیتها به دست می‌آیند.

1. INTRODUCTION

Project Scheduling has been a major objective of most models and methods proposed to aid planning and management of projects. Initially, the study of project scheduling has been done considering just the duration and precedence conditions and ignoring the resource requirements. The most important method to schedule a project assuming deterministic durations is the well-known CPM – Critical Path Method. However, most durations have the random natures and therefore, PERT was proposed to determine the distribution of the total duration, T.

This method is based on the substitution of the network by the CPAD – critical path assuming that each activity has a fixed duration equal to its mean (critical path using average durations). The mean and the variance of the CPAD are given by the sum of the means and of the variances of its activities, respectively, and therefore these results considered the mean and the variance of the total duration of the network.

Unfortunately, this is an optimistic assumption as the real mean, E(T), is greater than or equal to such estimate. Thus, many authors have studied:
1. Analytical approximations of the cumulative distribution function of T, F(T). Charnes, Cooper and Thompson [1] developed a chance–constrained

programming approach to PERT problems. They assume exponential activity durations. Martin [2] provides a systematic way of analyzing PERT networks through series–parallel reductions. Kulkarni and Adlakha [3] developed a continuous-time Markov process approach to PERT problems.

2. Upper or lower bounds of F(T). Elmaghraby [4] provides lower bounds for the true, expected project completion time. Fulkerson [5], Clingen [6], Robillard [7] and Perry and Creig [8] have done the similar works.

3. Monte-Carlo simulation to estimate F(T). Several authors have used conditional sampling to achieve variance reduction (see Burt and Garman [9], Garman [10], and Sigal, Pritsker and Solberg [11]). Fishman [12] achieves further variance reduction by using a combination of quasirandom points and conditional sampling to estimate the distribution and mean of project completion time.

In CPM networks, activity duration is viewed either as a function of cost or a function of resources committed to it. The well-known time-cost trade-off problem (TCTP) in CPM network takes the former view. Studies on TCTP have been done using various kinds of cost function such as linear (Fulkerson [13], Kelly [14]), discrete (Demeulemeester, Herroelen and Elmaghraby [15]), convex (Lamberson and Hocking [16], Berman [17]) and concave (Falk and Horowitz [18]). When the cost functions are arbitrary (still non-increasing), a dynamic programming (DP) approach was suggested by Robinson [19]. A powerful approach for solving this problem on dynamic programming was presented by Elmaghraby [20].

In the TCTP, the objective is to determine the duration of each activity in order to achieve the minimum total direct and indirect costs of the project. Tavares [21] has presented a general model based on the decomposition of the project into a sequence of stages and the optimal solution can be easily computed for each practical problem as it is shown for a real case study.

Laslo [22] described a stochastic extension of the critical path method and the time-cost trade-off model. He developed several ideas for formulating the relationship between time-cost trade-offs and two chance constraints for a single activity, the time chance constraint and the cost chance constraint. Golenko-Ginzburg, Gonik and Sitniakovski [23] developed an optimization procedure to maximize the probability confidence for project due-dates (the probability that milestones in the project will be completed on or before their schedule date) under budget constraints or to minimize the project budget under due-date chance constraints. Time-cost optimization methodologies under time and cost chance constraints have been formulated and preliminary simulations optimizing budget allocation among project activities have been implemented by Laslo and Goldberg [24].

Weglarz [25] studied this problem using optimal control theory and assuming that the processing speed of each activity at time t is a continuous, non-decreasing function of the amount resource allocated to the activity at that instant of time. This means that also time is here considered as a continuous variable. Unfortunately, it seems that this approach is not applicable to networks with a reasonable size (>10). The stochastic assumption introduces additional difficulties and then the experimental approach has to be adopted. There are also some other papers related to the applications of optimal control theory in stochastic networks, but we could not find any paper corresponding to the application of this issue in PERT networks. For example, Jordan and Ku [26] investigated the optimal control of two multiserver loss queues with two types of customers. Tseng and Hsiao [27] analyzed the optimal control of the arrival rate to a two-station network of queues for the objective of maximum system throughput under a system time-delay constraint optimality criterion. Shioyama [28] developed an optimal control problem in a queuing network system with two types of customers and two stages. Azaron and Fatemi Ghomi [29] developed a new model for optimal control of service rates of the service stations and also the arrival rates to these service stations in a class of Jackson networks, in which the expected value of shortest path of the network and also the total operating costs of the service stations of the network per period are minimized.

In this paper, we develop a new analytical model for the time-cost trade-off problem via optimal control theory in Markov PERT networks. It is assumed that the activity durations are independent random variables with either exponential or Erlang distributions. Initially, we use the method developed by Kulkarni and Adlakha [3] to obtain the distribution function of project completion time in Markov PERT networks. This is done through solving a system of linear differential equations, which is obtained from a relevant continuous-time Markov process.

It is also assumed that the amount of resource allocated to each activity is controllable, in which the mean of activity duration is a non-increasing function of this control variable.

If we increase the resources allocated to the activities, the mean project completion time will be decreased, but the total direct costs of the project, which is a non-decreasing function of the resources allocated to the activities, will be increased, which is undesirable. Therefore, we try to solve a bicriteria problem, in which the first criterion is the minimization of the total direct costs of the project and the second criterion is the minimization of the mean project completion time (note that the project completion time is a random variable itself, and because of that we use the mean of this quantity as a proper criterion).

We can apply one of the multiple objective techniques to obtain the optimal values of resources allocated to the activities, in this bicriteria optimal control problem, after discretizing this continuous-time problem and converting that one to a related bicriteria nonlinear programming. We use the goal attainment method to solve this bicriteria problem.

Therefore, we present a new analytical method for solving the time-cost trade-off problem in Markov PERT networks, via optimal control theory, Markov processes and multiple objective programming. This method can be applied for the management and control of the projects with the stochastic natures.

The remainder of this paper is organized in the following way. In section 2, an analytical method for obtaining the distribution function of project completion time in Markov PERT networks is presented. In section 3, we present the bicriteria optimal control problem. Section 4 includes some numerical examples, and finally we draw the conclusion of the paper in section 5.

2. DISTRIBUTION FUNCTION OF PROJECT COMPLETION TIME IN MARKOV PERT NETWORKS

In this section, we present an analytical method to obtain the distribution function of project completion time in PERT networks, or in fact the distribution function of longest path from the source to the sink node of a directed acyclic stochastic network, in which the arc lengths or activity durations are mutually independent random variables with either exponential or Erlang distributions. To do that, we develop the method of Kulkarni and Adlakha [3].

Let G = (V,A) be a PERT network with set of nodes $V=\{v_1, v_2, ..., v_m\}$ and set of activities $A=\{a_1, a_2, ..., a_n\}$. Duration of activity $a \in A$ is a random variable with either exponential distribution with the parameter λ_a, or Erlang distribution with the parameters (λ_a, n_a).

We know an exponential density function is a special case of Erlang density function with n_a =1, and consequently a random variable with Erlang density function and the parameters (λ_a, n_a) can be decomposed to n_a series of exponentially random variables with the parameter λ_a.

First, we transform the original PERT network into a new one, in which all activity durations have exponential distributions. To do that, we transform each Erlang activity with the parameters (λ_a, n_a) to n_a series of exponential activities with the parameter λ_a. Now, Let $G' = (V', A')$ be the transformed network, in which V' represents the set of nodes and A' represents the set of arcs of the transformed network. The source and sink nodes are denoted by s and t, respectively. For $a \in A'$, let $\alpha(a)$ be the starting node of arc a, and $\beta(a)$ be the ending node of arc a.

In the case of general PERT networks, we can approximate the general distributions of activity durations by the appropriate Erlang distributions, which is a special class of PH distribution, by matching the first two moments.

Definition 1

$$I(v) = \{a \in A : \beta(a) = v\} \quad (v \in V) \tag{1}$$

$$O(v) = \{a \in A : \alpha(a) = v\} \quad (v \in V) \tag{2}$$

I(v) is the set of arcs ending at node v, and O(v) is

TABLE 1. State Space for the Example Network.

1. (1,2)	5. $(1,4^*,6)$	9. $(3^*,4,6)$	13. $(3,4^*,6^*)$	17. (ϕ,ϕ)
2. (2,3)	6. $(1,4,6^*)$	10. $(3,4^*,6)$	14. (5,6)	
3. $(2,3^*)$	7. $(1,4^*,6^*)$	11. $(3,4,6^*)$	15. $(5^*,6)$	
4. (1,4,6)	8. (3,4,6)	12. $(3^*,4,6^*)$	16. $(5,6^*)$	

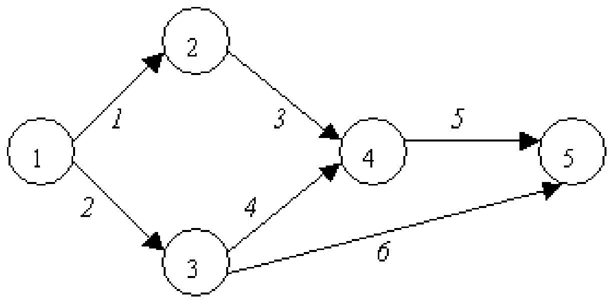

Figure 1. The example network.

the set of arcs starting at node v.

Definition 2 If $X \subset V$ such that $s \in X$ and $t \in \overline{X}$=V-X, then an (s,t) cut is defined as:

$$(X,\overline{X}) = \{a \in A : \alpha(a) \in X, \beta(a) \in \overline{X}\} \tag{3}$$

An (s,t) cut $(X,\overline{X})$ is called a uniformly directed cut (UDC) if $(\overline{X},X)$ is empty.

Definition 3 A pair (E,F) of subsets of A is called an admissible 2-partition of a UDC D if $E \cup F = D$ $E \cap F = \phi$, and $I(\beta(a)) \not\subset F$ for any $a \in F$.

We assume that the project modeled by network G starts at time zero and ends at a random time T. During the project execution, each activity can be in one of the following states:

(i) Active: an activity 'a' is active at time 't' if it is being executed at time 't'.

(ii) Dormant: an activity 'a' is dormant at time 't' if it has finished but there is at least one unfinished activity in $I(\beta(a))$ at time 't'. If an activity 'a' is dormant at time 't', then its successor activities in $O(\beta(a))$ cannot begin.

(iii) Idle: an activity 'a' is idle at time 't' if it is neither active nor dormant at time 't'.

Definition 4

$$Y(t)=\{a\in A: \text{a is active at time t}\} \tag{4}$$

$$Z(t)=\{a\in A: \text{a is dormant at time t}\} \tag{5}$$

$$X(t)=\{Y(t),Z(t)\} \tag{6}$$

Let S denote the set of all admissible 2-partitions of all UDCs of the network, and let $\overline{S} = S \cup \{(\phi,\phi)\}$. It can be proved that $\{X(t),\ t\geq 0\}$ is a continuous-time Markov process with sate space $\overline{S}$. The infinitesimal generator matrix of this process is denoted by $Q=[q\{(E,F),(E',F')\}]$ (E,F) and $(E',F') \in \overline{S}$ where

$$q\{(E,F),(E',F')\}=\begin{cases} \lambda_a & \text{if } a\in E, I(\beta(a))\not\subset F\cup\{a\}, E'=E-\{a\}, F'=F\cup\{a\}; \\ \lambda_a & \text{if } a\in E, I(\beta(a))\subset F\cup\{a\}, E'=(E-\{a\})\cup O(\beta(a)), \\ & F'=F-I(\beta(a)); \\ -\sum_{a\in E}\lambda_a & \text{if } E'=E, F'=F \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

(See Kulkarni and Adlakha [3] for the details of proof).

It can be concluded that $\{X(t),t\geq 0\}$ is a finite-state absorbing continuous-time Markov process with single absorbing state (ϕ,ϕ), since $q\{(\phi,\phi),(\phi,\phi)\}=0$. It can also be concluded that all sates in S are transient and moreover that it is possible to number the states in $\overline{S}$ so that under this order the Q matrix is upper triangular. We assume that the states are numbered 1, 2,

…, N = $|\bar{S}|$. State 1 is the initial state, namely, $(O(v_1),\phi)$; and state N is the absorbing state, namely (ϕ,ϕ).

Let T represent the length of the longest path in the network, or the project completion time. It is clear that T = min {t > 0: X(t) = N/X(0) = 1}. Thus T is the time until {X(t), t$\geq$0} gets absorbed in the final state starting from state 1.

Before proceeding, we illustrate the material by an example. Consider the network shown in Figure 1. Table 1 presents the state space for this network. We use a superscript star to denote a dormant activity. All others are active.

Chapman-Kolmogorov backward or forward equations can be applied to compute F(t)=P{T$\leq$t} or the distribution function of project completion time in the Markov PERT network. Using the backward algorithm, we define:

$$P_i(t) = P\{X(t) = N/X(0) = i\} \quad i = 1, 2, \ldots, N \tag{8}$$

Therefore, $F(t) = P_1(t)$.

The system of differential equations for the vector $P(t) = [P_1(t), P_2(t), \ldots, P_N(t)]^T$ is given by:

$$P'(t) = Q.P(t)$$

$$P(0) = [0,0,\ldots,1]^T \tag{9}$$

where P(t) represents the state vector of the system and Q is the infinitesimal generator matrix of the stochastic process {X(t), t $\geq$ 0}. By taking advantage of the upper triangular nature of Q, the differential Equations 9 can be easily solved.

Now, let explain how the system of differential equations with constant coefficients 9 is solved. Let M be the modal matrix of Q. That is, M is the N$\times$N matrix whose N columns are the eigenvectors of Q. Let $\lambda(1),\lambda(2),\ldots,\lambda(N)$ be the eigenvalues of Q, which are the diagonal elements of Q owning to its upper triangular nature. We can compute P(t), and finally $F(t)=P_1(t)$ from Equation 10.

$$P(t) = M\, e^{\Lambda t}\, M^{-1}P(0) \tag{10}$$

where $e^{\Lambda t}$ is the diagonal matrix with *i*th diagonal element $1+\lambda(i)t+\ldots$ (see Luenberger [30] for the details), in this form

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda(1)t} & 0 & . & 0 \\ 0 & e^{\lambda(2)t} & . & . \\ . & . & . & . \\ 0 & . & . & e^{\lambda(N)t} \end{bmatrix} \tag{11}$$

3. BICRITERIA OPTIMAL CONTROL PROBLEM

In this section, we develop an analytical model to optimally control the resources allocated to the activities in the Markov PERT networks. It is assumed that the activity durations are independent random variables with either exponential or Erlang distributions, in which the mean of each activity duration is a non-increasing function of the amount of resource allocated to it. We may decrease the total direct costs of the project, by decreasing the resources allocated to the activities. However, clearly it causes the mean project completion time to be increased. Consequently, an appropriate trade-off between the total direct costs, and the project completion time is required.

To achieve the above-mentioned goals, we develop a bicriteria optimal control model, in which the first objective is the minimization of the total direct costs, and the second objective is the minimization of the mean project completion time. We use the goal attainment method, which is one of the multi objective techniques with priori articulation of preference information given, to solve the problem and obtain the optimal values of the resources allocated to the activities after discretizing this continuous-time problem and converting that one to a related bicriteria nonlinear programming.

The mean of activity duration $a \in A$, which is equal to n_a/λ_a (n_a=1 for exponential and n_a>1 for Erlang distribution), is assumed to be a non-increasing function $g_a(x_a)$ of the amount of resource x_a allocated to it. Let U_a represent the amount of resource available to allocate to the activity a, and L_a represent the minimum amount of resource required to achieve the activity a. It is also assumed that the mean of activity duration a cannot be smaller than a specific value c_a, even if we allocate it the maximum amount of resource or U_a.

The direct cost of activity $a \in A$ is assumed to be a non-decreasing function $d_a(x_a)$ of the amount of resource x_a allocated to it. Therefore, the total direct costs of the project would be equal to $\sum_{a \in A} d_a(x_a)$. The mean of project completion time would be equal to:

$$E(T) = \int_0^\infty (1 - P_1(t))dt \tag{12}$$

where $P_1(t)$ is computed from Equation 9.

Taking into account the above assumptions, the infinitesimal generator matrix, Q, is not constant, but it is a function of the vector $\lambda = [\lambda_1, \lambda_2, ..., \lambda_n]^T$. Therefore, the system of differential equations for the vector $P(t)=[P_1(t),P_2(t),\ldots,P_N(t)]^T$ is given by:

$$P'(t) = Q(\lambda).P(t)$$
$$P_i(0) = 0 \quad i = 1,2,\ldots,N-1 \tag{13}$$
$$P_N(t) = 1$$

Theorem 1 Assuming $X = [x_1, x_2, ..., x_n]^T$ as the control vector, the appropriate bicriteria optimal control problem would be

$$\text{Min } f_1(X,\lambda)=\sum_{a \in A} d_a(x_a) \tag{14a}$$

$$\text{Min } f_2(X,\lambda)=\int_0^\infty (1 - P_1(t))dt \tag{14b}$$

s.t.

$$P'(t) = Q(\lambda).P(t) \tag{14c}$$

$$P_i(0) = 0 \quad i = 1, 2, \ldots, N-1 \tag{14d}$$

$$P_N(t) = 1 \tag{14e}$$

$$\lambda_a \le {n_a}/{c_a} \qquad a \in A \tag{14f}$$

$$\lambda_a \le {n_a}/{g_a(x_a)} \qquad a \in A \tag{14g}$$

$$x_a \le U_a \qquad a \in A \tag{14h}$$

$$x_a \ge L_a \qquad a \in A \tag{14i}$$

$$P(t),\ \lambda \ge 0 \tag{14j}$$

Proof The objective Function 14a is the sum of direct costs of all activities, or the total direct costs of the project. The objective Function 14b is the mean of project completion time, as indicated. The constraints 14c, 14d and 14e determine the dynamic of this continuous-time system. Taking into account the constraints 14f and 14g, If we allocate x_a to the activity $a \in A$, and $g_a(x_a)$ does not become smaller than c_a, then λ_a gets its maximum value and becomes equal to ${n_a}/{g_a(x_a)}$, or the mean of activity duration a becomes equal to $g_a(x_a)$, because the second objective, or the mean project completion time, will be satisfied, and if $g_a(x_a)$ becomes smaller than c_a, then because of the same reason, λ_a gets its maximum value and becomes equal to ${n_a}/{c_a}$, or the mean of activity duration a becomes equal to c_a. Constraints 14h result from the capacity limitation on the resources, and the minimum amount of resource required to achieve each activity is insured by constraints 14i.

This continuous-time problem is so complicated to solve by analytical methods, and therefore we try to solve it numerically. To do that, we discretize this continuous-time system and convert the optimal control problem into an equivalent nonlinear programming. In other words, we transform the differential equations to the equivalent difference equations as well as transform the integral term into equivalent summation term. To follow this approach, the time interval is divided into K equal portions with length Δt. If Δt is sufficiently small, it can be assumed that P(t) varies only in times 0, Δt, …, (K-1)Δt. It should be noticed that the accuracy of the discrete-time model is guaranteed by using a small value for Δt, and a great value for K. Assuming $P(k\Delta t)$ or the kth value of P as P(k), the appropriate bicriteria nonlinear programming can be obtained from Corollary 1.

Corollary 1 The appropriate bicriteria nonlinear programming would be

$$\text{Min } f_1(X,\lambda) = \sum_{a\in A} d_a(x_a) \quad (15a)$$

$$\text{Min } f_2(X,\lambda) = \sum_{k=0}^{K} \left(1 - P_1(k)\right)\Delta t \quad (15b)$$

s.t.

$$P(k+1) = P(k)+Q(\lambda).P(k)\,\Delta t \quad k = 0, 1, .., K-1 \quad (15c)$$

$$P_i(0) = 0 \quad i = 1,2,\ldots,N-1 \quad (15d)$$

$$P_N(k) = 1 \quad k = 0,1,\ldots,K \quad (15e)$$

$$\lambda_a \le {n_a}/{c_a} \quad a \in A \quad (15f)$$

$$\lambda_a \le {n_a}/{g_a(x_a)} \quad a \in A \quad (15g)$$

$$x_a \le U_a \quad a \in A \quad (15h)$$

$$x_a \ge L_a \quad a \in A \quad (15i)$$

$$P_i(k) \le 1 \quad i = 1,2,\ldots,N-1 \text{ , } k = 1,2,\ldots,K \quad (15j)$$

$$P(k),\ \lambda \ge 0 \quad (15k)$$

Proof The integral term in 14b, is easily transformed into the equivalent summation term 15b, by considering P(k) instead of $P(k\Delta t)$. The differential Equations 14c, 14d and 14e are also easily transformed into the equivalent difference Equations 15c, 15d and 15e, with the same reason. Since each $P_i(k)$, for i = 1, 2,…,N-1 , k = 1,2,…,K is a distribution function, then the constraints 15j should be also included in this nonlinear programming.

For solving this multiple objective programming, we use the goal attainment method, which is one of the multi objective techniques with priori articulation of preference information given. This method is a variation of the goal programming. The method requires a goal vector, b, and a vector of weight w relating to the relative under attainment of the desired goals. The smaller weighting coefficient is associated with the more important objectives.

Let b_1 and b_2 represent the goals for the total direct costs of the project, and the mean of project completion time. Let w_1 and w_2 represent the weights relating the under attainments of these desired goals, in which $\sum_{i=1}^{2} w_i = 1$ Now, Assuming z as a scalar variable unrestricted in sign, the appropriate mathematical formulation of the problem can be obtained from Corollary 2.

Corollary 2 The appropriate nonlinear programming would be

$$\text{Min } z \quad (16a)$$

s.t.

$$P(k+1) = P(k)+Q(\lambda).P(k)\,\Delta t \quad k = 0,1,\ldots,K-1 \quad (16b)$$

$$P_i(0) = 0 \quad i = 1,2,\ldots,N-1 \quad (16c)$$

$$P_N(k) = 1 \quad k = 0,1,\ldots,K \quad (16d)$$

$$\lambda_a \le {n_a}/{c_a} \quad a \in A \quad (16e)$$

$$\lambda_a \le {n_a}/{g_a(x_a)} \quad a \in A \quad (16f)$$

$$x_a \le U_a \quad a \in A \quad (16g)$$

$$x_a \ge L_a \quad a \in A \quad (16h)$$

$$P_i(k) \le 1 \quad i = 1,2,\ldots,N-1 \text{ , } k = 1,2,\ldots,K \quad (16i)$$

$$\sum_{a\in A} d_a(x_a) - w_1 z \le b_1 \quad (16j)$$

$$\sum_{k=0}^{K} \left(1 - P_1(k)\right)\Delta t - w_2 z \le b_2 \quad (16k)$$

$$P(k),\ \lambda \ge 0 \quad (16l)$$

TABLE 2. Characteristics of the Activities of Case I.

a	$d_a(x_a)$	$g_a(x_a)$	c_a	L_a	U_a
1	$3x_1+2$	$24-5x_1$	5	1	4
2	$2x_2+1$	$20-3x_2$	4	1	6
3	x_3+3	$15-2x_3$	3	1	7
4	x_4^2+2	$10-x_4$	2	1	9

TABLE 3. Matrix Q(λ) Corresponding to Case I.

State	1	2	3	4	5	6	7
1	$-\lambda_1$	λ_1	0	0	0	0	0
2	0	$-(\lambda_2+\lambda_3)$	λ_2	λ_3	0	0	0
3	0	0	$-\lambda_3$	0	λ_3	0	0
4	0	0	0	$-(\lambda_2+\lambda_4)$	λ_2	λ_4	0
5	0	0	0	0	$-\lambda_4$	0	λ_4
6	0	0	0	0	0	$-\lambda_2$	λ_2
7	0	0	0	0	0	0	0

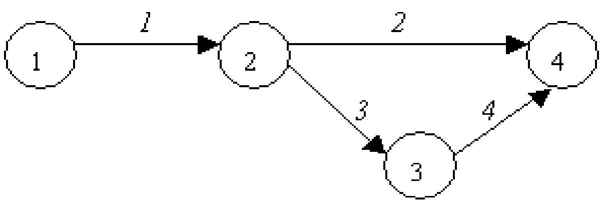

Figure 2. PERT network of Case I.

Proof Taking into account the desired goals for the total direct costs of the project and the mean of project completion time, and also the weights relating the under attainment of these desired goals, by minimizing z, the both objectives are satisfied.

If we consider the constraints 16c and 16d implicitly, and replace each $P_i(0)$ with zero and each $P_N(k)$ with one in another constraints, the nonlinear programming 16 would have 3K(N-1)+5n+2 constraints including the constraints 16l, and 3K(N-1)+7n+3 variables including the slack variables.

4. NUMERICAL EXAMPLES

For showing the numerical stability of the theoretical developments of the paper, we solve 3 numerical examples. Case I is depicted in Figure 2. Table 2 shows the characteristics of the activities. In this case, all activity durations are exponentially distributed random variables. We want to obtain the optimal values of the resources allocated to the activities in this time-cost trade-off problem.

$\overline{S} = \{(1),(2,3),(2^*,3),(2,4),(2^*,4),(2,4^*),(\phi,\phi)\}$.

Table 3 shows matrix Q(λ).

The appropriate nonlinear programming model for obtaining the optimal value of the control vector $X=[x_1,x_2,x_3,x_4]^T$ would be

Min z
s.t.

$P_1(k+1) = P_1(k) - \lambda_1 P_1(k)\Delta t + \lambda_1 P_2(k)\Delta t$;
k = 0, 1, …, K-1

$P_2(k+1) = P_2(k) - \lambda_2 P_2(k)\Delta t - \lambda_3 P_2(k)\Delta t + \lambda_2 P_3(k)\Delta t + \lambda_3 P_4(k)\Delta t$;
k = 0,1,…,K-1

$P_3(k+1) = P_3(k) - \lambda_3 P_3(k)\Delta t + \lambda_3 P_5(k)\Delta t$;
k=0,1,…,K-1

TABLE 4. Optimal values of x_a and λ_a for a=1,2,3,4 Corresponding to Case I.

a	1	2	3	4
x_a	3.8	1	4.441	1
λ_a	0.2	0.059	0.163	0.111

TABLE 5. $P_1(k\Delta t)$ for k=1,2,…,10 Corresponding to Case I.

k	0	1	2	3	4	5	6	7	8	9	10
$P_1(k)$	0	0	0	0	0.401	0.633	0.771	0.852	0.902	0.934	0.954

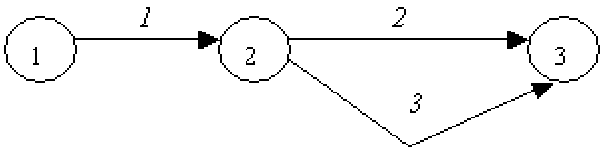

Figure 3. PERT network of Case II.

$P_4(k+1) = P_4(k)-\lambda_2 P_4(k)\Delta t -$
$\lambda_4 P_4(k)\Delta t+\lambda_2 P_5(k)\Delta t+\lambda_4 P_6(k)\Delta t$;
k = 0,1,…,K-1

$P_5(k+1) = P_5(k)-\lambda_4 P_5(k)\Delta t+\lambda_4 \Delta t$;
k = 0,1,…,K-1

$P_6(k+1) = P_6(k)-\lambda_2 P_6(k)\Delta t+\lambda_2 \Delta t$;
k = 0,1,…,K-1

$P_i(0) = 0$ i = 1,2,…,6

$\lambda_a \le n_a / c_a$ a = 1,2,3,4

$\lambda_a \le n_a / g_a(x_a)$ a = 1,2,3,4

$x_a \le U_a$ a = 1,2,3,4

$x_a \ge L_a$ a = 1,2,3,4

$P_i(k) \le 1$ i=1,2,…,6 , k=1,2,…,K

$\sum_{a=1}^{4} d_a(x_a) - w_1 z \le b_1$

$\sum_{k=0}^{K} (1-P_1(k))\Delta t - w_2 z \le b_2$

$P_i(k) \ge 0$ i = 1,2,…,6 , k = 1,2,…,K

$\lambda_a \ge 0$ a = 1,2,3,4

(17)

In nonlinear model 17, $P_N(k) = P_7(k) = 1$ for k = 0, 1, …, K, and we replaced $P_7(k)$ with 1 in this model. It is also clear that $n_a = 1$ for a = 1, 2, 3, 4. We set the goals for the total direct costs of the project and the mean of project completion time as b_1 = 15 and b_2 = 10, respectively. The values of other parameters are w_1 = 0.4, w_2 = 0.6, K = 10 and Δt = 5.

We use LINGO to solve nonlinear programming 17. Table 4 shows the optimal value of the amount of resource allocated to each activity a, or x_a for a = 1, 2, 3, 4, and also the optimal values of λ_a for a = 1, 2, 3, 4. Table 5 shows the distribution function of project completion time, or $P_1(k\Delta t)$, for k = 1, 2, …, 10.

Finally, the optimal values of z or the objective function, $f_1(X,\lambda)$ or the total direct costs of the project, and $f_2(X,\lambda)$ or the mean project completion time are obtained as follows:

$$z = 29.602$$
$$f_1(X,\lambda) = 26.841 \quad (18)$$
$$f_2(X,\lambda) = 27.761$$

In Case II, which is depicted in Figure 3, we test

TABLE 6. Characteristics of the Activities of Case II.

a	Distribution	Parameters	$d_a(x_a)$	$g_a(x_a)$	c_a	L_a	U_a
1	Exponential	λ_1	$3x_1+2$	$24-5x_1$	5	1	4
2	Exponential	λ_2	$2x_2+1$	$20-3x_2$	4	1	6
3	Erlang	$(\lambda_3, n_3=2)$	x_3^2+2	$15-2x_3$	3	1	7

TABLE 7. Optimal values of x_a for a=1,2,3 corresponding to Case II.

a	1	2	3
x_a	3.728	1	1

our model by a more general PERT network with Erlang distributions of activity durations. Table 6 shows the characteristics of the activities.

In this case, after transforming the network into an equivalent network with exponentially distributed activity durations and obtaining matrix $Q(\lambda)$, we construct the appropriate nonlinear programming in order to obtain the optimal resources allocated to the activities. It is assumed that $b_1=12$, $b_2 = 10$, $w_1 = 0.4$, $w_2 = 0.6$, $K = 10$ and $\Delta t = 5$, respectively. Table 7 shows the optimal values of x_a for a = 1, 2, 3.

The minimum values of the total direct costs of the project and the mean project completion time, in this case, are obtained as follows:

$$f_1(X, \lambda) = 19.185$$

$$f_2(X, \lambda) = 20.777 \tag{19}$$

In Case III, which is depicted in Figure 4, we test our model by a relatively large-scale system with 10 activities. Table 8 shows the characteristics of the activities. In this case, all activity durations are exponentially distributed random variables.

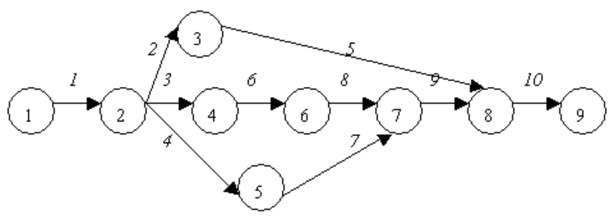

Figure 4. PERT network of Case III.

In this case, after obtaining matrix $Q(\lambda)$, we construct the appropriate nonlinear programming in order to obtain the optimal amount of resource allocated to the activities. It is assumed that $b_1=70$, $b_2 = 2.5$, $w_1 = 0.9$, $w_2 = 0.1$, $K = 10$ and $\Delta t = 0.5$, respectively. Table 9 shows the optimal values of x_a for a = 1, 2, …, 10. The minimum values of the total direct costs of the project and the mean project completion time, in this PERT network, are obtained as follows:

$$f_1(X, \lambda) = 114.853$$

$$f_2(X, \lambda) = 4.832 \tag{20}$$

5. CONCLUSION

In this paper, we developed a new bicriteria model for the time-cost trade-off problem via optimal control theory in Markov PERT networks, in which the first criterion or the total direct costs of the project, and also the second criterion or the mean project completion time are both analytically minimized. Our method can be applied for the management and control of the projects with the stochastic natures.

TABLE 8. Characteristics of the Activities of Case III.

a	$d_a(x_a)$	$g_a(x_a)$	c_a	L_a	U_a
1	$2x_1$	$0.7-0.1x_1$	0	1	5
2	$3x_2+1$	$1.5-0.2x_2$	0	1	6
3	x_3+2	$1-0.1x_3$	0	1	9
4	x_4	$1.5-0.3x_4$	0	1	4
5	$3x_5+4$	$1.3-0.2x_5$	0	1	5
6	x_6+3	$1.1-0.1x_6$	0	1	6
7	$2x_7+5$	$1.5-0.2x_7$	0	1	7
8	$4x_8+1$	$1-0.2x_8$	0	1	4
9	$5x_9+2$	$2-0.4x_9$	0	1	4
10	$2x_{10}+3$	$2.25-0.25x_{10}$	0	1	5

TABLE 9. Optimal values of x_a for a = 1, 2, …, 10 corresponding to Case III.

a	1	2	3	4	5	6	7	8	9	10
x_a	4.969	5.953	1	1	4.953	1	5.418	3.937	3.922	1

It was assumed that the mean of each activity duration is a non-increasing function, and the direct cost of each activity is a non-decreasing function of the resources allocated to this activity, or the controller of the problem. In application, these functions can be estimated using linear regression.

The corresponding continuous problem was so complicated to solve analytically. Therefore, we solved it numerically, by discretizing the relevant continuous-time system and converting the optimal control problem into an equivalent nonlinear programming.

To solve the relevant multiple objective programming, we used the goal attainment method, which is a variation of the goal programming technique. Goal attainment method is one of the multi objective techniques with priori articulation of preference information given. This method has the same disadvantages as those of goal programming; namely, the preferred solution is sensitive to the goal vector and the weighting vector given by the decision maker. However, the goal attainment method has fewer variables to work with, so it will be computationally faster, and therefore is a good method to solve our problem, which is complicated to solve even with the presented numerical method.

We could also use an interactive method like surrogate worth trade-off method, method of satisfactory goals or SEMOPS method for solving this bicriteria problem.

In the case of general PERT networks, we can approximate the general distributions of activity durations by the appropriate Erlang distributions, which is a special class of PH distribution, by matching the first two moments, and then use our proposed method to obtain the optimal resources allocated to the activities.

The limitation of this model is that the number of variables and constraints of the nonlinear programming 16 can grow exponentially with the network size. As the worst-case example, consider a complete directed acyclic network with n nodes and $n(n-1)/2$ arcs. The size of the state space for this network is given by $N(n) = U_n - U_{n-1}$, where

$$U_n = \sum_{k=0}^{n} 2^{k(n-k)} \tag{21}$$

Refer to Kulkarni and Adlakha [3].

Consequently in this case, the number of constraints of the nonlinear model would be equal to 3K(N(n)-1) + 2.5n(n-1) + 2, and the number of variables would be 3K(N(n)-1) + 3.5n(n-1) + 3. Therefore, the number of constraints and variables of the nonlinear programming 16 grows exponentially with n. In practice, the number of arcs of the PERT networks is generally much less than $n(n-1)/2$. Even large sparse networks generally produce a reasonable size state space.

Our model can be extended in the following directions:

1. The model can include the other objective functions like the total indirect costs of the project.
2. The model can be extended to the general PERT networks.
3. The control variable, or the amount of resource allocated to the activity, can be considered as a function of time.

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