A SOLUTION OF RICCATI NONLINEAR DIFFERENTIAL EQUATION USING ENHANCED HOMOTOPY PERTURBATION METHOD (EHPM)

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Abstract Homotopy Perturbation Method is an effective method to find a solution of a nonlinear differential equation, subjected to a set of boundary condition. In this method a nonlinear and complex differential equation is transformed to series of linear and nonlinear and almost simpler differential equations. These set of equations are then solved secularly. Finally a linear combination of the solutions completes the answer if the convergence is maintained. HPM based solution incorporates some reasonable assumptions. These are inspired from the boundary condition and a separation mechanism. In this paper, the need for stability verification is shown trough some examples. The novel idea is to keep the inherent stability of nonlinear dynamic in whole term, even if the selected linear part is not stable. Consequently, HPM is enhanced by a preliminary assumption. The proposed method is applied to Riccati equation as well as some other problems. The simulation result verifies the significance of the method whilst numerical and the exact solutions confirm the achievement.

Keywords Homotopy Perturbation Method, Secular Terms, Nonlinear Dynamic, Nonlinear Differential Equation, Enhanced HPM (EHPM), Optimal Control

چکیده روش هموتوپی پرتوربیشن یکی از روش های موثر در حل معادلات دیفرانسیلی است. در این روش ابتدا یک معادله دیفرانسیل غیر خطی پیچیده به یک سری معادلات دیفرانسیل خطی ساده تر تبدیل می‌شود. این دسته از معادلات دیفرانسیل بصورت زنجیره ای حل شده تا اینکه در نهایت مجموعه جواب های بدست آمده، به پاسخ معادله دیفرانسیل غیر خطی اولیه همگرا می‌شود. در این مقاله، با چند مثال اهمیت بررسی پایداری نشان داده می‌شود. ایده جدید، همگرایی جواب معادله بدست آمده از روش HPM به جواب واقعی معادله را حتی با ناپایدار بودن قسمت خطی حفظ می‌نماید. این روش پیشنهادی همانند مسائل مشابه بر روی معادله ریکاتی پیاده سازی شده است. کیفیت روش ارائه شده در همگرائی جواب، از طریق شبیه سازی عددی و مقایسه با حل دقیق آن اثبات شده است.

1. INTRODUCTION

In the last two decades with the rapid development of nonlinear dynamics, there has appeared an ever-increasing interest of scientists and engineers in the analytical techniques for nonlinear problems. The widely applied perturbation technique has been of interest to be used in control systems [1,2]. To eliminate the limitation of "small parameter" assumption a new technique, based on homotopy in terminology, was proposed [3-5]. According to this method, a nonlinear problem is transformed into an infinite number of simple problems without using the perturbation techniques. Effectively, letting the small parameter float and converge to the unity, the problem will be converted into a special perturbation problem. This method was given a name; the Homotopy Perturbation Method (HPM). The effectiveness of the new technique presented [6,7]. This method can take full advantage of the traditional perturbation methods and homotopy analysis method. It has successfully been applied to linear, nonlinear ordinary and partial differential equations, which almost describing a system dynamic incorporating the perturbation value (Called Homotopy Perturbation Method, i.e. HPM) [3,8,9]. It has also been applied to a wide class of differential equations, such as:

Duffing equation in [6], the area of numerical and algebraic methods [10-12], Gelfand equation [13], autonomous systems [1,14,15], system dynamic [16] and Heat transfer [9,17-19]. However, Homotopy Perturbation Method (HPM) is an asymptotical based Method. The method relies on some assumptions which is ignoring them makes the solution unreliable [20]. The need for the stability verification has been illustrated through some examples. HPM is then subjected to keep the stability of nonlinear dynamics. This method is investigated and restricted to such a situation to maintain the inherent convergence of the main problem. It will be done by keeping the stability of the linear part by adding an extra term to it. The fundamental idea will be introduced through some examples. The idea is to keep the original stability situation as an important criterion to ensure the global convergence.

Let us introduce HPM via following nonlinear equation:

$$A(x) - f(r) = 0\ ,\ \ u \in R^n \tag{1}$$

Subjected to boundary conditions:

$$B(x, \partial x/\partial t) = 0\ ,\ \ x \in R^n \tag{2}$$

Where A is a general differential operator, B is a boundary operator, x is known as analytic and n dimensional function (here, state) and u is an m dimensional input (independent variable). The differential part A(x) can be generally divided into two linear L, and nonlinear N, parts. Equation 1 can therefore, be rewritten as:

$$L(x) + N(x) - f(r) = 0 \tag{3}$$

A homotopy statement H(ν,p) by using an auxiliary variable ν(x,p) with $p \in [0,1]$ can be defined as:

$$\begin{aligned} &H(v,p) = (1-p)[L(v) - L(x_0)] + \\ &p[A(v) - f(r)] = 0\ ,\ p \in [0,1] \end{aligned} \tag{4}$$

P is called homotopy parameter (which is inspired from the “small parameter” in perturbation terminology). The idea behind using a small parameter p is smart. By p equals 0.0, Equation 4 is being completely linear whereas p equals to 1.0 the linear part in Equation 4 completely vanishes and Equation 4 will become the same as Equation 1. With a simple manipulation Equation 4 is reduced to the following Equation 5:

$$\begin{aligned} &H(v,p) = L(v) - L(x_0) + pL(x_0) + \\ &p[N(v) - f(r)] = 0\ ,\ p \in [0,1] \end{aligned} \tag{5}$$

The initial guess x_0 (u_0 in the literature) needs to be a good initial approximation for the solution of Equation 1. Where $p \in [0,1]$ is an embedding parameter, x_0 is an initial guess approximation of Equation 1, which satisfies the boundary conditions.

$$\begin{aligned} &H(v,0) = L(v) - L(x_0) = 0 \\ &H(v,1) = A(v) - f(r) = 0 \end{aligned}$$

However, in the system area, it is a property of the system and can be meaningfully assigned. We use the embedding parameter p as a small parameter and assume that the solution of Equation 4 can be written as a power series in p:

$$v = p^0 v_0 + p^1 v_1 + p^2 v_2 + \cdots \tag{6}$$

By substituting (6) in (5) and rearranging the function in terms of ascending powers of P, an infinite number of differential equations in terms of ν, is achieved. A special attention must be given to avoid the secular terms to produce boundedness [1]. This set of almost simple differential equation with proper initial conditions is then solved. Finally an approximate solution of (1) can be written as:

$$x = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \tag{7}$$

The convergence of HPM is discussed in the literature [7,20,28]. Due to practical restrictions, a shorter term of v_i , i = 0, 1, 2, … is of interest. So the accuracy of the solution and therefore the rate of convergence is directly reduced. Among v_i , i = 0,1,2,… functions, and due to secular property of series in (6), the function v_0, which is obtained from p^0 term in Equation 6, affects all the other v_i , i = 0,1,2,… functions. It means that the possible discrepancy is additive. However, a way of treatment and the source of an error will be clarified through some case studies.

Case 1. The heat transfer equation with radiation [17,25], to describe the capability of the HPM, consider the following nonlinear differential equation:

$$\frac{dy}{dt} + y + y^4 = 0 \ , \ y(0) = 0.5 \qquad (8)$$

The linear and nonlinear parts are chosen respectively as:

$$L(y) = \dot{y} + y \quad , \quad N(y) = y^4 \qquad (9)$$

Then the Homotopy statement is built as follows:

$$H(v,p) = \dot{v} + v - L(y_0) + pL(y_0) + p[v^4] \qquad (10)$$

By substituting $v = p^0 v_0 + p^1 v_1 + p^2 v_2 + \cdots$ in (10) and assuming $y_0 = 0$, leads us to:

$$\begin{aligned} &(p^0 v_0 + p^1 v_1 + p^2 v_2 + ...) + \\ &(p^0 v_0 + p^1 v_1 + p^2 v_2 + ...) + \\ &p[(p^0 v_0 + p^1 v_1 + p^2 v_2 + ...)^4] = 0, v(0) = 0.5 \end{aligned} \qquad (11)$$

Rearranging the equation in terms of powers op p, the following equations are achieved:

$$p^0 : \dot{v}_0 + v_0 = 0 \ , \ v_0(0) = 0.5 \qquad (12)$$

$$p^1 : \dot{v}_1 + v_1 + v_0^4 = 0 \ , \ v_1(0) = 0 \qquad (13)$$

$$p^2 : \dot{v}_2 + v_2 + 4v_1 v_0^3 = 0 \ , \ v_2(0) = 0 \qquad (14)$$

and so forth. By solving the simple equations in 12 to 14, we have the solution as:

$$\begin{aligned} &v_0 = 0.5e^{-t} \\ &v_1 = \frac{1}{48}(e^{-3t} - 1)e^{-t} \\ &v_2 = \frac{1}{576}(e^{-6t} - 2e^{-3t} + 1)e^{-t} \end{aligned} \qquad (15)$$

In order to have a satisfactory result, a 3rd order Homotopy is chosen. Therefore:

$$y = \lim_{p \to 1} (v_0 + pv_1 + p^2 v_2) \qquad (16)$$

This leads to:

$$\begin{aligned} &y(t) = \frac{1}{2}e^{-t} + \frac{1}{48}(e^{-4t} - e^{-t}) + \\ &\frac{1}{576}(e^{-7t} - 2e^{-4t} + e^{-t}) \end{aligned} \qquad (17)$$

However, to assess the efficiency of HPM, the result is compared with the numerical method presented in Figure 1. This graph shows that the two graphs are similar and the error is not significant. The initial conditions and partitioning of Equation 1 into two parts as Equation 3 have met the requirements. The secular property of the function $v_0 = e^{-t}$ in Equations 12 to 14 is

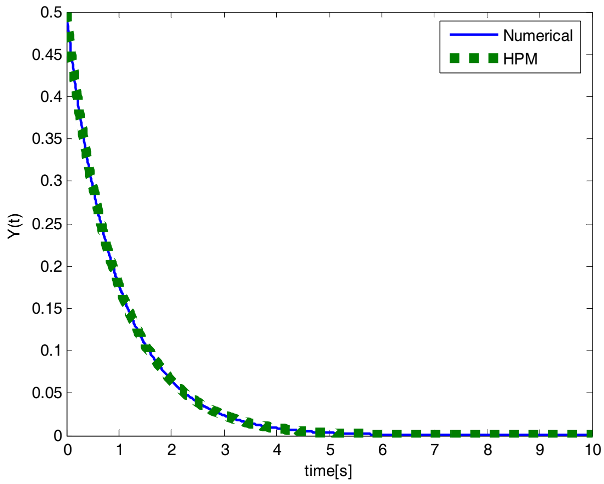

Figure 1. HPM and the numerical solutions of the nonlinear differential equation.

unavoidable. So instability of this term affects all of the rest of functions. Although this statement is true for the other v_i's, $i = 0,1,2,\cdots$, the role of function v_0 is more crucial. This is because it comes from the linear part and the operator may design it improperly. In other words, the operator has a degree of freedom to select the linear part such that a stable answer v_0 from the corresponding equation i.e. (12) is achieved. To highlight the significance of the method, consider another differential equation.

Case 2. To introduce the possible source of the error in the HPM, consider the following nonlinear differential equation. It should be noted that this equation is similar to Equation 8 but the sign of y in the linear term.

$$\frac{dy}{dt} - y + y^4 = 0 \text{ , } y(0) = 0.5 \tag{18}$$

In essence, this equation differs from the similar one by a negative sign on y(t).The linear and the nonlinear parts can be distinguished as:

$$L(y) = \dot{y} - y \quad , \quad N(y) = y^4 \tag{19}$$

Using a similar procedure and considering the same situation for y_0 as $y_0 = 0$ alters the homotopy expression to:

$$H(v,p) = \dot{v} - v - L(y_0) + pL(y_0) + p[v^4] \tag{20}$$

Again substituting $v = p^0 v_0 + p^1 v_1 + p^2 v_2 + \cdots$ in (19) yields:

$$\begin{aligned} &(p^0 v_0 + p^1 v_1 + p^2 v_2 + \ldots.) \\ &(p^0 v_0 + p^1 v_1 + p^2 v_2 + \ldots.) + \\ &p[(p^0 v_0 + p^1 v_1 + p^2 v_2 + \ldots.)^4] = 0, \ v(0) = 0.5 \end{aligned} \tag{21}$$

To equalize Equation 21 to zero, the statements including the ascending powers of p must be all together equal to zero. This means:

$$p^0 : \dot{v}_0 - v_0 = 0 \ , \ v_0(0) = 0.5 \tag{22}$$

$$p^1 : \dot{v}_1 - v_1 + v_0^4 = 0 \ , \ v_1(0) = 0 \tag{23}$$

$$p^2 : \dot{v}_2 - v_2 + 4 v_1 v_0^3 = 0 \ , \ v_2(0) = 0 \tag{24}$$

Solving Equations 22 to 24 by HPM yields:

$$\begin{aligned} &v_0 = 0.5 e^t \\ &v_1 = -\frac{1}{48}(e^{4t} - e^t) \\ &v_2 = \frac{1}{576}(e^{7t} - 2e^{4t} + e^t) \end{aligned} \tag{25}$$

When a 3rd order v is of interest as:

$$y_{HPM} = \lim_{p \to 1} (v_0 + p v_1 + p^2 v_2) \tag{26}$$

We achieve:

$$\begin{aligned} &y(t) = \frac{1}{2} e^t - \frac{1}{48}(e^{4t} - e^t) + \\ &\frac{1}{576}(e^{7t} - 2e^{4t} + e^t) \end{aligned} \tag{27}$$

The solution of functions Equation 17 and 27 are then plotted in Figure 2 in a shorter time (a typical

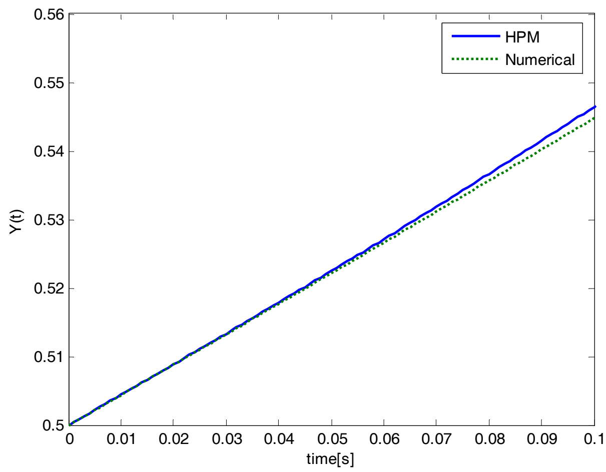

Figure 2. HPM and the numerical Solutions of the nonlinear differential equation, case no. 2 and in a short time.

time in some literature). As it can be seen, the discrepancy is small but growing. If someone increases the time of simulation, the error will also increase (Figure 3).

The cause of divergence is in the equation's nature, specifically on the stability property of the linear part in Equations 9 and 19. The L(y) in Equation 9 is stable, whereas in (19) is unstable.

An unstable linear part (it means the solutions of linear part have finite value in infinite time) e.g. L(y), produces a generative function v_0, which in return produces generative and unstable v_i function secularly. To achieve the convergence of the resultant equation with unstable linear part, a higher order v may cope with the error. This in turn, needs much work to extract the higher term for v_i. So it is not investigated but an alternative option is suggested. It should be noted that the stability criteria for ordinary differential equations (i.e. the linear systems) can easily be verified (for example by checking the roots of the characteristic polynomial). A way of treating the lack of HPM will be presented by modification of the linear part of the HPM. The method is explained through some illustrative examples.

2. EHPM (ENHANCED HOMOTOPY PERTURBATION METHOD)

Using HPM in some equations may lead to a different and unexpected answer. This problem specifically occurs when a stable nonlinear differential equation has an unstable linear part. This problem has to be treated to match the actual solution. A possible way of treatment will be briefly introduced. In this method, the stability of the nonlinear differential equation will be examined. If it is provided, the stability of the first seen linear part (the usual chosen) will be tested. It is hoped that the usual HPM leads to a satisfactory result (e.g. case 1). If not, there is no guarantee to have a well behaved answer (e.g. case 2). Hopefully a zero term (an extra term with zero effect in global i.e. $\eta(x) \in \Omega(x)$, space of linear functions with real coefficients) may cope with the problem. At this stage, an extra term will be added to the linear part and subtracted from the rest of the statement at the same time, such that a linear part is stabilized.

$$\underbrace{L(x)+\eta(x)}_{\text{new } L(v)}\underbrace{-\eta(x)+N(x)}_{\text{new } N(v)}-f(u)=0 \tag{28}$$

The rest of the statement establishes the nonlinear part i.e. N(v). This alteration and using HPM reduces the method to so called EHPM. This new method stabilizes v_0 and as a result, the whole solution. In order to interpret the procedure, let us again consider the Equation 18 to have further description of the proposed method. The usual linear and nonlinear part is again written for ease of referring.

$$L(y)=\dot{y}-y \ , \ N(y)=y^4 \tag{29}$$

By adding and subtraction 3y in Equation 17 we have:

$$\underbrace{\frac{dy}{dt}+3y}_{L(y)}\underbrace{-3y-y+y^4}_{N(y)}=0 \ , \ y(0)=0.5 \tag{30}$$

The new linear L(y) and N(y) parts are underlined respectively, as follows:

$$L(y)=\dot{y}+3y \quad , \quad N(y)=-4y+y^4 \tag{31}$$

According to Equation 5 the homotopy function can be written as:

$$H(v,p)=\dot{v}+3v-L(y_0)+pL(y_0)+p[v^4-4v] \tag{32}$$

Assuming $y_0 = 0$ as an initial condition and by substituting v as

$v=p^0v_0+p^1v_1+p^2v_2+\cdots$ in Equation 32, leads us to:

$$\begin{aligned}&(p^0v_0+p^1v_1+p^2v_2+....)+\\&3(p^0v_0+p^1v_1+p^2v_2+....)+\\&p[(p^0v_0+p^1v_1+p^2v_2+....)^4-\\&4(p^0v_0+p^1v_1+p^2v_2+....)]=0, v(0)=0.5\end{aligned} \tag{33}$$

The constant term is caused due to the effect of $y_0 = 1$. By rearranging the equations in terms of ascending powers of p, one obtains:

$$p^0 : \dot{v}_0 + 3v_0 - 3 = 0 \ , \ v_0(0) = 0.5 \tag{34}$$

$$p^1 : \dot{v}_1 + 3v_1 + 3 + (v_0^4 - 4v_0) = 0 \ , \ v_1(0) = 0 \tag{35}$$

$$p^2 : \dot{v}_2 + 3v_2 + (4v_1 v_0^3 - 4v_1) = 0 \ , \ v_2(0) = 0 \tag{36}$$

With the same situation, again a 3rd order response was considered. The solutions of Equations 34 to 36 are found as:

$$v_0 = 1 - 0.5e^{-3t}$$

$$v_1 = (\frac{1}{2}e^{-3t} - \frac{1}{12}e^{-6t} + \frac{1}{144}e^{-9t} - \frac{61}{144})e^{-3t} \tag{37}$$

$$v_2 = (-\frac{1}{864}e^{-12t} + \frac{61}{216}e^{-3t} - \frac{1}{6}e^{-6t} + \frac{1}{54}e^{-9t} - \frac{115}{864})e^{-3t}$$

and finally when p according to Equation 16 approaches the unity, the complete solution will be achieved. The graph of two HPM and EHPM based solutions are then plotted in the next figure (Figure 3) together with the numerical one.

It can be seen in Figure 4 that EHPM based response is more satisfactory with respect to HPM, assuming the numerical solution as an actual response. The convergence of EHPM solution is also shown while HPM one is not converging. The significance of the method is investigated through another example.

2.1. HPM Based Solution Consider the following differential equation as another case to interpret both capability of HPM and EHPM:

$$\frac{dy}{dt} + e^t y^2 - y = 0 \ , \ y(0) = 1 \tag{38}$$

The linear and nonlinear parts can be respectively chosen as:

$$L(y) = \dot{y} - y \ , \ N(y) = e^t y^2 \tag{39}$$

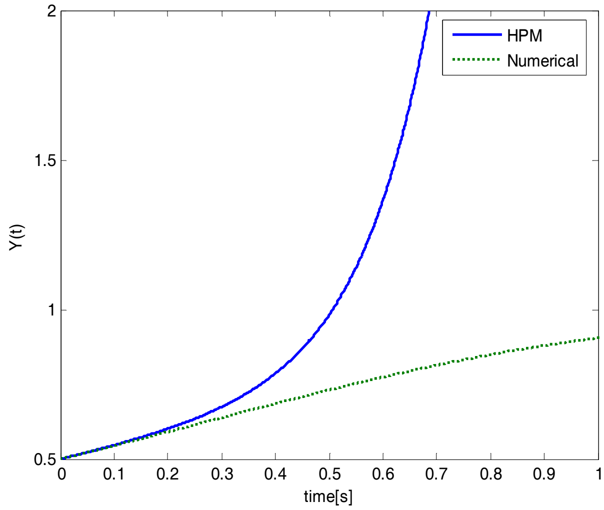

Figure 3. HPM and the numerical Solution of the nonlinear differential equation, case no. 2 in a longer time.

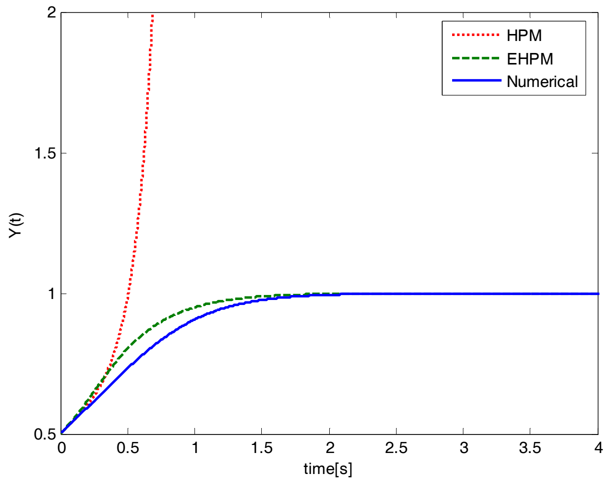

Figure 4. Comparison of EHPM, HPM and the numerical solutions.

In order to construct the homotopy statement in (5), HPM is applied. This leads us to:

$$H(v,p) = \dot{v} - v - L(y_0) + pL(y_0) + p[e^t v^2] \tag{40}$$

Considering $y_0 = 0$ as an initial condition together with assumption $v = p^0 v_0 + p^1 v_1 + p^2 v_2 + \cdots$, the following set of equations is yielded:

$$p^0 : \dot{v}_0 - v_0 = 0 \;,\; v_0(0) = 1 \quad (41)$$

$$p^1 : \dot{v}_1 - v_1 + (e^t v_0^2) = 0 \;,\; v_1(0) = 0 \quad (42)$$

$$p^2 : \dot{v}_2 - v_2 + 2e^t v_0 v_1 = 0 \;,\; v_2(0) = 0 \quad (43)$$

The solutions are:

$$v_0 = e^t$$

$$v_1 = \frac{1}{2}(e^t - e^{3t}) \quad (44)$$

$$v_2 = \frac{1}{4}(e^{5t} - 2e^{3t} + e^t)$$

Therefore a 3rd order answer of Equation 38 by using Equation 16 can simply be obtained by summation of v_i, $i = 1,2,3$. The dynamic behavior in two situations; the numerical solution and HPM based, are shown in Figure 5.

Since the linear part of Equation 38 is unstable, graphs in Figure 5 are completely erroneous. Thus EHPM as a powerful tool will be used.

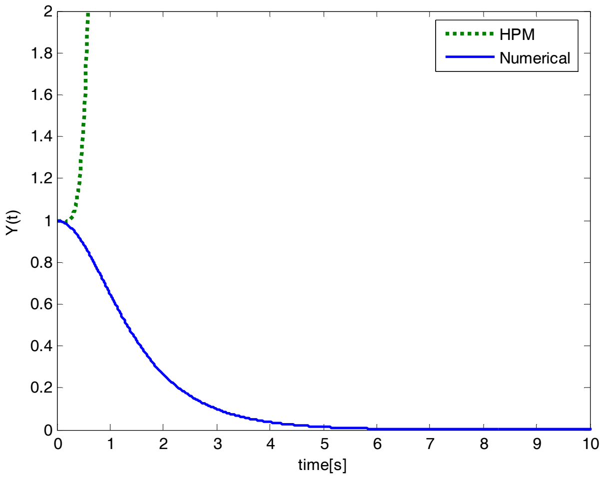

Figure 5. Comparison of HPM and the numerical simulation of Equation 38.

2.2. EHPM Based Solution Since nonlinear differential Equation 38 is stable whereas the linear term is not, EHPM is used. Adding and subtracting $\eta(x) = y$ in (38), we have:

$$\dot{y} + y + e^t y^2 - 2y = 0 \;,\; y(0) = 1 \quad (45)$$

Consequently $\dot{y} + y$ is chosen as linear part. Where as the rest of Equation 38 is chosen as the nonlinear section. As a result of applying HPM, the homotopy statement can be established as:

$$H(v,p) = \dot{v} + v - L(y_0) + pL(y_0) + p[e^t v^2 - 2v] \quad (46)$$

Considering $y_0 = 0$ as an initial condition together assumption $v = p^0 v_0 + p^1 v_1 + p^2 v_2 + \cdots$, yields the following set of equations:

$$p^0 : \dot{v}_0 + v_0 = 0 \;,\; v_0(0) = 1 \quad (47)$$

$$p^1 : \dot{v}_1 + v_1 + e^t v_0^2 - 2v_0 = 0 \;,\; v_1(0) = 0 \quad (48)$$

$$p^2 : \dot{v}_2 + v_2 + 2e^t v_0 v_1 - 2v_1 = 0 \;,\; v_2(0) = 0 \quad (49)$$

Those equations lead us to:

$$v_0 = e^{-t}$$

$$v_1 = te^{-t} \quad (50)$$

$$v_2 = 0$$

With the same situation and by choosing a 3rd order of approximation as an answer of Equation 38, the following results are achieved:

It can be seen from Figure 6 that both EHPM and the numerical results are quite similar; whereas HPM's counterpart completely differs.

2.3. HPM Based Riccati Equation Riccati equation has frequently been used in the engineering field e.g. in optimal control [21,22]. So many attempts have been made to solve this problem [23,24]. Perturbation technique, homotopy perturbation method are also efficient approach which are used for solving riccati equation [23]. It should be noted that improper use

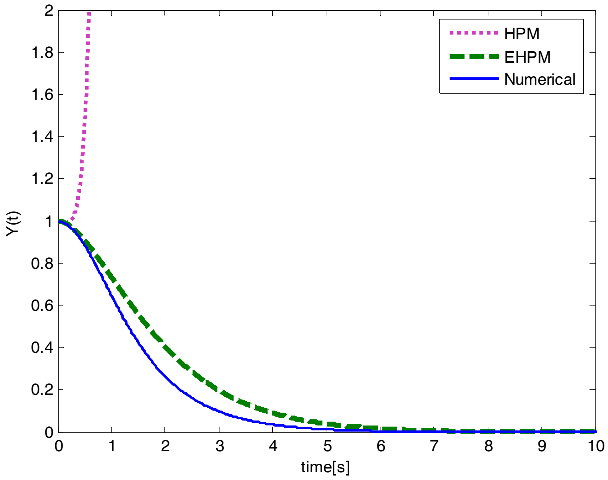

Figure 6. Comparison of HPM and the numerical simulation of Equation 38.

of HPM diverges the result. The reader is advised to have a tour on MATLAB software to find more information about riccati equation. An alternative solution is presented here. Consider the following riccati equation [23,24]:

$$\dot{y}-2y+y^2-1=0\ ,\ y(0)=0 \tag{51}$$

The linear and nonlinear parts can be respectively chosen as:

$$L(y)=\dot{y}-2y\ ,\ \ N(y)=y^2-1 \tag{52}$$

By using Equation 5 the Homotopy statement can be written as:

$$H(v,p)=\dot{v}-2v-L(y_0)+pL(y_0)+p[v^2-1] \tag{53}$$

Initial guess is chosen as $y_0 = 2.4$. By substituting v as $v=p^0v_0+p^1v_1+p^2v_2+\cdots$ in Equation 53 the following equation in terms of ascending power of p can be written as:

$$p^0:\dot{v}_0+v_0=0\ ,\ \ v_0(0)=0 \tag{54}$$

$$p^1:\dot{v}_1-2v_1-4.8+v_0^2-1=0\ ,\ v_1(0)=0 \tag{55}$$

$$p^2:\dot{v}_2-2v_2+2v_0v_1=0\ ,\ v_2(0)=0 \tag{56}$$

Therefore the solution can be found as:

$$v_0=\frac{12}{5}(1-e^{2t})$$

$$v_1=(-\frac{1}{50}e^{-2t}+\frac{288}{25}t-\frac{72}{25}e^{2t}+\frac{29}{10})e^{2t} \tag{57}$$

$$v_2=\frac{432}{125}e^{2t}-\frac{6}{125}(292t+e^{-2t}-576te^{2t}-e^{2t}+576t^2+72e^{4t})e^{2t}$$

When the limit in (7) takes place, a 3rd order answer will be achieved by the summation of v_i, as follows:

$$v=v_0+v_1+v_2 \tag{58}$$

The resultant is plotted together with the numerical solution in Figure 7, in a short time interval.

Since the time of representation is short, the discrepancy of graphs is not shown very well. However one might notice the growth of an error between two responses. This notification inspires to increase the time of simulation, as it is done in Figure 8.

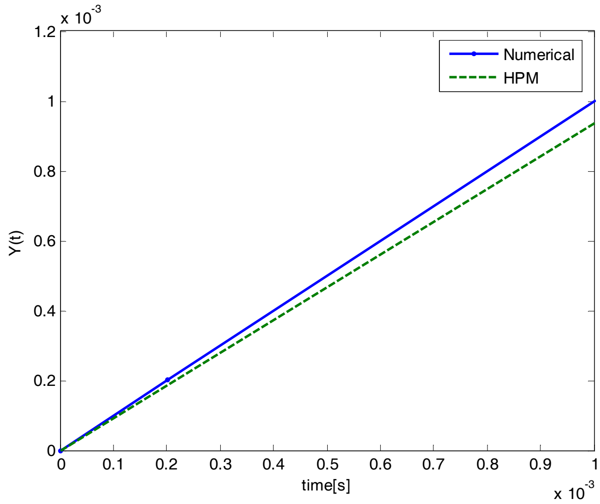

Figure 7. An HPM based solution of Riccati equation with respect of the numerical solution.

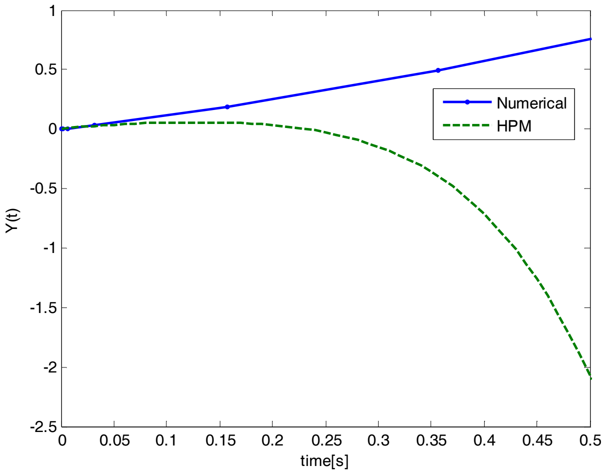

Figure 8. An HPM based solution of Riccati equation with respect of the numerical solution in a longer time with respect to Figure 7.

The divergence of HPM with respect to the real transient behavior even in 0.5 seconds emphasizes the use of EHPM.

3. EHPM (ENHANCED HOMOTOPY PERTURBATION METHOD)

Consider again the following riccati equation:

$$\dot{y} - 2y + y^2 - 1 = 0 \ , \ y(0) = 0 \tag{59}$$

A nonlinear stability test shows that Riccati equation in 59 is stable in sense of large. But the selected linear part in (52) is unstable. So EHPM is applied to find a stable solution for Riccati equation. An extra term of $\eta(x) = 2y$ term is added to the linear part and is deduced from the nonlinear part of Equation 52.

$$\underbrace{\dot{y} + 2y}_{L(y)} + \underbrace{y^2 - 1 - 4y}_{N(y)} = 0, y(0) = 0 \tag{60}$$

or in a separation format:

$$L(y) = \dot{y} + 2y \ , \ N(y) = y^2 - 4y - 1 \tag{61}$$

Consequently the new linear part is stable as well as Riccati equation in 59.

Equation 5 is applied and a homotopy interpretation is built which is as follows:

$$H(v,p) = \dot{v} + 2v - L(y_0) + pL(y_0) + p[v^2 - 4v - 1] \tag{62}$$

Again and similar to HPM case, the initial condition is selected as $y_0 = 2.4$. v is substituted by $v = p^0 v_0 + p^1 v_1 + p^2 v_2 + \cdots$. The rest of algorithm is the same as HPM. Therefore the following set of differential equation in terms of ascending power of p is achieved:

$$p^0 : \dot{v}_0 + 2v_0 - 4.8 = 0 \ , \ v_0(0) = 0 \tag{63}$$

$$p^1 : \dot{v}_1 + 2v_1 + 4.8 + v_0^2 - 4v_0 - 1 = 0 \ , \ v_1(0) = 0 \tag{64}$$

$$p^2 : \dot{v}_2 + 2v_2 + 2v_0 v_1 - 2v_1 = 0 \ , \ v_2(0) = 0 \tag{65}$$

Those equations conduct us to the following answers:

$$v_0 = \frac{12}{5}(1 - e^{-2t})$$

$$v_1 = (\frac{1}{50}e^{-2t} + \frac{48}{25}t + \frac{72}{25}e^{-2t} - \frac{29}{10})e^{-2t} \tag{66}$$

$$v_2 = -\frac{293}{125}e^{-2t} - \frac{1}{125}e^{2t} + 96t^2 - 726e^{-2t} - 302t + 576te^{-2t} + 432e^{-4t})e^{-2t}$$

When Equation 7 with assuming a 3[rd] order approximation, are considered, an EHPM solution of Riccati equation is achieved. The result is then plotted in Figure 9, to compare the significance of EHPM.

The efficiency of EHPM is significant, whereas HPM based one is not satisfactory.

3.1. The Exact Solution [27]

With an initial condition y(0) = 0, the exact solution of Equation 51 was found [27] as:

$$y(t) = 1 + \sqrt{2}\tanh\left\{\sqrt{2}t + \frac{1}{2}\ln(\frac{\sqrt{2}-1}{\sqrt{2}+1})\right\} \tag{67}$$

EHPM solution is again compared with the exact solution presented in Figure 10 to show the existing error.

4. CONCLUSION

A well-defined homotopy perturbation method was used to solve nonlinear differential equations. In some circumstances, especially when the chosen linear part is unstable; the results are not satisfactory. It was shown that the stability of linear part is more important and must be verified. Therefore, an alternative enhancement approach to improve HPM performance is proposed. A method was proposed to stabilize the unstable linear part by adding and subtracting an extra linear term. Accordingly, the solution of the linear part, i.e.v_0 is approaching the stable position. This

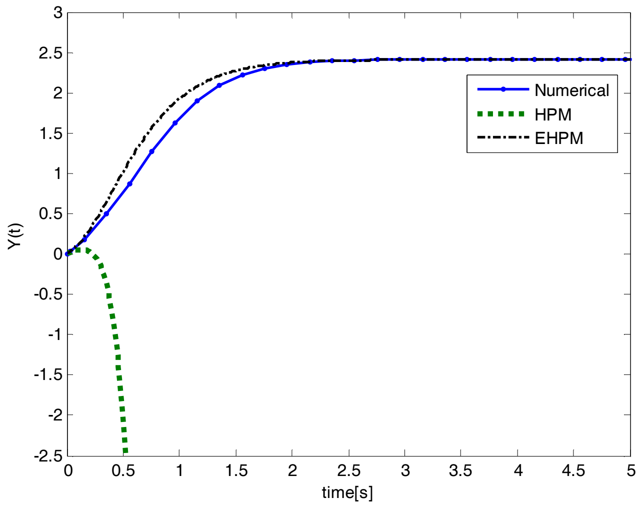

Figure 9. Comparison of EHPM, HPM and the numerical solution of Riccati equation in a longer duration.

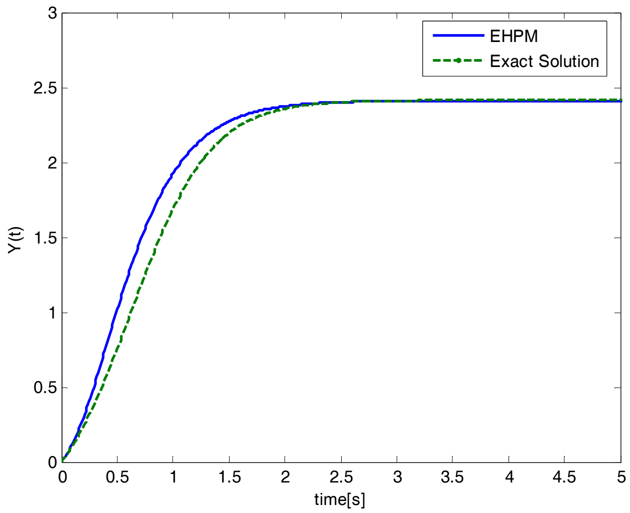

Figure 10. Comparison of Riccati exact solution and EHPM based response.

solution establishes a stable basis for the rest of the successive v_i differential equations. This method was successfully applied to Riccati differential equation by stabilizing the linear part. The simulation result and the exact solution confirm the significance of EHPM method. Although the results are satisfactory, one may investigate the role of the extra term in the convergence and the transient behavior by choosing other extra terms.

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