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Length Scale Effect on Vibration Analysis of Functionally Graded Kirchhoff and Mindlin Micro-plates

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ABSTRACT

In this paper, the modified couple stress theory is used to study vibration analysis of functionally graded rectangular micro-plates. Considering classical and first order plate theories, the couple governing equations of motion are obtained using the Hamilton's principle. Using an assumed mode method, the accurate size dependent natural frequencies are established for simply supported functionally graded rectangular micro-plates. To show the accuracy of the formulations, present results in specific cases are compared with available results in literature and a good agreement is seen. It is found that the natural frequency parameter of micro-plates will decrease as thickness-length ratio increases especially for lower length scale values. The effects of length scale, functionally graded parameter and plate theories on natural frequencies of functionally graded micro-plates are discussed in details.

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1. INTRODUCTION

Experimental results show that as length scales of a material are reduced, the influences of intermolecular cohesive forces on the mechanical properties become important and cannot be neglected. It is well known that size effects often become prominent at micro-length scales, the causes of which need to be explicitly addressed especially with an increasing interest in the general area of micro-structures. Couple stress theory is one of the higher order continuum theories which contain material length scale parameters and can capture the small size effects of microstructure. The classical couple stress theory contains two classical and two additional material constants for isotropic elastic materials.

Yang et al. [1] developed an additional equilibrium relation to govern the behavior of the material in couple stress theory of continuum. The relation constrained the couple stress tensor to be symmetric. The symmetric curvature tensor became the only properly conjugated high order strain measures in the theory to have a real contribution to the total strain energy of the system. They introduced a new modified couple stress theory that contains only one additional material length scale parameter. After this pioneer work, many studies have been extensively used the modified couple stress theory to study the behavior of micro structures such as micro-beams and micro-plates.

Park and Gao [2] used modified couple stress theory to study the bending effect of a cantilever Bernoulli-Euler beam. Their model contained an internal material length scale parameter that can capture the size effect. Ma et al. [3] developed Timoshenko beam model based on the modified couple stress theory to study bending and axial deformations of micro-beams. Lazopoulos [4] considered Kirchhoff theory of plates to study bending analysis of strain gradient elastic thin plates. In addition, Kirchhoff plate model was considered for investigating the static analysis of isotropic micro-plates with arbitrary shapes based on a modified couple stress theory by Tsiatas [5]. Asghari et al. [6] studied static and vibration behavior of micro-beams made of functionally graded materials. They used basis of modified couple stress theory in the elastic range and

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showed that size effect is more prominent in micro scales. Nonlinear Timoshenko beam model was also established by Asghari et al. [7] to study the bending and free vibration of hinged micro-beams. Gheshlaghi et al. [8] investigated the torsional vibration of nanotubes using modified couple stress theory. Yin et al. [9] considered couple stress theory to study non-classical Kirchhoff plate model for the dynamic analysis of microscale plates. They investigated natural frequencies of rectangular plates with at least two edges simply supported. The equations of motion and boundary conditions were obtained through a variational formulation for a Mindlin plate based on a couple stress theory by Ma et al. [10]. Jomehzadeh et al. [11] used couple stress theory to establish free vibration analysis of rectangular and circular micro-plates based on the Levy type solution. Based on a new modified couple stress theory a model for composite laminated Reddy plate was developed by Chen et al. [12]. Ke et al. [13] established nonlinear free vibration of micro-beams made of functionally graded materials based on modified couple stress theory. Recently, Reddy and Kim [14] presented the mechanical analyses of micro-plates using the third order shear deformation plate theory. Furthermore, Thai and Kim [15] developed a size-dependent model for bending and free vibration of functionally graded Reddy plates.

In the recent studies on new performance materials, new materials have addressed which known as functionally graded materials (FGMs). These are high performance, heat resistance materials able to withstand ultra high temperature and extremely large thermal gradients used in aerospace industries. Vibration analysis of structures made of functionally graded materials has been considered by engineers as a new field for researches. Zhao et al. [16] considered free vibration of moderately thick functionally graded rectangular plates using kp-Ritz method. The decoupling of bending–stretching governing equations of FG rectangular plates was first investigated by Saidi and Jomehzadeh [17]. Hosseini-Hashemi et al. [18] studied the free vibration of FG rectangular plates using first-order shear deformation plate theory. They neglected the effects of in-plane displacement on free vibration of rectangular plates. Using the classical plate theory, Liu et al. [19] studied the free vibration of FG rectangular plates by assuming the in-plane variation of material properties of the plate in which the bending/stretching equations are not coupled. Hasani Baferani et al. [20] found a new exact analytical solution for free vibration characteristics of thin functionally graded rectangular plates with different boundary conditions. Neves et al. [21] derived a higher-order shear deformation theory for modeling functionally graded plates accounting for extensibility in the thickness direction.

Because of unique properties of functionally graded materials, they are widely used in micro structures such as thin films in micro-electromechanical systems (MEMS). Therefore, the vibration analysis of FG micro-films is considered to be necessary. In the present article, size dependent vibration analysis of functionally graded rectangular micro-plates is studied using the modified couple stress theory. The governing equations of motion are obtained for both classical and first order shear deformation micro-plate theories. The partial differential equations of motion are converted into algebraic equations using the admissible functions for displacement components. The nondimensional natural frequency parameter is determined for different powers of FGM and various length scale parameters for both theories.

2. MATERIAL PROPERTIES

Let us consider a rectangular micro-plate of length a, width b and thickness h in x_1, x_2 and x_3 directions, respectively which is made of a functionally graded material (FGM). Although functionally graded materials are locally homogeneous; but, they are globally inhomogeneous due to spatial variations of volume fraction of the components. Mechanical properties of FGMs depend on the volume fraction of each constituent. FGMs are typically made from composition of metals and ceramics or a combination of different metals. Here, it is assumed that the micro-plate is made of a mixture of ceramic and metal in which the properties of functionally graded micro-plate vary smoothly and continuously through the thickness by the power law as follows [22]:

$$\begin{aligned} E(z) &= E_m + (E_c - E_m)\left(\frac{1}{2} - \frac{z}{h}\right)^n \\ \rho(z) &= \rho_m + (\rho_c - \rho_m)\left(\frac{1}{2} - \frac{z}{h}\right)^n \end{aligned} \tag{1}$$

where, E and ρ are the Young modulus and density of the micro-plate, respectively, n is the power of FGM or FGM parameter and subscripts m and c refer to metal and ceramic, respectively. Since the variation of Poisson's ratio is not considerable, the Poisson's ratio (ν) of the micro-plate is assumed to be constant.

3. FUNCTIONALLY GRADED KIRCHHOFF MICRO-PLATES

The Kirchhoff or classical plate theory is an extension of Euler-Bernoulli beam theory to thin plates. It is assumed that the straight lines normal to the midsurface remain straight and normal to the mid-plane after deformation. According to this theory, the displacement components in the x_1, x_2 and x_3 directions are considered as

$$
\begin{aligned}
u^k(x_1,x_2,x_3,t) &= u_0^k(x_1,x_2,t) - x_3 w_{0,1}^k(x_1,x_2,t)\\
v^k(x_1,x_2,x_3,t) &= v_0^k(x_1,x_2,t) - x_3 w_{0,2}^k(x_1,x_2,t)\\
w^k(x_1,x_2,x_3,t) &= w_0^k(x_1,x_2,t)
\end{aligned} \quad (2)
$$

where, u_0^k and v_0^k are the midplane displacements of the micro-plate in x_1 and x_2 directions, respectively, w_0^k is the transverse displacement, comma subscript shows differentiate with respect to coordinate and superscript k denotes Kirchhoff plate theory. Using strain-displacement relations, the strain components at distance x_3 from the middle plane are expressed as

$$
\begin{aligned}
&\varepsilon_{11}^k = u_{0,1}^k - x_3 w_{0,11}^k \qquad \varepsilon_{22}^k = v_{0,2}^k - x_3 w_{0,22}^k\\
&\varepsilon_{33}^k = 0 \qquad 2\varepsilon_{12}^k = u_{0,2}^k + v_{0,1}^k - 2x_3 w_{0,12}^k\\
&2\varepsilon_{23}^k = 0 \qquad 2\varepsilon_{13}^k = 0
\end{aligned} \quad (3)
$$

It can be found that the out of plane strain components are ignored in this theory. Based on the displacement field of classical plate theory, the rotation components of the micro-plate can be expressed as

$$\theta_1^k = w_{0,2}^k \qquad \theta_2^k = -w_{0,1}^k \qquad \theta_3^k = (v_{0,1}^k - u_{0,2}^k)/2 \quad (4)$$

The classical continuum elasticity, which is a scale free theory, can not predict the size effects in micro-structures, whereas the modified couple stress theory allows one to account the small length scale effect. It becomes significant when dealing with microstructures by considering only one material constant. According to the modified couple stress theory [1], the strain energy density is a function of both strain and gradient of the rotation vector as

$$U = 1/2\int_V (\sigma_{ij}\varepsilon_{ij} + m_{ij}\chi_{ij})\,dV \quad (5)$$

where, m_{ij} and χ_{ij} are deviatory part of couple stress and symmetric curvature components which are defined as

$$\chi_{ij} = (\theta_{i,j} + \theta_{j,i})/2 \qquad m_{ij} = E(z)l^2\chi_{ij}/(1+\nu) \quad (6)$$

where, l is a length scale parameter. This length scale parameter is a material property which carried with all of the difference between classical and couple stress elasticity theories. This parameter is small in comparison with the body dimensions. Therefore, its influence might become important as dimensions of a body diminish to the order of the length scale parameter. The equations of motion for a micro-plate based on the modified couple stress theory can be obtained using the Hamilton's principle. The Hamilton's principle states that

$$\int_0^t (\delta T - \delta U + \delta W)\,dt = 0 \quad (7)$$

where, T and W are kinetic energy of micro-plate and potential energy of the external loads, respectively. Expressing these parameters and strain energy based on the displacement field of Kirchhoff plate theory and considering the modified couple stress theory, the equations of motion for a micro-plate can be obtained as follows [11]:

$$\delta u_0^k: N_{11,1}^k + N_{12,2}^k + 1/2\,Y_{13,12}^k + 1/2\,Y_{23,22}^k = I_0\ddot{u}_0^k - I_1\ddot{w}_{0,1}^k \quad (8a)$$

$$\delta v_0^k: N_{12,1}^k + N_{22,2}^k - 1/2\,Y_{13,11}^k - 1/2\,Y_{23,12}^k = I_0\ddot{v}_0^k - I_1\ddot{w}_{,2}^k \quad (8b)$$

$$
\begin{aligned}
\delta w_0^k: &M_{11,11}^k + 2M_{12,12}^k + M_{22,22}^k - Y_{11,12}^k - Y_{12,22}^k + Y_{12,11}^k + Y_{22,12}^k\\
&- p_0 = I_0\ddot{w}_0^k + I_1(\ddot{u}_{0,1}^k + \ddot{v}_{0,2}^k) - I_2(\ddot{w}_{0,11}^k + \ddot{w}_{0,22}^k)
\end{aligned} \quad (8c)
$$

where, δ represents the variational symbol and p_0 is the transverse loading function. In addition, force resultant (N_{ij}^k), moments resultant (M_{ij}^k) and couple resultants (Y_{ij}^k) are defined as

$$
\begin{aligned}
N_{ij}^k &= \int_{-h/2}^{h/2} \sigma_{ij}^k\, dx_3\\
M_{ij}^k &= \int_{-h/2}^{h/2} \sigma_{ij}^k x_3\, dx_3\\
Y_{ij}^k &= \int_{-h/2}^{h/2} m_{ij}\, dx_3
\end{aligned} \quad (9)
$$

also, the inertia terms are expressed as

$$(I_0, I_1, I_2) = \int_{-h/2}^{h/2} \rho(z)(1, x_3, x_3^2)\, dx_3 \quad (10)$$

By assuming the plane stress state for the FG micro-plate, the force and moment resultants are obtained in the matrix form in terms of displacement components as

$$
\begin{bmatrix} N_{11}^k\\ N_{22}^k\\ N_{12}^k\\ M_{11}^k\\ M_{22}^k\\ M_{12}^k \end{bmatrix}
=
\begin{bmatrix}
A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0\\
A_{12} & A_{22} & 0 & B_{12} & B_{22} & 0\\
0 & 0 & A_{33} & 0 & 0 & B_{33}\\
B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0\\
B_{12} & B_{22} & 0 & D_{12} & D_{22} & 0\\
0 & 0 & B_{33} & 0 & 0 & D_{33}
\end{bmatrix}
\begin{bmatrix} u_{0,1}^k\\ v_{0,2}^k\\ u_{0,2}^k + v_{0,1}^k\\ -w_{0,11}^k\\ -w_{0,22}^k\\ -2w_{0,12}^k \end{bmatrix} \quad (11a)
$$

$$
\begin{bmatrix} Y_{11}^k\\ Y_{22}^k\\ Y_{12}^k\\ Y_{13}^k\\ Y_{23}^k \end{bmatrix}
= A_{33}l^2
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 2\\
0 & 0 & 0 & 0 & 0 & 0 & -2\\
0 & 0 & 0 & 0 & -1 & 1 & 0\\
0 & -1/2 & 1/2 & 0 & 0 & 0 & 0\\
-1/2 & 0 & 0 & 1/2 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} u_{0,22}^k\\ u_{0,12}^k\\ v_{0,11}^k\\ v_{0,12}^k\\ w_{0,11}^k\\ w_{0,22}^k\\ w_{0,12}^k \end{bmatrix} \quad (11b)
$$

where, the properties coefficients A_{ij} and B_{ij} $(i,j=1,2,3)$ are defined as

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} (1, z, z^2) Q_{ij}\, dz \quad i,j = 1,2,3$$
$$A_{ij} = \int_{-h/2}^{h/2} Q_{ij}\, dz \quad i,j = 1,2,3 \tag{12}$$

Based on the definition of Young modulus in Equation (1), Q_{ij}'s can be found for FG micro-plates as

$$Q_{ij} = \frac{E_m}{1-\nu^2} + \frac{E_{cm}}{1-\nu^2}\left(\frac{1}{2} - \frac{z}{h}\right)^n \qquad i = j = 1,2$$
$$Q_{ij} = \frac{E_m}{2(1+\nu)} + \frac{E_{cm}}{2(1+\nu)}\left(\frac{1}{2} - \frac{z}{h}\right)^n \qquad i = j = 3,4,5 \tag{13}$$
$$Q_{ij} = \nu\left(\frac{E_m}{1-\nu^2} + \frac{E_{cm}}{1-\nu^2}\left(\frac{1}{2} - \frac{z}{h}\right)^n\right) \qquad i \neq j$$

The governing equations of motion of a FG micro-plate based on the Kirchhoff theory can be obtained by substituting Equations (11) into equilibrium Equations (8) as follows:

$$\begin{aligned} &A_{11} u^k_{0,11} + A_{33} u^k_{0,22} + (A_{11} - A_{33}) v^k_{0,12} + \\ &A_{33} l^2 (v^k_{0,1112} - u^k_{0,1122} + v^k_{0,1222} - u^k_{0,2222})/4 - \\ &B_{11}(w^k_{0,111} + w^k_{0,122}) = I_0 \ddot{u}^k_0 - I_1 \ddot{w}^k_{0,1} \end{aligned} \tag{14a}$$

$$\begin{aligned} &A_{11} v^k_{0,22} + A_{33} v^k_{0,11} + (A_{11} - A_{33}) u^k_{0,12} - \\ &A_{33} l^2 (v^k_{0,1122} - u^k_{0,1222} + v^k_{0,1111} - u^k_{0,1112})/4 - \\ &B_{11}(w^k_{0,112} + w^k_{0,222}) = I_0 \ddot{v}^k_0 + I_1 \ddot{w}^k_{0,1} \end{aligned} \tag{14b}$$

$$\begin{aligned} &D_{11}(w^k_{0,111} + 2w^k_{0,1122} + w^k_{0,2222}) + \\ &A_{33} l^2 (2w^k_{0,1122} + w^k_{0,1111} + w^k_{0,2222}) - \\ &B_{11}(u^k_{0,111} + v^k_{0,222} + v^k_{0,112} + u^k_{0,122}) + \\ &p_0 = I_0 \ddot{w}^k_0 + I_1(\ddot{u}^k_{0,1} + \ddot{v}^k_{0,2}) - I_2(\ddot{w}^k_{0,11} + \ddot{w}^k_{0,22}) \end{aligned} \tag{14c}$$

It can be seen that, unlike the classical Kirchhoff plate theory, the current FG Kirchhoff micro-plate model based on the modified couple stress theory contains a material length scale parameter l and can capture the size effect. Furthermore, because of the nonhomogeneous properties of functionally graded materials, the stretching and bending equations of FG micro-plates are couple to each other. In fact B_{11} parameter, which is zero in isotropic case, causes a coupling between in-plane and out of plane equations.

4. FUNCTIONALLY GRADED MINDLIN MICRO-PLATES

The Mindlin or first order shear deformation plate theory assumes that the plane sections originally perpendicular to the longitudinal plane of the plate remain plane, but not necessarily perpendicular to the longitudinal plane. This theory accounts for the shear strains in the thickness direction of the micro-plate and is based on the displacement field.

$$\begin{aligned} u^m(x_1,x_2,x_3,t) &= u^m_0(x_1,x_2,t) + x_3 \psi_1^m(x_1,x_2,t) \\ v^m(x_1,x_2,x_3,t) &= v^m_0(x_1,x_2,t) + x_3 \psi_2^m(x_1,x_2,t) \\ w^m(x_1,x_2,x_3,t) &= w^m_0(x_1,x_2,t) \end{aligned} \tag{15}$$

where, u^m_0 and v^m_0 are the midplane displacements of the micro-plate in the x_1 and x_2 directions, respectively, w^m_0 is the transverse displacement, ψ_1^m and ψ_2^m are rotation functions of the middle surface in the x_1 and x_2 directions, respectively and superscript m denotes the Mindlin plate theory. Using strain-displacement relations, the strain components at distance x_3 from the middle plane which are expressed as

$$\begin{aligned} &\varepsilon^m_{11} = u^m_{0,1} + x_3 \psi^m_{1,1} \qquad \varepsilon^m_{22} = v^m_{0,2} + x_3 \psi^m_{2,2} \\ &\varepsilon^m_{33} = 0 \qquad 2\varepsilon^m_{12} = u^m_{0,2} + v^m_{0,1} + x_3(\psi^m_{1,2} + \psi^m_{2,1}) \\ &2\varepsilon^m_{23} = \psi_2^m + w^m_{0,2} \qquad 2\varepsilon^m_{13} = \psi_1^m + w^m_{0,1} \end{aligned} \tag{16}$$

It can be seen that, unlike the Kirchhoff plate theory, the out of plane strain components are not zero in this theory. The rotation components based on the Mindlin plate theory can be also obtained as

$$\begin{aligned} &\theta_1^m = (w^m_{0,2} + \psi_2^m)/2 \qquad \theta_2^m = -(w^m_{0,1} + \psi_1^m)/2 \\ &\theta_3^m = (v^m_{0,1} - u^m_{0,2} - x_3 \psi^m_{2,1} + -x_3 \psi^m_{1,2})/2 \end{aligned} \tag{17}$$

As the Kirchhoff micro-plate, the equations of motion for FG Mindlin micro-plates can be obtained using the Hamilton's principle. Expressing the strain, kinetic and potential energies in terms of the components of Mindlin theory and substituting the results into Equation (7), the equations of motion of FG Mindlin micro-plates can be defined as

$$\delta u_0: \quad N^m_{11,1} + N^m_{12,2} + 1/2 Y^m_{13,12} + 1/2 Y^m_{23,22} = I_0 \ddot{u}^m_0 - I_1 \ddot{\psi}_1^m \tag{18a}$$

$$\delta v_0: \quad N^m_{12,1} + N^m_{22,2} - 1/2 Y^m_{13,11} - 1/2 Y^m_{23,12} = I_0 \ddot{v}^m_0 - I_1 \ddot{\psi}_2^m \tag{18b}$$

$$\begin{aligned} \delta\psi_1: \quad &M^m_{11,1} + M^m_{12,2} - Q^m_{13} + 1/2 Y^m_{12,1} + 1/2 Y^m_{22,2} - 1/2 Y^m_{33,2} + \\ &1/2 H^m_{23,22} + 1/2 H^m_{13,12} = I_1 \ddot{u}^m_0 - I_2 \ddot{\psi}_1^m \end{aligned} \tag{18c}$$

$$\begin{aligned} \delta\psi_2: \quad &M^m_{12,1} + M^m_{22,2} - Q^m_{23} - 1/2 Y^m_{11,1} - 1/2 Y^m_{12,2} + 1/2 Y^m_{33,1} - \\ &1/2 H^m_{13,11} - 1/2 H^m_{23,12} = I_1 \ddot{v}^m_0 - I_2 \ddot{\psi}_2^m \end{aligned} \tag{18d}$$

$$\begin{aligned} \delta w_0: \quad &N^m_{13,1} + N^m_{23,2} - 1/2 Y^m_{11,12} + 1/2 Y^m_{22,12} + 1/2 Y^m_{12,11} - \\ &1/2 Y^m_{12,22} - p_0 = I_0 \ddot{w}^m_0 \end{aligned} \tag{18e}$$

While there are three equations in case of Kirchhoff theory, it can be seen that the equations of FG Mindlin micro-plates are five couple equations. Assuming the plane stress state for FG micro-plate, the force and moment resultants are obtained in the matrix form as

$$\begin{bmatrix} N^m_{11} \\ N^m_{22} \\ N^m_{12} \\ M^m_{11} \\ M^m_{22} \\ M^m_{12} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 \\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & 0 \\ 0 & 0 & A_{33} & 0 & 0 & B_{33} \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 \\ B_{12} & B_{22} & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & B_{33} & 0 & 0 & D_{33} \end{bmatrix} \begin{bmatrix} u^m_{0,1} \\ v^m_{0,2} \\ u^m_{0,2} + v^m_{0,1} \\ \psi^m_{1,1} \\ \psi^m_{2,2} \\ \psi^m_{1,2} + \psi^m_{2,1} \end{bmatrix} \tag{19a}$$

$$\begin{bmatrix} Q_{13}^m \\ Q_{23}^m \end{bmatrix} = \begin{bmatrix} K_s^2 A_{33} & 0 \\ 0 & K_s^2 A_{33} \end{bmatrix} \begin{bmatrix} w_{0,1}^m + \psi_1^m \\ w_{0,2}^m + \psi_2^m \end{bmatrix} \tag{19b}$$

The couple resultant parameters can also be expressed as

$$\begin{aligned} &Y_{11}^m = l^2 A_{33}(w_{0,12}^m + \psi_{2,1}^m) \qquad Y_{22}^m = -l^2 A_{33}(w_{0,12}^m + \psi_{1,2}^m) \\ &Y_{13}^m = l^2 (A_{33} v_{0,11}^m - B_{33}\psi_{2,11}^m - A_{33} u_{0,12}^m + B_{33}\psi_{1,12}^m)/2 \\ &Y_{23}^m = l^2 (A_{33} v_{0,12}^m - B_{33}\psi_{2,12}^m - A_{33} u_{0,22}^m + B_{33}\psi_{1,22}^m)/2 \\ &Y_{33}^m = l^2 A_{33}(-\psi_{2,1}^m + \psi_{1,2}^m) \\ &Y_{12}^m = l^2 A_{33}(w_{0,22}^m + \psi_{2,2}^m - w_{0,11}^m - \psi_{1,1}^m)/2 \end{aligned} \tag{20a}$$

$$\begin{aligned} &H_{13}^m = \int_{-h/2}^{h/2} m_{13} x_3 \, dx_3 = l^2 (B_{33} v_{0,11}^m - D_{33}\psi_{2,11}^m - \\ &B_{33} u_{0,12}^m + D_{33}\psi_{1,12}^m)/2 \\ &H_{23}^m = \int_{-h/2}^{h/2} m_{23} x_3 \, dx_3 = l^2 (B_{33} v_{0,12}^m - D_{33}\psi_{2,12}^m - \\ &B_{33} u_{0,22}^m + D_{33}\psi_{1,22}^m)/2 \end{aligned} \tag{20b}$$

where, A_{ij}, B_{ij} and D_{ij} are the same as Equations (12) and K_s^2 is the shear correction factor.

The governing equations of motion of FG Mindlin micro-plate are obtained by substituting Equations (19) and (20) into Equation (18) as

$$\begin{aligned} &A_{11}(u_{0,11}^m + v_{0,12}^m) + A_{33}(u_{0,22}^m - v_{0,12}^m) - B_{11}(\psi_{1,11}^m + \psi_{2,12}^m) \\ &+ B_{33}(\psi_{2,12}^m - \psi_{1,22}^m) - 1/4 A_{33} l^2 (u_{0,2222}^m + u_{0,1122}^m - v_{0,1112}^m \\ &- v_{0,1222}^m) + 1/4 B_{33} l^2 (\psi_{1,2222}^m + \psi_{1,1122}^m - \psi_{2,1112}^m \\ &- \psi_{2,1222}^m) = I_0 \ddot{u}_0^m - I_1 \ddot{\psi}_1^m \end{aligned} \tag{21a}$$

$$\begin{aligned} &A_{11}(u_{0,12}^m + v_{0,22}^m) - A_{33}(u_{0,12}^m - v_{0,11}^m) - B_{11}(\psi_{1,12}^m + \psi_{2,22}^m) \\ &+ B_{33}(\psi_{1,12}^m - \psi_{2,11}^m) + 1/4 A_{33} l^2 (u_{0,1112}^m + u_{0,1222}^m - v_{0,1111}^m \\ &- v_{0,1122}^m) - 1/4 B_{33} l^2 (\psi_{1,1112}^m + \psi_{1,1222}^m - \psi_{2,1111}^m \\ &- \psi_{2,1122}^m) = I_0 \ddot{v}_0^m - I_1 \ddot{\psi}_2^m \end{aligned} \tag{21b}$$

$$\begin{aligned} &B_{11}(u_{0,11}^m + v_{0,12}^m) + A_{33} K_s^2 (\psi_1^m - w_{0,1}^m) - A_{33} l^2 (\psi_{1,11}^m / 2 + \psi_{1,22}^m \\ &- 3\psi_{2,12}^m / 4 + w_{0,111}^m / 2 - w_{0,122}^m / 2)/2 + B_{33}(u_{0,22}^m - v_{0,12}^m) - \\ &B_{33} l^2 (u_{0,2222}^m + u_{0,1122}^m - v_{0,1112}^m - v_{0,1222}^m)/4 - D_{11}(\psi_{1,11}^m + \\ &\psi_{2,12}^m) - D_{33}(\psi_{1,22}^m - \psi_{2,12}^m) + D_{33} l^2 (\psi_{1,2222}^m + \psi_{1,1122}^m - \\ &\psi_{2,1112}^m - \psi_{2,1222}^m)/4 = I_1 \ddot{u}_0^m - I_2 \ddot{\psi}_1^m \end{aligned} \tag{21c}$$

$$\begin{aligned} &B_{11}(u_{0,12}^m + v_{0,22}^m) + A_{33} K_s^2 (\psi_2^m - w_{0,2}^m) - A_{33} l^2 (\psi_{2,22}^m / 2 + \psi_{2,11}^m \\ &- 3\psi_{1,12}^m / 4 + w_{0,112}^m / 2 - w_{0,222}^m / 2)/2 - B_{33}(u_{0,12}^m - v_{0,11}^m) + \\ &B_{33} l^2 (u_{0,1112}^m + u_{0,1222}^m - v_{0,1111}^m - v_{0,1122}^m)/4 - D_{11}(\psi_{1,12}^m + \\ &\psi_{2,22}^m) + D_{33}(\psi_{1,22}^m - \psi_{2,11}^m) - D_{33} l^2 (\psi_{1,1112}^m + \psi_{1,1222}^m - \\ &\psi_{2,1111}^m - \psi_{2,1122}^m)/4 = I_1 \ddot{v}_0^m - I_2 \ddot{\psi}_2^m \end{aligned} \tag{21d}$$

$$\begin{aligned} &A_{33} K_s^2 (w_{0,11}^m + w_{0,22}^m - \psi_{1,1}^m - \psi_{2,2}^m) - A_{33} l^2 (\nabla^4 w_0^m + \\ &\psi_{1,111}^m + \psi_{1,122}^m + \psi_{2,112}^m + \psi_{2,222}^m)/4 + p_0 = I_0 \ddot{w}_0^m \end{aligned} \tag{21e}$$

It can be found that as the length scale parameter is ignored, five couple equations of motion of Mindlin micro-plate convert into equations of classical Mindlin plate $(l = 0)$.

5. SOLUTION USING NAVIER'S APPROACH

The developed governing differential equations of motion of Sections 3 and 4 are solved by Navier's approach for simply supported boundary conditions. For free vibration analysis, the transverse loading function p_0 is considered to be zero. The simply supported boundary conditions for Kirchhoff and Mindlin theories based on the couple stress theory are written as

$$\left.\begin{aligned} &v_0^k = 0 \\ &w_0^k = 0 \\ &2N_{11}^k + Y_{13,2}^k = 0 \\ &2M_{11}^k + Y_{12}^k = 0 \\ &Y_{12}^k = 0 \\ &Y_{13}^k = 0 \end{aligned}\right\} \quad \text{at } x_1 = 0, a \tag{22a}$$

$$\left.\begin{aligned} &u_0^k = 0 \\ &w_0^k = 0 \\ &2N_{22}^k + Y_{23,1}^k = 0 \\ &2M_{22}^k + Y_{12}^k = 0 \\ &Y_{12}^k = 0 \\ &Y_{23}^k = 0 \end{aligned}\right\} \quad \text{at } x_2 = 0, b \tag{22b}$$

$$\left.\begin{aligned} &v_0^m = 0 \\ &w_0^m = 0 \\ &\psi_2^m = 0 \\ &2N_{11}^m + Y_{13,2}^m = 0 \\ &2M_{11}^m + H_{13,2}^m + Y_{12}^m = 0 \\ &Y_{12}^m = 0 \\ &Y_{13}^m = 0 \\ &H_{13}^m = 0 \end{aligned}\right\} \quad \text{at } x_1 = 0, a \tag{22c}$$

$$\left.\begin{aligned} &u_0^m = 0 \\ &w_0^m = 0 \\ &\psi_1^m = 0 \\ &2N_{22}^m + Y_{23,1}^m = 0 \\ &2M_{22}^m + H_{23,1}^m + Y_{12}^m = 0 \\ &Y_{12}^m = 0 \\ &Y_{23}^m = 0 \\ &H_{23}^m = 0 \end{aligned}\right\} \quad \text{at } x_2 = 0, b \tag{22d}$$

TABLE 1. Comparison of nondimensional natural frequency $(\omega_{pq}\pi^2(a^2/h)\sqrt{\rho_m/E_m})$ of FG Kirchhoff plates $(l=0)$

n	b/a		Mode 1	Mode 2	Mode 3
0	1	Present	115.9251(1,1)	289.7771(1,2)	463.5861(2,2)
		[20]	115.8695(1,1)	289.7708(1,2)	463.4781(2,2)
	2	Present	72.4554(1,1)	115.9251(2,1)	188.3686(3,1)
		[20]	72.3942(1,1)	115.8695(2,1)	188.2637(3,1)
0.5	1	Present	98.1594(1,1)	245.3676(1,2)	392.5390(2,2)
		[20]	98.0136(1,1)	245.3251(1,2)	392.4425(2,2)
	2	Present	61.3515(1,1)	98.1594(2,1)	159.5006(3,1)
		[20]	61.3313(1,1)	98.0136(2,1)	159.3448(3,1)
1	1	Present	88.4500(1,1)	221.0595(1,2)	353.7056(2,2)
		[20]	88.3093(1,1)	221.0643(1,2)	353.6252(2,2)
	2	Present	55.2831(1,1)	88.4500(2,1)	143.7232(3,1)
		[20]	55.1205(1,1)	88.3093(2,1)	143.6239(3,1)

TABLE 2. Comparison of Nondimensional natural frequency $(\omega_{11}h\sqrt{\rho_c/E_c})$ of FG Mindlin plates $(l=0)$

h/a		n		
		0.5	**1**	**4**
0.05	Present	0.01254	0.01131	0.00982
	[19]	0.01281	0.01150	0.01013
	[16]	0.01241	0.01118	0.0097
0.1	Present	0.04900	0.04419	0.03823
	[19]	0.04920	0.04454	0.03825
	[16]	0.04818	0.04346	0.03757
0.2	Present	0.1805	0.1631	0.1397
	[19]	0.1806	0.1650	0.1371
	[16]	0.1757	0.1587	0.1356

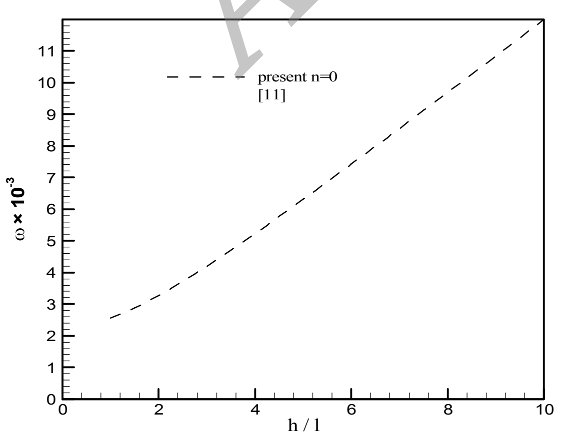

Figure 1. Comparison of natural frequencies of isotropic Kirchhoff micro plates $(n=0)$

The following expressions of various generalized displacements have been assumed:

$$\begin{aligned} u_0^k(x_1,x_2,t) &= u_{pq}^k \cos(\beta_p x_1)\sin(\eta_q x_2)e^{i\omega_{pq}t} \\ v_0^k(x_1,x_2,t) &= u_{pq}^k \sin(\beta_p x_1)\cos(\eta_q x_2)e^{i\omega_{pq}t} \\ w_0^k(x_1,x_2,t) &= w_{pq}^k \sin(\beta_p x_1)\sin(\eta_q x_2)e^{i\omega_{pq}t} \end{aligned} \tag{23a}$$

$$\begin{aligned} u_0^m(x_1,x_2,t) &= u_{pq}^m \cos(\beta_p x_1)\sin(\eta_q x_2)e^{i\omega_{pq}t} \\ v_0^m(x_1,x_2,t) &= u_{pq}^m \sin(\beta_p x_1)\cos(\eta_q x_2)e^{i\omega_{pq}t} \\ \psi_1^m(x_1,x_2,t) &= \psi_{1pq}^m \cos(\beta_p x_1)\sin(\eta_q x_2)e^{i\omega_{pq}t} \\ \psi_2^m(x_1,x_2,t) &= \psi_{2pq}^m \sin(\beta_p x_1)\cos(\eta_q x_2)e^{i\omega_{pq}t} \\ w_0^m(x_1,x_2,t) &= w_{pq}^m \sin(\beta_p x_1)\sin(\eta_q x_2)e^{i\omega_{pq}t} \end{aligned} \tag{23b}$$

where, β_p and η_q denote $p\pi/a$ and $q\pi/b$, respectively. In addition, ω_{pq} is the natural frequency of the micro-plate. It can be found that these admissible functions exactly satisfy the boundary conditions of simply supported micro-plates in Equation (22). Substituting proposed series (23a) and (23b) into governing equations of motion of FG Kirchhoff and Mindlin micro-plates in Equations (14) and (21) respectively, a system of homogeneous equations is obtained. In the case of Kirchhoff micro-plates, this system contains three homogeneous equations and in the case of Mindlin micro-plates, five homogeneous equations are obtained. Setting the determinant of coefficient matrix in each case equal to zero, the natural frequencies can be determined for both Kirchhoff and Mindlin micro-plates.

TABLE 3. Nondimensional natural frequency parameter ($\omega_{11} a^2 \sqrt{\rho_m / E_m h^2}$) of FG Kirchhoff micro-plates ($a/b=1$)

k	h/a	h/l				
		0.25	**0.5**	**1**	**2**	**4**
0	0.05	96.8087	49.4575	26.7315	16.7841	13.1716
	0.1	72.4849	49.1565	26.5688	16.6820	13.0914
0.5	0.05	86.7983	44.2432	23.7331	14.6528	11.2945
	0.1	65.0460	43.9647	23.5856	14.5620	11.2246
1	0.05	80.3939	40.9385	21.8856	13.4062	10.2426
	0.1	60.2786	40.6581	21.7404	13.3177	10.1752
2	0.05	72.6106	36.9853	19.7876	12.1428	9.2963
	0.1	54.4534	36.7003	19.6431	12.0550	9.2292

TABLE 4. Nondimensional natural frequency parameter ($\omega_{11} a^2 \sqrt{\rho_m / E_m h^2}$) of FG Mindlin micro-plates ($a/b=1$)

k	h/a	h/l				
		0.25	**0.5**	**1**	**2**	**4**
0	0.05	86.7045	47.8202	26.3910	16.6515	13.0782
	0.1	68.7814	43.6820	25.3412	16.1900	12.7411
0.5	0.05	77.7758	42.8024	23.4455	14.5473	11.2229
	0.1	61.7225	39.1416	22.5472	14.1702	10.9555
1	0.05	72.0592	39.6162	21.6265	13.3139	10.1811
	0.1	57.1987	36.2418	20.8068	12.9754	9.9443
2	0.05	65.0917	35.7907	19.5526	12.0584	9.2398
	0.1	51.6711	32.7320	18.8001	11.7434	9.0180

6. NUMERICAL RESULTS AND DISCUSSION

To verify the accuracy of the present formulations, three comparisons have been carried out with available results for all edges simply supported plates. Figure 1 compares the variations of natural frequencies of a homogenous Kirchhoff micro-plate ($n=0$) with Ref. [11]. The natural frequencies of classical FG plates ($l=0$) for both Kirchhoff and Mindlin plates are also compared in Tables 1 and 2 with available results in literature. The comparisons show that the present results agree well with those in the literature. For numerical computation, the considered functionally graded material is composed of Aluminum ($E_m = 70\,Gpa, \rho_m = 2707 Kg/m^3$) and Alumina ($E_c = 380\,Gpa, \rho_c = 3800 Kg/m^3$). Since the variation of Poisson ratio is not considerable changes for different materials, their values are considered to be constants and equal to 0.3 [23]. Furthermore, for Mindlin micro-plates the shear correction factor (K_s^2) has been considered to be $\pi^2/12$. The accurate non-dimensional natural frequency parameter ($\Omega = \omega_{pq} a^2 \sqrt{\rho_m / E_m h^2}$) of FG Kirchhoff and Mindlin square micro-plates are tabulated in Tables 3 and 4 for various values of non-dimensional length scale parameter (h/l) and different powers of FGM. Since no experimental date is available for FG micro-plates, the value of material length scale parameter for homogeneous epoxy micro-plate ($l = 17.6\,\mu m$) is approximately used for FG micro-plates [6, 15]. It can be seen that for a constant nondimensional length scale parameter, the frequency parameter decreases for both theories as the power of FGM (n) increases. The reason is that with increasing the power of FGM, the stiffness of the micro-plate decreases and results in decreasing the natural frequency of the FG rectangular micro-plate.

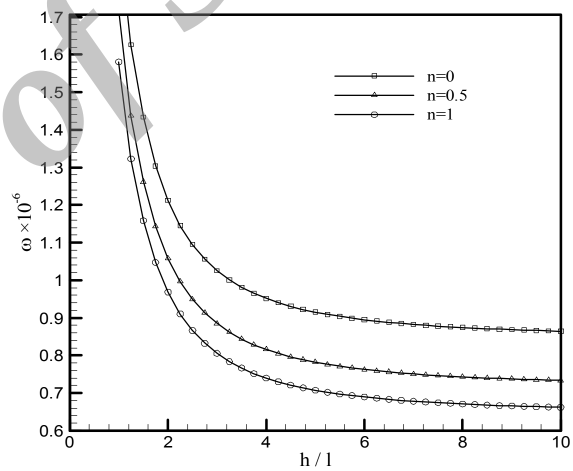

Figure 2. Variation of fundamental natural frequency of FG Kirchhoff micro-plates

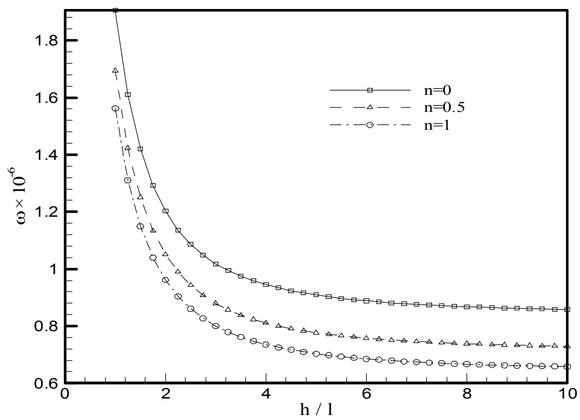

Figure 3. Variation of fundamental natural frequency of FG Mindlin micro-plates

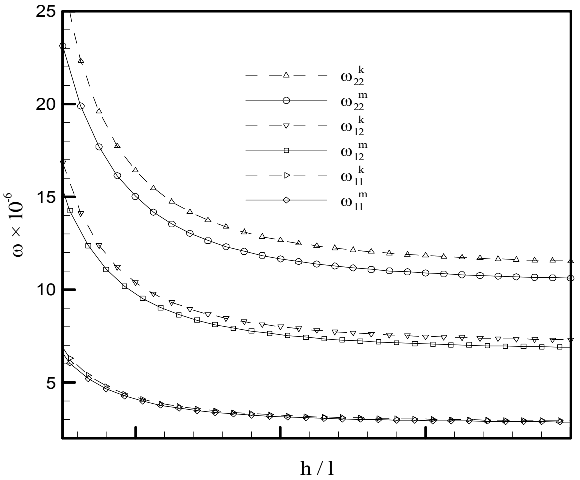

Figure 4. Variation of natural frequency of FG Kirchhoff and Mindlin micro-plates for first three modes $(n = 0.5)$

In addition, it can be found that as the thickness-length ratio (h/a) increases, the nondimensional frequency parameter will decrease and this effect is more significant for lower nondimensional scale values (h/l) especially for Mindlin micro-plates.

In order to study the effect of power of FGM on natural frequencies of FG micro-plates, the variation of fundamental natural frequencies with respect to length scale parameter is depicted for different FGM parameters in Figures 2 and 3 for Kirchhoff and Mindlin micro-plates, respectively. The thickness of the FG micro-plates is assumed to be constant and equal to $h = 17.6\,\mu m$ in all figures. It can be concluded that as the isotropic micro-plates, the natural frequencies of FG micro-plates decreases as decreasing of length scale parameter (l). The variations of natural frequencies of first three modes are shown in Figure 4 for both FG Kirchhoff and Mindlin micro-plates. The power of FGM is $n = 0.5$ and the dimensions of the micro-plate is considered to be $a = b = 176\,\mu m$. It can be seen that the difference of two theories is considerable in higher modes.

7. CONCLUSION

In this paper, vibration analysis of functionally graded Kirchhoff and Mindlin micro-plates has been studied by considering the small length scale effect. Using the modified couple stress theory, the governing equations of motion have been obtained for both Kirchhoff and Mindlin micro-plates. The accurate natural frequencies have been determined for a simply supported functionally graded rectangular micro-plate. The nondimensional natural frequency parameter has been determined for different powers of FGM and various length scale parameters for both theories. In addition, the effects of material property, thickness to length ratio and length scale parameter on the vibrational behavior of the FG rectangular micro-plates have been investigated.

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Length Scale Effect on Vibration Analysis of Functionally Graded Kirchhoff and Mindlin Micro-plates

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چکیده

در این مقاله با استفاده از تئوری اصلاح شده تنش کوپلی، ارتعاشات آزاد میکرو ورق‌های مستطیلی هدفمند تابعی مورد مطالعه قرار می‌گیرد. با در نظر گرفتن تئوری‌های کلاسیک و مرتبه اول ورق، معادلات حرکت با استفاده از اصل همیلتون بدست می‌آید. فرکانس‌های طبیعی میکرو ورق‌های هدفمند تابعی با استفاده از روش مد فرضی برای شرایط مرزی تکیه‌گاه ساده تعیین می‌گردند. جهت بررسی صحت نتایج، نتایج کنونی در حالت‌های خاص با نتایج موجود در تحقیقات دیگر مقایسه شده است و دقت مناسبی مشاهده گردید. با بررسی نتایج مشاهده می‌شود که با کاهش نسبت اثر مقیاس اندازه به ضخامت ورق، پارامتر فرکانس طبیعی میکرو ورق‌ها به‌خصوص برای مقادیر کوچک مقیاس اندازه افزایش می‌یابد. اثرات پارامتر مقیاس کوچک، پارامتر تابع هدفمند و تئوری‌های ورق بر روی فرکانس طبیعی میکرو ورق های هدفمند تابعی نشان داده شده است.

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