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Free Vibration Analysis of Functionally Graded Materials Non-uniform Beams

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ABSTRACT

In this article, nonuniformity effects on free vibration analysis of functionally graded beams is discussed. Variation in material properties is modeled after power law and exponential models and the nonuniformity is assumed to be exponentially varying in the width along the beams with constant thickness. Analytical solution is achieved for free vibration with simply supported conditions. Results show that functionally graded material (FGM) accompanying section variation have a significant effect on natural frequencies of FGM nonuniform beams. In order to show this dependency, Al/Al_2O_3 composite beam is modeled and the first three natural frequencies with simply supported boundary conditions is obtained for different power law and exponential parameters which shows a great sensitivity to nonuniformity in different shape modes.

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NOMENCLATURE

E_t	Young's modulus of top surface (pa)	U	Strain energy(J)
E_b	Young's modulus of bottom surface(pa)	T	Kinetic ecergy (J)
ρ_t	Density of the material of top surface (kg/m^3)	m_i	Mass parameter (kg)
ρ_b	Density of the material of bottom surface (kg/m^3)	G_i	Rigidity parameter (pa.m^4)
P	Power law index parameter	w	Transvere deflection (m)
Φ	Exponential parameter	δ	Variation term
h	Thickness of beam (m)	γ	Frequency parameter
L	Length of beam (m)	C_i	Constant coefficients
A	Cross sectional area of the beam (m^2)	λ_i	Frequency index parameters
b_0	Width of the beam (m)	ε	Strain tensor
η	Exponential index for nonuniformity	σ	Stress tensor (pa)
ω	Natural frequency term		

1. INTRODUCTION

One of the most important key steps in engineering design is using smart materials with modified properties in order to reach the requirement of the machine, system, etc. Composite materials have the potential to act more productive in one dimension with respect to others. This ability made a significant change in engineering designs by using them as beams, plates and shells in different composite structures in macro- and nanoscales. FGMs are a class of composite materials which have attracted a lot attention in last decades. This

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kind of materials have the ability to change their properties and behavior in one direction with respect to a specific function. In order to be able to use them in different fields of engineering, it is important to fully understand their behavior under different conditions. To this end, many researches have been done to understand the behavior of beams and is still one of the most important subjects between researchers and scientists by focusing on bending [1, 2], buckling [3, 4], free vibration [5, 6] and deflection due to moving loads or forced vibration [7, 8] in different nano- and macroscale problems.

On the other hand, varying geometrical properties of beams and plates used in structures could be a great help in designing an optimized system. Optimizing geometrical properties makes a system to occupy less space and be more reliable. Therefore, many researchers studied different kinds of nonuniformities in beams and plates to present the benefits of using them.

Cem Ece et al. [9] studied the effects of nonuniformity on natural frequency and mode shapes in isotropic beams. Nonuniformity was modeled by exponentially varying thickness, and Euler–Bernoulli beam theory was used to model the beam. Verification was done by neglecting the nonuniform parameters and comparing the results to those presented in literature for uniform beams. It has been shown that nonuniformity in beams has a great effect in varying the frequency parameter. Hosseini Hashemi and Bakhshi Khaniki [10] analyzed the same nonuniform beam in nanoscale model. Small scale effects were modeled after Eringens nonlocal theory and different boundary conditions were discussed. Analytical solution for free vibration of such kind of nanobeams were presented and the results were verified in two different ways: first, by neglecting the nonlocal effects and comparing to those achieved by Cem Ece et al., and second, by neglecting the nonuniformity parameter and comparing to the results in literatures for free vibration in uniform nanobeam. They have shown that nonlocal and nonuniform parameters have significant effect on dynamic behavior and natural frequency of nanobeams. Mirzabeigy [11] studied the vibration on nonuniform beam on an elastic foundation. The beam was under axial tensile force and modeled after Euler beam theory. The same type of exponential varying nonuniformity was assumed in this model. Differential transform method was employed to solve the problem and the results were verified by comparing to Cem Ece. Effects of elastic foundation, axial force and nonuniformity on natural frequency was presented.

To the knowledge of the authors, there is no analytical study done in order to investigate the free vibration analysis of nonuniform FGM beams which is the main idea of this study. General solution is presented for this problem and for simply supported nonuniform FGM beams results are achieved and the solving process could also be used for all kinds of boundary conditions. FGM materials are assumed to vary as power law and exponential functions. For both cases, nondimensional frequency parameter is achieved by analytically solving the problem and the effects of material variation with respect to a function and nonuniformity effects are studied.

2. PROBLEM FORMULATION

According to the definition of FGM material theory and assuming that material variation occurs in the thickness direction, two kinds of power law and exponential variation in FGM material properties are presented as:

2. 1. Power Law Function Model For power law material variation method, Young's modulus and the density of the beam could be expressed as:

$$\begin{cases} E(z) = (E_t - E_b)\left(\dfrac{h-2z}{2h}\right)^p + E_b \\ \rho(z) = (\rho_t - \rho_b)\left(\dfrac{h-2z}{2h}\right)^p + \rho_b \end{cases} \tag{1}$$

in which the E_t, E_b, ρ_t and ρ_b are Young's modulus and density of the material in the top and bottom surface of the beam, h is the thickness of the beam which is assumed to be constant through the beam and P is the power law index of material.

2. 2. Exponential Function Model In this kind of FGM materials, properties change in an exponential function as:

$$\begin{cases} E(z) = E_b e^{\varnothing z} \\ \rho(z) = \rho_b e^{\varnothing z} \end{cases} \tag{2}$$

where Φ describes the material gradation along the thickness as an exponential function index

2. 3. Nondimensional Euler Beam Model for FGM Materials This class of nonuniform Euler beam was modeled [9] for a homogenous nonlocal material. To apply this nonuniformity in this formulation, displacements, strain and stress are assumed as those presented in [12] and the strain energy which is stored in the beam due to the stress and strain in the longitudinal direction of the nonuniform beam is defined as:

$$U = \frac{1}{2}\int_0^L\int_A \sigma_{xx}\varepsilon_{xx}\,dA\,dx = \frac{1}{2}\int_0^L\int_A E(z)\varepsilon_{xx}^2\,dA\,dx = \frac{1}{2}\int_0^L\int_A E(z)z^2\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dA\,dx \tag{3}$$

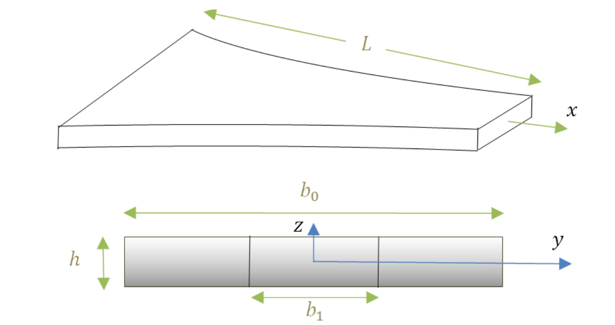


Figure 1. Schematic isometric and front view of nonuniform FGM beam with appearance of material variation

As shown in Figure 1, L is the length and A the cross sectional area of the beam which is assumed to vary in x direction by having a constant thickness and exponential variation in the width of the beam as:

$$\begin{cases} b(x) = b_0 e^{\eta x} \\ b_1 = b_0 e^{\eta l} \end{cases} \tag{4}$$

where η is the exponential index for nonuniformity in width of the beam and b_0 and $b_1$are the widths of the beam in the right and left end.
Also, the Kinetic energy in the beam by neglecting the rotary inertia effects is written as:

$$T = \frac{1}{2}\int_0^L\int_A \rho(z)\left(\frac{\partial w}{\partial t}\right)^2 dA\,dx \tag{5}$$

Mass inertia and rigidity parameters with respect to nonuniformity in the beam could be defined as:

$$\begin{cases} m_1(x) = \int_A \rho(z)dA = b_0 e^{Nx} m_0 \\ m_0 = \int_A \rho(z)dz \end{cases} \tag{6}$$

$$\begin{cases} G_1(x) = \int_A E(z)z^2 dA = b_0 e^{Nx} G_0 \\ G_0 = \int_0^h E(z)z^2 dz \end{cases} \tag{7}$$

where G_0 and m_0 are defined for each type of FGM. Deflection in the beam is assumed to be harmonic as:

$$w(x,t) = W(x)e^{i\omega t} \tag{8}$$

By inserting Equations (3) to (8) into Lagrange-Hamilton principle:

$$\begin{aligned} &\delta(T - U) = 0 \\ &\int_0^L m_1(x)\omega^2 W\,\delta W + G_1(x)\frac{d^2W}{dx^2}\frac{d^2\delta W}{dx^2}dx = 0 \end{aligned} \tag{9}$$

With integrating by parts, we have:

$$\begin{aligned} &\int_0^L \left[m_1(x)\omega^2 W\,\delta W + \frac{d^2}{dx^2}\left(G_1(x)\frac{d^2W}{dx^2}\right)\delta W \right]dx \\ &+ \left[G_1(x)\frac{d\,\delta W}{dx} - \frac{d\,G_1(x)}{dx}\delta W \right]_0^L = 0 \end{aligned} \tag{10}$$

δW is arbitrary in $0 < x < L$, so the terms inside the integral should be zero and the governing equation of motion is obtained as:

$$\begin{aligned} &G_0\frac{d^4W}{dx^4} + \eta G_0\frac{d^3W}{dx^3} + \eta^2 G_0\frac{d^2W}{dx^2} + \\ &m_0\omega^2 W = 0 \end{aligned} \tag{11}$$

It can be seen that neglecting the nonuniformity parameter in Equation (11) leads to the equation of motion of classical beam theory for FGM materials. This equation could be rewritten in nondimensional form as:

$$\begin{aligned} &\frac{d^4W}{dx^4} + \eta\frac{d^3W}{dx^3} + \eta^2\frac{d^2W}{dx^2} + \\ &\gamma^2 W = 0 \end{aligned} \tag{12}$$

in which γ^2 is the frequency parameter defined as

$$\gamma^2 = m_0\omega^2 / G_0 \tag{13}$$

3. SOLUTION PROCEDURE

In the present study, both ends of the beam is considered to be simply supported (SS), which the boundary conditions are:

$$\begin{aligned} \text{BC's}: \quad &W(0) = 0,\ W''(0) = 0, \\ &W(L) = 0,\ W''(L) = 0 \end{aligned} \tag{14}$$

In order to solve Equation (12) with the conditions given in Equation (14), the general form of solution is assumed as:

$$W(X) = C_1 e^{(\lambda_1 X)} + C_2 e^{(\lambda_2 X)} + C_3 e^{(\lambda_3 X)} + e^{(\lambda_4 X)} \tag{15}$$

where $\lambda_1$to λ_4 are depended on ω and defined as:

$$\begin{cases} \lambda_1 = -\frac{b}{4a} - S + \frac{1}{2}\sqrt{-4S^2 - 2p + \frac{q}{S}} \\ \lambda_2 = -\frac{b}{4a} - S - \frac{1}{2}\sqrt{-4S^2 - 2p + \frac{q}{S}} \\ \lambda_3 = -\frac{b}{4a} + S + \frac{1}{2}\sqrt{-4S^2 - 2p - \frac{q}{S}} \\ \lambda_4 = -\frac{b}{4a} + S - \frac{1}{2}\sqrt{-4S^2 - 2p - \frac{q}{S}} \end{cases} \tag{16}$$

where p, q, S and Q are:

$$\begin{cases} p = \dfrac{8ac-3b^2}{8a^2} \\ q = \dfrac{b^3-4abc+8a^2d}{8a^3} \\ S = \dfrac{1}{2}\sqrt{-\dfrac{2}{3}p+\dfrac{1}{3a}\left(Q+\dfrac{\Delta_0}{Q}\right)} \\ Q = \sqrt[3]{\dfrac{\Delta_1+\sqrt{\Delta_1^{\,2}-4\Delta_0^{\,3}}}{2}} \end{cases} \tag{17}$$

Also, Δ_0 and Δ_1 are defined as:

$$\begin{cases} \Delta_0 = c^2 - 3bd + 12ae \\ \Delta_1 = 2c^3 - 9bcd + 27b^2e + 27ad^2 - 72ace \end{cases} \tag{18}$$

a, b, c, d and e are the coefficients of quartic function. Applying boundary conditions in this case leads to an equation for the determination of the natural frequency. The natural frequency equation and the coefficients C_1 to C_3 are given below for simply-supported boundary condition case.

$$W(0)=0: \quad C_3 = -1 - C_1 - C_2 \tag{19}$$

$$W''(0)=0: \quad C_2 = -\frac{\lambda_1^2 C_1 - \lambda_3^2 C_1 - \lambda_3^2 + \lambda_4^2}{\lambda_2^2 - \lambda_3^2} \tag{20}$$

$$W''(1)=0: \quad C_1 = -K_{11}/K_{12} \tag{21}$$

By using the last boundary condition and applying C1, C2 and C3 from the above equations we get:

$$W^{''(1)}=0: \ \frac{e^{\lambda_1}K_{11}}{K_{12}} - \frac{e^{\lambda_2}}{\lambda_2^2-\lambda_3^2}\left(\frac{\left(\lambda_1^2-\lambda_3^2\right)K_{11}}{K_{12}}+\left(\lambda_4^2-\lambda_3^2\right)\right) + e^{\lambda_3}\left(-1-\frac{K_{11}}{K_{12}}+\frac{(\lambda_1^2+\lambda_3^2)}{\lambda_2^2-\lambda_3^2}\left(\frac{K_{11}}{K_{12}}\right)-\frac{\lambda_3^2-\lambda_4^2}{\lambda_2^2-\lambda_3^2}\right) + e^{\lambda_4}=0 \tag{22}$$

where K_{11} and K_{12} are defined as:

$$K_{11} = \lambda_2^2\lambda_3^2\left(e^{\lambda_2}-e^{\lambda_3}\right)+\lambda_2^2\lambda_4^2\left(e^{\lambda_4}-e^{\lambda_2}\right) + \lambda_3^2\lambda_4^2\left(e^{\lambda_3}-e^{\lambda_4}\right) \tag{23}$$

$$K_{12} = \lambda_1^2\lambda_2^2\left(e^{\lambda_1}-e^{\lambda_2}\right)+\lambda_1^2\lambda_3^2\left(e^{\lambda_3}-e^{\lambda_1}\right) + \lambda_2^2\lambda_3^2\left(e^{\lambda_2}-e^{\lambda_3}\right) \tag{24}$$

4. RESULTS AND DISCUSSIONS

In this study, analytical solution for free vibration of nonuniform FGM beams is presented. to verify our procedure in solving the problem, results are achieved for homogenous material model by having $P = 0$ in power law function materials which leads to an isotropic material with $E = E_b$and $\rho \ = \rho_b$ and in the other way by having $E_t = E_b$ and $\rho_t \ = \rho_b$, power law function models could become a homogenous model and at last by having $\emptyset = 0$ for exponential model, isotropic material model is achieved. These results are calculated for isotropic nonuniform beams for $P = 0$ and $\emptyset = 0$ which is presented in Tables 1 and 2 for $|\eta| = 1$ and compared with those presented by Cem Ece [9] which are in excellent agreement and shows the verification of current study.

In order to clarify the effects of nonuniformity in different kinds of FGM materials, Al/Al_2O_3 composite is selected for modeling and it is assumed that the property of FGM materials in Power law functional type is: $E_{tP} = 380Gpa$,$E_{bP} = 70Gpa$, $\rho_{tP} = 3{,}800\ kg/m^3$, $\rho_{bP} = 2{,}707\ kg/m^3$ also for exponential model of FGM materials:$E_{bE} = 70Gpa$, $\rho_{bE} = 2{,}702\ kg/m^3$ [13]. Calculation is done for different values of nonuniformity parameters as: $|\eta| = 0\ to\ 2.5$.

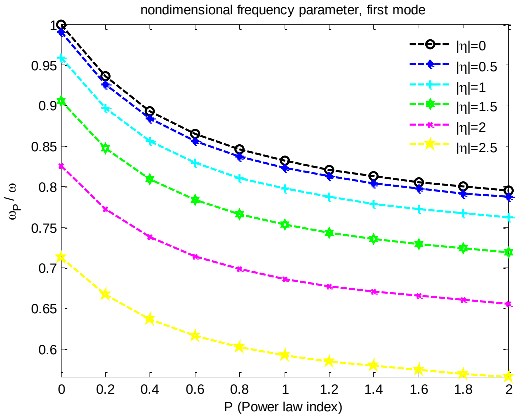
Results for Power law model for first three modes of vibration are shown in Figure 2. It can be seen that for all power law FGM beams, increasing the nonuniformity parameter leads to a smaller natural frequencies in all first three modes.

TABLE 1. Natural frequency parameters for a simply supported beam with exponentially width variation $\eta = 1$

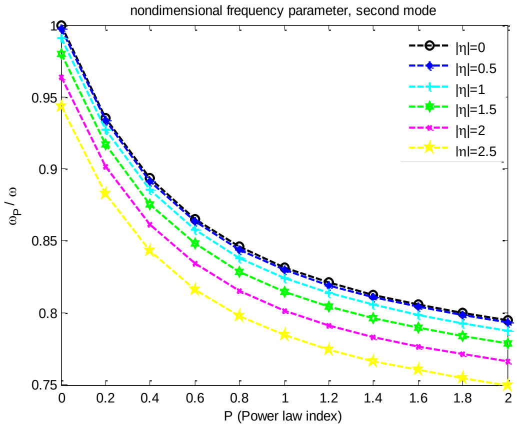
Mode Number	Natural frequencies ($\eta = 1$)		
	Power Law (P=0)	Exponential (Φ=0)	Ref [9]
1	3.12618	3.12618	3.1262
2	6.29045	6.29046	6.2905
3	9.42477	9.42477	9.4248
4	12.56640	12.56639	12.5664
5	15.70802	15.70802	15.7080

TABLE 2. Natural frequency parameters for a simply supported beam with exponentially width variation $\eta = 2$

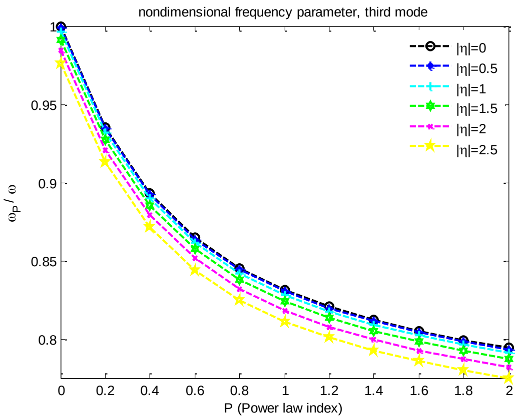
Mode Number	Natural frequencies ($\eta = 1$)		
	Power Law (P=0)	Exponential (Φ=0)	Ref [9]
1	3.08008	3.08008	3.0801
2	6.31288	6.31288	6.3129
3	9.45539	9.45540	9.4554
4	12.59354	12.59354	12.5935
5	15.73172	15.73171	15.7317



(a)



(b)

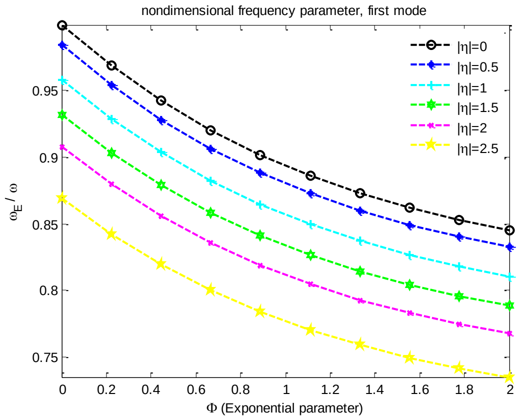


(c)

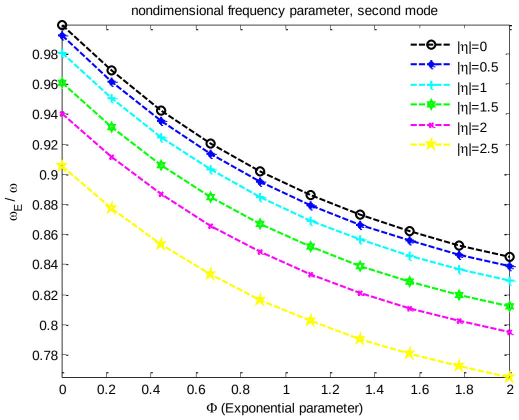
Figure 2. Nondimensional frequency parameter with respect to variation in nonuniformity and power law parameters (a) First mode (b) Second mode (c) Third mode

Also, increasing the power law index causes a smaller amount for natural frequency terms. For power law FGM materials, varying the power law index has a significant effect on frequency parameters. Also, for the nonuniformity parameter of the beam, it has its most effect on the first frequency term as shown in Figure 2.

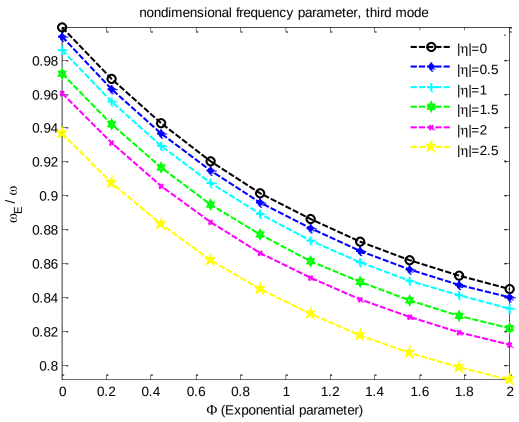
In the same way, for exponential varying FGM materials, nondimensional frequency parameter is presented for different number of exponential and nonuniformity parameters in Figure 3. It can be seen that in addition to the exponential parameter, nonuniformity parameter plays a great role in changing the frequency in exponential varying FGM beams.



(a)



(b)



(c)

Figure 3. Nondimensional frequency parameter with respect to variation in nonuniformity and exponential parameters (a) First mode (b) Second mode (c) Third mode

Frequency parameter decreases by increasing the nonuniform and exponential terms. Nonuniformity has the strongest effect in the first frequency mode and this effect decreases in higher modes of frequency.

5. CONCLUSION

In this study, by applying nonuniformity to FGM beams, free vibration analysis of this type of beams have been investigated. FGM materials were modeled after two type of power law and exponential functions and nonuniformity is assumed to be in the width direction by having exponential variation as a function of longitudinal direction. Also, material variation was assumed to be in the thickness direction of the beam. A general analytical solution for free vibration analysis of a nonuniform FGM beam is presented. Equations are solved with the focus on Al/Al_2O_3 composites for different type of material variation and nonuniformity. It is shown that nonuniformity in FGM beams has a significant effect in variating the frequency parameter for both type of power law and exponential FGM beams.

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چکیده

در این مقاله رفتار دینامیکی تیرهای هوشمند با سطح مقطع متغیر مورد بررسی قرار گرفته است. تغییرات در جنس تیر، در راستای ضخامت آن فرض شده و از دو مدل تغییرات توانی و نمایی مدل‌سازی استفاده شده است. همچنین، تغییرات سطح مقطع به شکل نمایی در راستای طولی باضخامت ثابت فرض شده است. معادلات حرکتی برای تیر اویلر به کمک روش همیلتون به دست آمده و حل تحلیلی برای آن ارائه شده است. در این تحقیق نشان داده‌شده است که تغییرات ایجاد شده در سطح مقطع تیرهای هوشمند، تأثیر بسزایی بر رفتار دینامیکی آن‌ها دارد. جهت درک بهتر این تأثیرات، از تیر با جنس Al/Al_2O_3 جهت بررسی بیشتر و مدل‌سازی تیر هوشمند استفاده شده است و تغییرات در جنس ماده در هر دو حالت نمایی و توانی انجام شده است. پس از انجام محاسبات، سه بسامد اول ارتعاشی تیر هوشمند اویلر برای هردو مدل مواد هدفمند با در نظرگیری سطح مقطع متغیر ارائه‌شده است. نتایج نشان از وابستگی بسامدهای ارتعاشی تیرهای هوشمند به تغییرات ایجاد شده در عبارت‌های هندسی سطح مقطع و نوع تغییر جنس ماده هوشمند در تیر است.

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