Response of Secondary Systems Subjected to Multicomponent Earthquake Input

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ABSTRACT: *A formulation for the response of the secondary systems subjected to multicomponent earthquake acceleration has been developed, using the random vibration theory. The method accounts for interaction between the primary and the secondary systems as well as the nonproportionality of the combined primary-seondary system damping. The required formulations for the calculation of the autocorrelation function, the power spectral density function, the response spectrum and the critical angle have been obtained. The formulation has been arranged in such a way that the floor response spectrum can be calculated directly from the earthquake response spectra of multicomponent input. The floor response spectra of torsional frames subjected to average response spectrum of 20 earthquake records of Iran have been calculated. Variations of the spectra to various structure parameters such as eccentricity, mass ratio, and nonproportional damping have been studied. Results show that for large eccentricities the effect of multicomponentness of earthquake becomes important and can not be neglected.*

Keywords: Secondary Systems; Multicomponent earthquake; Interaction; Nonproportional damping; Mass ratio; Tunning

1. Introduction

For the modal analysis of important secondary systems, it is common to generate the floor spectrum. The floor spectrum defines the maximum absolute acceleration response of a series of single degree of freedom systems with different natural frequencies and damping ratios, which are attached to the floor under consideration. In the random vibration based method, the spectral moments of the response of floor is determined, and the floor spectrum is calculated by multiplying its mean square by the appropriate peak factor. Singh [1] used this method and calculated the floor spectrum directly from the design spectrum. In his method the input of the secondary system was the response of the primary system, so the interaction between the two systems was neglected, and it was called decoupled (or cascaded) analysis.

Although the decoupled analysis is acceptable in most cases, but there are situations which results to significant overestimation, particularly when the mass ratio is not too small, and the frequency of the secondary system is tuned to one of the predominant frequencies of the primary system. To incorporate the effect of the dynamic interaction between the two systems in the seismic analysis, the secondary system can be considered as a part of the whole structure and analyzed the combined primary-secondary system by a conventional method. However, for light secondary systems, the mass, damping, and stiffness matrices of the combined system will have elements with much smaller magnitude, which can cause numerical instability in the dynamic analysis, resulting in the numerical errors in the eigenvalue problem and consequently on the response of the combined system. Suarez and Singh [2] attempted to overcome this shortcoming by presenting an exact approach to determine the frequencies and mode shapes of the combined system. Because the mass of the secondary system is usually small in comparison to the primary system, it changes the dynamic properties of the combined system slightly, so the perturbation method become convenient to calculate the dynamic properties and response of the combined system [3, 4, 5, 6]. Gupta and Tembulkar [7] studied the changes in both the frequencies and the response of the primary and secondary systems due to decoupling.

Another problem in the analysis of the combined system is due to the different damping characteristic of the primary and secondary systems, which results to the nonproportionality of the damping matrix. Thus the analytical methods for the combined system must be able to consider this effect. Direct and exact method for calculating eigenvalues and eigenvectors of nonproportionally damped systems, according to the state vector approach, has been developed by Foss [8], and after that has been used by many researchers [9, 10]. Recently, new approach has been developed by the first author, which substantially reduces computational time [11].

In the seismic analysis of the secondary systems, earthquake motion is commonly idealized as having a single horizontal component, while in the actual case there exist six correlated components (three translational and three rotational), which are felt by structure [12]. The rotational components can be expressed in terms of the spatial derivatives of the translational components [13, 14]. Three translational components of earthquake are generally correlated, but it can be found an orthogonal set of axis such that the translational components are independent along these axes. Kubo and Penzien [15] called these axes, the principal axes of earthquake. Wilson and Button [16] assumed that the two horizontal spectrum of translational acceleration component of earthquake are linearly dependent and proposed a complete quadratic combination method for evaluating the response of structure under these components. Ghafory-Ashtiany and Singh [14] considered all of the six components and calculated the response of structure using the random vibration theory. Lopez and Torres [17] considered structure as three-dimensional frame with rigid floor that subjected to the two horizontal component of the earthquake, and determined the critical angle and the maximum response of the structure.

A review of the existing literature shows that little attention has been directed toward the effect of torsion of the primary system on the response of the secondary systems. Yang and Huang [18] presented a complete quadratic combination rule to calculate the response of the secondary systems, which are attached to the torsional buildings, as well as the effect of the base isolation system [19]. Bernal [20] presented an example of a secondary system attached to a torsional building.

In this paper the autocorrelation function, the power spectral density function, and the floor spectrum of the secondary systems, for which their primary systems subjected to multicomponent earthquake have been formulated. In the presented method, the interaction between the two systems and the nonproportionality of damping of the combined system is considered. The input of the primary system can be in the form of time history with correlated horizontal and vertical components of the earthquake, power spectral density function or response spectrum of ground accelerations. However, for the practical purposes, the formulations are arranged such that the floor spectrum can be calculated directly from the design spectrum at the input level. Also the critical angle, i.e. the angle between the axis of structure and the principal axes of earthquake that produce maximum floor spectrum, is calculated. In order to illustrate the applicability of the method, torsional frames are considered and parameters such as eccentricity, interaction, mass ratio, and elevation of the secondary system from ground, as well as nonproportional damping effect have been studied.

To simplify the complex formulation for design purposes, variation or sensitivity of each terms in the formulation for different structural parameters have been studied. The results of this study have been presented in reference [21].

2. Formulation

Consider an N-degrees-of-freedom primary system with a mass $[M_p]$, damping $[C_p]$, and stiffness matrix $[K_p]$, attached by a single degree of freedom secondary system with a mass (M_s), a damping (C_s), and a stiffness (K_s), to its m^{th} degree of freedom, which have been subjected to a multicomponent ground excitation, $\{\ddot{x}'_g(t)\}$. The equations of motion of the combined system become:

$$[M]\{\ddot{Y}\}+[C]\{\dot{Y}\}+[K]\{Y\}=-\begin{bmatrix}[M_p] & [r_p] \\ & M_s[r_s]\end{bmatrix}\{\ddot{x}'_g(t)\} \qquad (1)$$

Where $\{Y\}$= relative displacement response of combined system and

$$[M]=\begin{bmatrix}[M_p] & \{0\}\\ [0] & M_s\end{bmatrix} \tag{2}$$

$$[K]=\begin{bmatrix}[K_p] & \{0\}\\ [0] & 0\end{bmatrix}+[K_c] \tag{3}$$

$$[C]=\begin{bmatrix}[C_p] & \{0\}\\ [0] & 0\end{bmatrix}+[C_c] \tag{4}$$

in which $[K_c]$ and $[C_c]$ are the coupling matrices associated with the stiffness and damping matrices, respectively, contain the stiffness and damping coefficient of the secondary system in the m^{th} and $N+1^{th}$ element.

$$[K_c]=\begin{bmatrix}0 & 0 & \cdots & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & K_s & \cdots & 0 & -K_s\\ \vdots & \vdots & & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & -K_s & \cdots & 0 & K_s\end{bmatrix} \tag{5}$$

$$[C_c]=\begin{bmatrix}0 & 0 & \cdots & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & C_s & \cdots & 0 & -C_s\\ \vdots & \vdots & & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & -C_s & \cdots & 0 & C_s\end{bmatrix} \tag{6}$$

In the present formulation it is assumed that the earthquake has three translational correlated components. Each row of $\{\ddot{x}'_g(t)\}$ corresponds to an acceleration component of earthquake along the principal axes of the primary system, two horizontal and one vertical axes [14, 15]. Also in Eq. (1), $[r_p]$ and $[r_s]$ are the displacement influence matrices (matrices whose elements are the displacements of degrees-of-freedom of the primary and the secondary system, respectively, due to a unit static displacement of the base of the structure in the directions of earthquake) of the primary and secondary systems. These matrices have three columns, each column corresponds to one component of the earthquake.

The eigenvalue problem of Eq. (1) in the case of classical damped system is:

$$([K]+\omega^2[M])[\phi]=[0] \tag{7}$$

where ω and $[\phi]$ are the frequency and mode shape of the combined system. For the light secondary systems, the elements of $[K]$ and $[M]$ matrices are not of the same order, which might cause numerical inaccuracy. To overcome this shortcoming the following transformation has been introduced [22]:

$$[\varphi]=[U][\overline{\varphi}] \tag{8}$$

where

$$[U]=\begin{bmatrix}[\varphi_p] & \{0\}\\ [0] & \varphi_s\end{bmatrix} \tag{9}$$

in which $[\phi_p]$ is the mode shape of the primary system normalized with respect to its mass matrix and $\phi_s=1/\sqrt{M_s}$. Substitution of Eq. (8) into Eq. (7) gives:

$$\left([\overline{K}]-\omega^2[\overline{M}]\right)[\overline{\varphi}]=[0] \tag{10}$$

where

$$[\overline{M}]=[U]^T[M][U] \tag{11}$$

$$[\overline{K}]=[U]^T[K][U] \tag{12}$$

The numerical inaccuracy of Eq. (7) has been eliminated through the above mentioned transformation, since all diagonal elements of the matrices in Eq. (10) are of the same order. However, two sets of eigenvalue problems should be solved. First for the primary system, and second the eigenvalue Eq. (10).

Once the $N+1$ eigenvalues and eigenvectors of the combined system are obtained, the equation of motion, Eq. (1), can be solved using the normal mode approach with the help of the following standard transformation:

$$\{Y\}=[\phi]\{V\} \tag{13}$$

where $\{V\}$ is the vector of principal coordinates. Since $[\phi]$ is the normal mode shape of the system, only the mass and the stiffness matrices could be decoupled. Considering that the damping characteristic of the primary and secondary systems are different, the nonproportionality of damping matrix is an inherent property of the coupled system. The nonproportionality effect is particularly important for the primary-secondary systems in tuning or nearly tuning, with small values of mass ratio and large differences in their damping constants. Substituting Eq. (13) into Eq. (1) and premultiplying by $[\phi_i]$ leads to Eq. (14):

$$\ddot{V}_j+\sum_{i=1}^{N+1}C_{ij}\dot{V}_i+\omega_j^2V_j=-[\gamma_j]\{\ddot{x}'_g\}\,;\quad j=1,2,\ldots,N+1 \tag{14}$$

where

$$C_{ij} = \{\phi_i\}^T [C] \{\phi_j\} \tag{15}$$

and

$$[\gamma_j] = \{\varphi_j\}^T \begin{bmatrix} [M_p][r_p] \\ M_s[r_s] \end{bmatrix} \tag{16}$$

Eq. (14) is a coupled equation which can be decoupled by using the state vector approach. The state vector is defined as:

$$\{Z\} = \begin{Bmatrix} \{\dot{V}\} \\ \{V\} \end{Bmatrix} \tag{17}$$

By using Eq. (17), Eq. (14) can be written as:

$$[A]\{\dot{Z}\} + [B]\{Z\} = -[Q]\begin{bmatrix}[0] \\ [\gamma]\end{bmatrix}\{\ddot{x}_g'\} \tag{18}$$

where

$$[A] = \begin{bmatrix} [0] & [I] \\ [I] & [C] \end{bmatrix} \quad [B] = \begin{bmatrix} -[I] & [0] \\ [0] & [O^2] \end{bmatrix} \quad [Q] = \begin{bmatrix} [0] & [0] \\ [0] & [I] \end{bmatrix} \tag{19}$$

where $[\Omega^2]$ is a diagonal matrix of combined frequencies, and $[I]$ is the unit matrix of order $N+1$. To solve Eq. (18) by modal analysis, one should find its eigenvalues and eigenvectors. The eigenvalue problem of the Eq. (18) is:

$$(\lambda[A] + [B])[\psi] = [0] \tag{20}$$

Eq. (20) gives $2(N+1)$ eigenvalues (λ) and corresponding eigenvectors $[\psi]$, which occur in the pairs of complex and conjugate, due to negative-definiteness of matrix $[B]$. Using the expansion theorem, the vector $\{Z\}$ is written as:

$$\{Z\} = [\psi]\{x\} \tag{21}$$

where $\{Z\}$ is a vector of complex principal coordinates. This vector is obtained by solving the Eq. (22):

$$\dot{x}_a - \lambda_a x_a = [F_a]\{\ddot{x}_g'\}; \qquad a = 1, 2, \ldots, 2(N+1) \tag{22}$$

where

$$[F_a] = -\left(\{\psi_a\}^T[Q]\begin{bmatrix}[0] \\ [\gamma]\end{bmatrix}\right) \Big/ \left(\{\psi_a\}^T[A]\{\psi_a\}\right) \tag{23}$$

in which F_{al} corresponds to direction l of the earthquake.

The main response of the secondary system, which is used for the generation of the floor spectrum, is the absolute acceleration $(\ddot{U}_s)$ which can be written in terms of its relative acceleration $(\ddot{Y}_s)$ and the ground acceleration component affecting it as:

$$\ddot{U}_s = \ddot{Y}_s + [r_s]\{\ddot{x}_g'\} \tag{24}$$

Using Eq. (13), the relative acceleration of the secondary system can be written as:

$$\ddot{Y}_s = \sum_{j=1}^{N+1} \varphi_{N+1,j} \ddot{V}_j \tag{25}$$

Also by using Eqs. (17) and (21), we have:

$$\ddot{V}_j = \dot{Z}_j = \sum_{a=1}^{2(N+1)} \psi_{ja} \dot{x}_a \tag{26}$$

Obtaining $\dot{x}_a$ from Eq. (22) and substituting it into Eqs. (26), (25) and (24), $\ddot{U}_s$ will become:

$$\ddot{U}_s = \sum_{j=1}^{N+1} \varphi_{N+1,j}\left[\sum_{a=1}^{2(N+1)} \psi_{ja}\lambda_a x_a(t)\right] + \left([r_s]\{\ddot{x}_g'\} + \sum_{j=1}^{N+1}\varphi_{N+1,j}\left[\sum_{a=1}^{2(N+1)}\psi_{ja}[F_a]\{\ddot{x}_g'\}\right]\right) \tag{27}$$

Considering that the second terms in the right hand side of Eq. (27) is zero, see Appendix (I), the absolute acceleration of the secondary system will become:

$$\ddot{U}_s(t) = \sum_{j=1}^{N+1}\varphi_{N+1,j}\left[\sum_{a=1}^{N+1}\left(\psi_{ja}\lambda_a x_a(t) + \psi_{ja}^*\lambda_a^* x_a^*(t)\right)\right] \tag{28}$$

where superscript (*) means complex conjugate. Eq. (28) forms the basis for the generation of floor response spectrum. This will be used to calculate the autocorrelation and power spectral density function of the floor acceleration, which in turn is required to define floor spectrum.

The autocorrelation function of the absolute acceleration of the secondary system can be obtained from Eq. (28) as:

$$\begin{aligned} E\left[\ddot{U}_s(t_1)\ddot{U}_s(t_2)\right] = \sum_{j=1}^{N+1}\sum_{k=1}^{N+1}\varphi_{N+1,j}\varphi_{N+1,k} \times \\ \left\{\sum_{a=1}^{N+1}\sum_{b=1}^{N+1}\psi_{ja}\lambda_a\psi_{kb}\lambda_b E\left[x_a(t_1)x_b(t_2)\right] + \right. \\ \sum_{a=1}^{N+1}\sum_{b=1}^{N+1}\psi_{ja}^*\lambda_a^*\psi_{kb}^*\lambda_b^* E\left[x_a^*(t_1)x_b^*(t_2)\right] + \\ \sum_{a=1}^{N+1}\sum_{b=1}^{N+1}\psi_{ja}\lambda_a\psi_{kb}^*\lambda_b^* E\left[x_a(t_1)x_b^*(t_2)\right] + \\ \left.\sum_{a=1}^{N+1}\sum_{b=1}^{N+1}\psi_{ja}\lambda_a\psi_{kb}^*\lambda_b^* E\left[x_a(t_1)x_b^*(t_2)\right]\right\} \end{aligned} \tag{29}$$

In this equation, there are four expected values which can be obtained in terms of autocorrelation or power spectral density function of ground acceleration. Here, the first expected value in Eq. (29) will be given as an example.

To obtain the autocorrelation function x_a, first we need to solve Eq. (22) for x_a in terms of Duhamel integral.

$$x_a(t) = \sum_{l=1}^{3} F_{al} \int_0^t \ddot{x}'_{gl}(\tau) e^{\lambda_a (t-\tau)} d\tau \qquad (30)$$

Using Eq. (30), $E[x_a(t_1)\, x_b(t_2)]$ becomes:

$$E\left[x_a(t_1) x_b(t_1)\right] = \sum_{l=1}^{3}\sum_{n=1}^{3} F_{al} F_{bn} \int_0^{t_1}\int_0^{t_2} \times E\left[\ddot{x}'_{gl}(t_1)\ddot{x}'_{gn}(t_2)\right] e^{\lambda_a (t_1 - t_1)} e^{\lambda_b (t_2 - t_2)} d\tau_1\, d\tau_2 \qquad (31)$$

The relationship between the correlated accelerations of earthquake along the principal axes of primary system, $\{\ddot{x}'_g\}$, and the uncorrelated accelerations along the principal axes of earthquake, $\{\ddot{x}_g\}$, is related by the cosine direction matrix, $[D]$, as [15]:

$$\{\ddot{x}'_g\} = [D]\{\ddot{x}_g\} \qquad (32)$$

where the acceleration cross correlation function can be written as:

$$E\left[\ddot{x}'_{gl}(\mathrm{t}_1)\, \ddot{x}'_{gn}(\mathrm{t}_2)\right] = \sum_{p=1}^{3}\sum_{q=1}^{3} d_{lp} d_{nq} E\left[\ddot{x}_{gp}(\mathrm{t}_1)\, \ddot{x}_{gq}(\mathrm{t}_2)\right] \qquad (33)$$

in which p and q corresponds to the cosine direction of the principal axes of the earthquake. Since the cross correlation between accelerations along the principal axes of earthquake is zero, the Eq. (33) can be reduced to:

$$E\left[\ddot{x}'_{gl}(\mathrm{t}_1)\ddot{x}'_{gn}(\mathrm{t}_2)\right] = \sum_{p=1}^{3} d_{lp} d_{np} E\left[\ddot{x}_{gp}(\mathrm{t}_1)\, \ddot{x}_{gp}(\mathrm{t}_2)\right] \qquad (34)$$

Expressing the autocorrelation function of the ground accelerations in the Eq. (34), in terms of its power spectral density function, and substituting it into Eq. (33) and the result into Eq. (31), it becomes:

$$E\left[x_a(t_1) x_b(t_2)\right] = \sum_{l=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3} F_{al} F_{bn} d_{lp} d_{np} \times \int_0^{t_1}\int_0^{t_2}\int_{-\infty}^{+\infty} S_p(\omega) e^{\lambda_a (t_1 - t_1)} e^{\lambda_b (t_2 - t_2)} e^{i\omega(t_1 - t_2)} d\mathrm{t}_1 d\mathrm{t}_2\, d\omega \qquad (35)$$

Where $S_p(\omega)$ is the power spectral density function of the ground accelerations along the p^{th} principal axes of earthquake. For the stationary processes, the limits of integrals in Eq. (35) can be extended to $-\infty$ and $+\infty$, and after some algebric calculation, the following is obtained:

$$E\left[x_a(t_1) x_b(t_2)\right] = \sum_{l=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3} F_{al} F_{bn} d_{lp} d_{np} \times \int_{-\infty}^{+\infty} \frac{S_p(\omega) e^{i\omega(t_1 - t_2)}}{(\lambda_a - i\omega)(\lambda_b + i\omega)} d\omega \qquad (36)$$

Eq. (36) is the first used in the Eq. (29). The other expected values, which can be obtained in similar manner, are as follows:

$$E\left[x_a^*(t_1) x_b^*(t_2)\right] = \sum_{l=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3} F_{al}^* F_{bn}^* d_{lp} d_{np} \times \int_{-\infty}^{+\infty} \frac{S_p(\omega) e^{i\omega(t_1 - t_2)}}{(\lambda_a^* - i\omega)(\lambda_b^* + i\omega)} d\omega \qquad (37)$$

$$E\left[x_a(t_1) x_b^*(t_2)\right] = \sum_{l=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3} F_{al} F_{bn}^* d_{lp} d_{np} \times \int_{-\infty}^{+\infty} \frac{S_p(\omega) e^{i\omega(t_1 - t_2)}}{(\lambda_a - i\omega)(\lambda_b^* + i\omega)} d\omega \qquad (38)$$

$$E\left[x_a^*(t_1) x_b(t_2)\right] = \sum_{l=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3} F_{al}^* F_{bn} d_{lp} d_{np} \times \int_{-\infty}^{+\infty} \frac{S_p(\omega) e^{i\omega(t_1 - t_2)}}{(\lambda_a^* - i\omega)(\lambda_b + i\omega)} d\omega \qquad (39)$$

Substituting Eqs. (36) to (39) into Eq. (29), the total acceleration autocorrelation function becomes:

$$E\left[\ddot{U}_s(t_1)\ddot{U}_s(t_2)\right] = \sum_{j=1}^{N+1}\sum_{k=1}^{N+1} \varphi_{N+1,j}\varphi_{N+1,k} \times \sum_{a=1}^{N+1}\sum_{b=1}^{N+1}\left\{\sum_{l=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3} d_{lp} d_{np}\left[\int_{-\infty}^{+\infty}\left(\frac{\lambda_a \psi_{ja} F_{al} \lambda_b \psi_{kb} F_{bn}}{(\lambda_a - i\omega)(\lambda_b + i\omega)} + \frac{\lambda_a^* \psi_{ja}^* F_{al}^* \lambda_b^* \psi_{kb}^* F_{bn}^*}{(\lambda_a^* - i\omega)(\lambda_b^* + i\omega)} + \frac{\lambda_a \psi_{ja} F_{al} \lambda_b^* \psi_{kb}^* F_{bn}^*}{(\lambda_a - i\omega)(\lambda_b^* + i\omega)} + \frac{\lambda_a^* \psi_{ja}^* F_{al}^* \lambda_b \psi_{kb} F_{bn}}{(\lambda_a^* - i\omega)(\lambda_b + i\omega)}\right) S_p(\omega) e^{i\omega(t_1 - t_2)} d\omega\right]\right\} \qquad (40)$$

With appropriate combination of the 1^{st} and 2^{nd} terms, and the 3^{rd} and 4^{th} terms, and after some algebraic simplification, Eq. (40) can be rewritten as:

$$E\left[\ddot{U}_s(t_1)\ddot{U}_s(t_2)\right] = \sum_{j=1}^{N+1}\sum_{k=1}^{N+1} \varphi_{N+1,j}\varphi_{N+1,k} \times \qquad (41)$$

$$\left\{\sum_{a=1}^{N+1}\left[\sum_{l=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3} d_{lp} d_{np} \int_{-\infty}^{+\infty}\left(A_{jkaln}\omega_a^4 + B_{jkaln}\omega_a^2\omega^2\right)\times |H_a|^2 S_p(\omega) e^{i\omega(t_1 - t_2} d?\right] + 2\sum_{a=1}^{N}\sum_{b=a+1}^{N+1}\left[\sum_{l=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3} d_{lp} d_{np} \times \int_{-\infty}^{+\infty}\left(\left(A_{jkabln}\omega_a^4 + B_{jkabln}\omega_a^2\omega^2\right)|H_a|^2 + \left(C_{jkabln}\omega_b^4 + D_{jkabln}\omega_b^2\omega^2\right)|H_b|^2\right) S_p(\omega) e^{i\omega(t_1 - t_2)} d\omega\right]\right\}$$

In which $|H_a|$ and $|H_b|$ are the complex frequency function of the modes "a" and "b" of the combined system, and the coefficients $A_{jkaln}\dots$, are defined in Appendix (II). Eq. (41) defines the autocorrelation function of the absolute acceleration of secondary system. The power spectral density function of this response will be:

$$S_{\ddot U_s}(\omega)=\sum_{j=1}^{N+1}\sum_{k=1}^{N+1}\varphi_{N+1,j}\,\varphi_{N+1,k}\left\{\sum_{a=1}^{N+1}\left[\sum_{l=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3}d_{lp}\,d_{np}\times\right.\right.$$
$$\left.(A_{jkaln}\omega_a^4+B_{jkaln}\omega_a^2\omega^2)|H_a|^2 S_p(\omega)\right]+$$
$$2\sum_{a=1}^{N}\sum_{b=a+1}^{N+1}\left[\sum_{l=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3}d_{lp}d_{np}\times\right. \tag{42}$$
$$\Big((A_{jkabln}\omega_a^4+B_{jkabln}\omega_a^2\omega^2)|H_a|^2+$$
$$\left.\left.(C_{jkabln}\omega_b^4+D_{jkabln}\omega_b^2\omega^2)|H_b|^2\Big)S_p(\omega)\right]\right\}$$

This equation defines the power spectral density function of the absolute acceleration of the secondary system in terms of the modal properties of the primary system, eigenproperties of the combined system, and the power spectral density function of the ground accelerations along the principal axes of earthquake.

For the stationary processes, the mean square response of the total acceleration of the secondary system can now be obtained through integration of the response power spectral density function, Eq. (42):

$$E[\ddot U_s^2]=\sum_{j=1}^{N+1}\sum_{k=1}^{N+1}\varphi_{N+1,j}\,\varphi_{N+1,k}\left\{\sum_{a=1}^{N+1}\left[\sum_{l=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3}d_{lp}\,d_{np}\times\right.\right.$$
$$\left.\left[(A_{jkaln}\omega_a^4 I_{0p}(\omega_a,\xi_a)+B_{jkaln}\omega_a^2 I_{1p}(\omega_a,\xi_a)\right]+$$
$$2\sum_{a=1}^{N}\sum_{b=a+1}^{N+1}\left[\sum_{l=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3}d_{lp}d_{np}\Big((A_{jkabln}\omega_a^4 I_{0p}(\omega_a,\xi_a)+\right. \tag{43}$$
$$B_{jkabln}\omega_a^2 I_{1p}(?_a,?_a)+C_{jkabln}\omega_b^4 I_{0p}(\omega_b,\xi_b)+$$
$$\left.\left.D_{jkabln}\omega_{bl}^2 I_{1p}(\omega_b,\xi_b)\Big)\right]\right\}$$

Where $I_{0p}(\omega_a,\ \xi_a)$ and $I_{1p}(\omega_a,\ \xi_a)$, respectively, represent the mean square values of the relative displacement and relative velocity response of an oscillator with frequency (ω_a) and damping ratio (ξ_a) excited by the ground motion in the p-direction. It is a good approximation to suppose that the mean square of the response are proportioned to the response spectrum [23]; i.e.:

$$I_{0p}(\omega_a,\xi_a)=\int_{-\infty}^{+\infty}S_p(\omega)\ \ d?\ =R_{dp}^2(\omega_a,\xi_a)/PF_d^2 \tag{44}$$

$$I_{0p}(\omega_a,\xi_a)=\int_{-\infty}^{+\infty}\omega^2 S_p(\omega)\ \ d\omega=R_{vp}^2(\omega_a,\xi_a)/PF_v^2 \tag{45}$$

In which $R_{dp}(\omega_a,\ \xi_a)$ and $R_{vp}(\omega_a,\ \xi_a)$, are respectively, relative displacement and relative velocity response spectrum of ground motions in p-direction, and PF_d and PF_v are the respective peak factors. Also, the root mean square values of the absolute acceleration of the secondary system when multiplied by the corresponding peak factor will results to its response spectrum:

$$R_{\ddot U_s}^2(\omega_s,\xi_s)=PF_{\ddot U_s}^2 E[\ddot U_s^2] \tag{46}$$

Where $R_{\ddot U_s}^2(\omega_s,\xi_s)$ defines the total acceleration floor spectra. Introducing Eqs. (43), (44), and (45) into Eq. (46), and assuming that all the peak factors are equal [24], the following result is obtained:

$$R_{\ddot U_s}^2(\omega_s,\xi_s)=\sum_{j=1}^{N+1}\sum_{k=1}^{N+1}\varphi_{N+1,j}\,\varphi_{N+1,k}\left\{\sum_{a=1}^{N+1}\left[\sum_{l=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3}d_{lp}\,d_{np}\times\right.\right.$$
$$\left.\left[(A_{jkaln}+B_{jkaln})R_{ap}^2(\omega_a,\xi_a)\right]+ \tag{47}$$
$$2\sum_{a=1}^{N}\sum_{b=a+1}^{N+1}\left[\sum_{l=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3}d_{lp}d_{np}\Big((A_{jkabln}+B_{jkabln})\times\right.$$
$$\left.\left.R_{ap}^2(\omega_a,\xi_a)+(C_{jkabln}+D_{jkabln})R_{ap}^2(\omega_b,\xi_b)\Big)\right]\right\}$$

Where $R_{dp}(\omega_a,\ \xi_a)$ is the pseudo-acceleration response spectrum of ground motions in p-direction. Eq. (47) gives the relation for calculating floor spectrum in terms of the cosine direction matrix [D] between the principal axes of earthquake and the principal axes of the primary system. That is, if the elements of this matrix are known, one can calculate the floor spectrum. It is known that for most of the tectonic regions, the ground motion can act along any horizontal direction; therefore, this implies the existence of a possible different direction of seismic incidence that would lead to an increase of floor spectrum. Thus for important secondary systems, the maximum response associated to the most critical directions of ground motions must be examined.

The presented formulation is general and can be applied to any structure and any direction of earthquake, but here for avoiding the complexity of the formulation, the primary system is restricted to torsional framed building with rigid floor that have two perpendicular horizontal and one vertical degree of freedom. Also, the earthquake is considered to have two horizontal and one vertical component. For this case the direction cosine matrix will become:

$$[D] = \begin{bmatrix} cos(\theta) & sin(\theta) & 0 \\ -sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{48}$$

where θ is the horizontal angle between the principal axes of earthquake and that of structure. Substituting the elements of matrix $[D]$ in Eq. (47) gives:

$$R^2_{\ddot{U}_s}(\omega_s,\xi_s) = R_c cos^2(\theta) + R_s sin^2(\theta) + R_{cs} cos(\theta) sin(\theta) + R_0 \tag{49}$$

where

$$R_c = \sum_{j=1}^{N+1}\sum_{k=1}^{N+1}\varphi_{N+1,j}\varphi_{N+1,k}\left\{\sum_{a=1}^{N+1}\left[(A_{jka11}+B_{jka11})\times S^2_{a1}(\omega_a,\xi_a) + (A_{jka22}+B_{jka22})\,S^2_{a2}(\omega_a,\xi_a)\right] + 2\sum_{a=1}^{N}\sum_{b=a+1}^{N+1}\left[(A_{jkab11}+B_{jkab11})\,S^2_{a1}(\omega_a,\xi_a) + (C_{jkab11}+D_{jkab11})\,S^2_{a1}(\omega_b,\xi_b) + (A_{jkab22}+B_{jkab22})\,S^2_{a2}(\omega_a,\xi_a) + (C_{jkab22}+D_{jkab22})\,S^2_{a2}(\omega_b,\xi_b)\right]\right\} \tag{50}$$

$$R_s = \sum_{j=1}^{N+1}\sum_{k=1}^{N+1}\varphi_{N+1,j}\varphi_{N+1,k}\left\{\sum_{a=1}^{N}\left[(A_{jka11}+B_{jka11})\times S^2_{a2}(\omega_a,\xi_a) + (A_{jka22}+B_{jka22})\,S^2_{a1}(\omega_a,\xi_a)\right] + 2\sum_{a=1}^{N}\sum_{b=a+1}^{N+1}\left[(A_{jkab11}+B_{jkab11})\,S^2_{a2}(\omega_a,\xi_a) + (C_{jkab11}+D_{jkab11})\,S^2_{a2}(\omega_b,\xi_b) + (A_{jkab22}+B_{jkab22})\,S^2_{a2}(\omega_a,\xi_a) + (C_{jkab22}+D_{jkab22})\,S^2_{a1}(\omega_b,\xi_b)\right]\right\} \tag{51}$$

$$R_{cs} = \sum_{j=1}^{N+1}\sum_{k=1}^{N+1}\varphi_{N+1,j}\varphi_{N+1,k}\left\{\sum_{a=1}^{N}\left[(A_{jka12}+B_{jka12} + A_{jka12}+B_{jka12})\left(S^2_{a2}(\omega_a,\xi_a) - S^2_{a1}(\omega_a,\xi_a)\right)\right] + 2\sum_{a=1}^{N}\sum_{b=a+1}^{N+1}\left[(A_{jkab12}+B_{jkab12}+A_{jkab21}+B_{jkab21})\times \left(S^2_{a2}(\omega_a,\xi_a) - S^2_{a1}(\omega_a,\xi_a)\right) + (C_{jkab12}+D_{jkab12}+C_{jkab21}+D_{jkab21})\times \left(S^2_{a2}(\omega_b,\xi_b) - S^2_{a1}(\omega_b,\xi_b)\right)\right]\right\} \tag{52}$$

$$R_0 = \sum_{j=1}^{N+1}\sum_{k=1}^{N+1}\varphi_{N+1,j}\varphi_{N+1,k}\left\{\sum_{a=1}^{N}\left[(A_{jka33}+B_{jka33})\times S^2_{a3}(\omega_a,\xi_a)\right] + 2\sum_{a=1}^{N}\sum_{b=a+1}^{N+1}\left[(A_{jkab33}+B_{jkab33})\times S^2_{a3}(\omega_a,\xi_a) + (C_{jkab33}+D_{jkab33})\,S^2_{a3}(\omega_b,\xi_b)\right]\right\} \tag{53}$$

Eq. (49) gives the floor spectrum as a function of the angle of incidence (θ). The critical angle (θ_{cr}) is defined as the angle of incidence that causes the maximum floor spectrum which can be obtained from the derivative of $R^2_{\ddot{U}s}(\omega_s,\xi_s)$ with respect to θ and by setting it equal to zero, it becomes:

$$tan(2\theta_{cr}) = \frac{R_{cs}}{R_c - R_s} \tag{54}$$

This equation gives two roots for θ_{cr}, with 90^o difference, which defines the maximum and the minimum values of the floor spectrum. It should be noted that the critical angle depends on the characteristics of the horizontal spectra and the horizontal dynamic properties of the structure.

From Eq. (52), it is known that if the two horizontal ground spectrum are equal ($R_{cs} = 0$), then $\theta_{cr} = 0$, which indicate that the floor spectrum does not depend on the angle of incidence. For this reason, it is enough to analyze the typical case of $\theta = 0$ in order to determine the maximum floor spectrum. In other words, if the two horizontal ground spectra are equal, the value of R_c is an upper bound of floor spectrum to any angle of incidence.

3. Numerical Results

To demonstrate the application of the presented method and to illustrate the importance of the multi-components of the earthquake, a 10-story torsional structure is considered to obtain the floor response spectra for various parameters. The dynamic properties of the structure are shown in Figure (1) and its natural frequencies in the case of zero eccentricities and without interaction and nonproportional damping effects are listed in Table (1). It is assumed that the eccentricity of stories are equal in both directions.

In the presented formulation, the seismic input of the primary system can be time history, power spectral density function, or response spectrum of ground accelerations. In this example the average response spectra of 20 earthquake with similar characteristic, normalized to $1.0g$, is considered as input. The two horizontal spectra are shown in

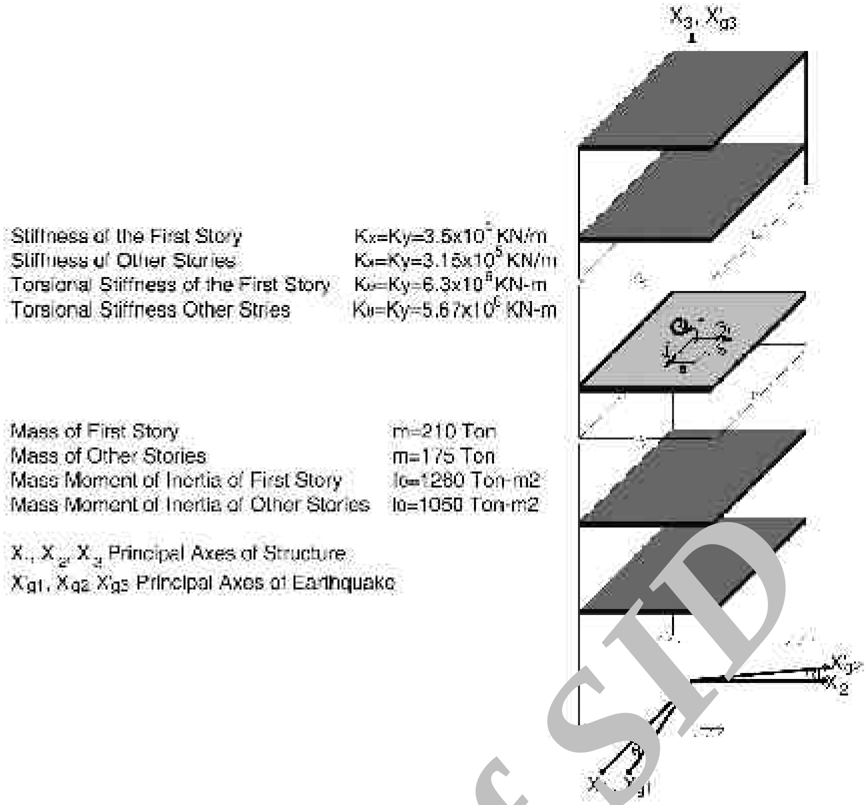

Figure 1. Ten-story example structure.

Table 1. Frequencies of the symmetric structure.

Mode Number	Lateral Mode Frequencies (rad/sec)	Torsional Mode Frequencies (rad/sec)
1	5.57	9.65
2	14.66	25.39
3	23.99	41.[illegible]
4	33.32	57.72
5	41.21	[illegible]
6	48.05	83.[illegible]
7	52.44	[illegible]0.83
8	61.16	105.93
9	67.28	116.53
10	77.9[illegible]	135.08

Figure (2). For this structure the effect of eccentricity, floor number, interaction, nonproportional damping, and point of connection of the secondary system on the floor, on the response of the secondary system subjected to multicomponent earthquake have been studied. The floor spectrum is obtained for these different inputs:

i) The multicomponent earthquake acted along the critical angle and its response will be denoted by R.

ii) The single-component earthquake acted along the critical angle and the floor spectrum and its relative difference with respect to R will be denoted by $R1$ and $e1$, respectively.

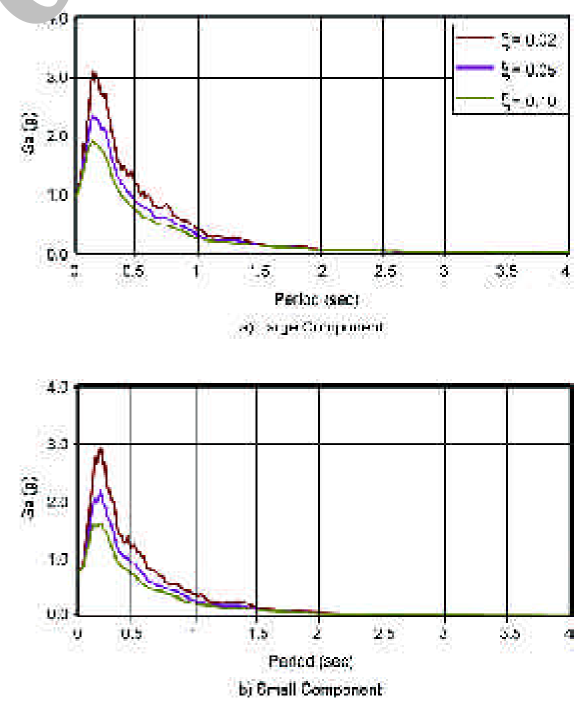

Figure 2. Input ground spectra.

iii) The single-component earthquake acted along the principal axes of the primary structure and the floor spectrum and its relative difference with respect to R will be denoted by R_0 and e_0, respectively.

3.1. Effect of Eccentricity

Figure (3) shows the floor spectra at 10^{th} floor for eccentricity ratios of 5, 10, 15, and 20%. The damping ratio of both the primary and the secondary systems is assumed to be 0.02, and the mass ratio is equal to 0.05. It is seen that as the eccentricity increases, the response of the secondary system and the difference between R, $R1$, and $R0$ will increase. The increase in the tuning frequencies is much larger than other frequencies. Thus, for large eccentricities and in the tuning frequencies, multicomponent effect should be considered.

3.2. Effect of Interaction

The most important factor in the interaction is the mass ratio of the secondary system to the floor. In order to study the effect of interaction, four different mass ratios is considered and floor spectrum of floor

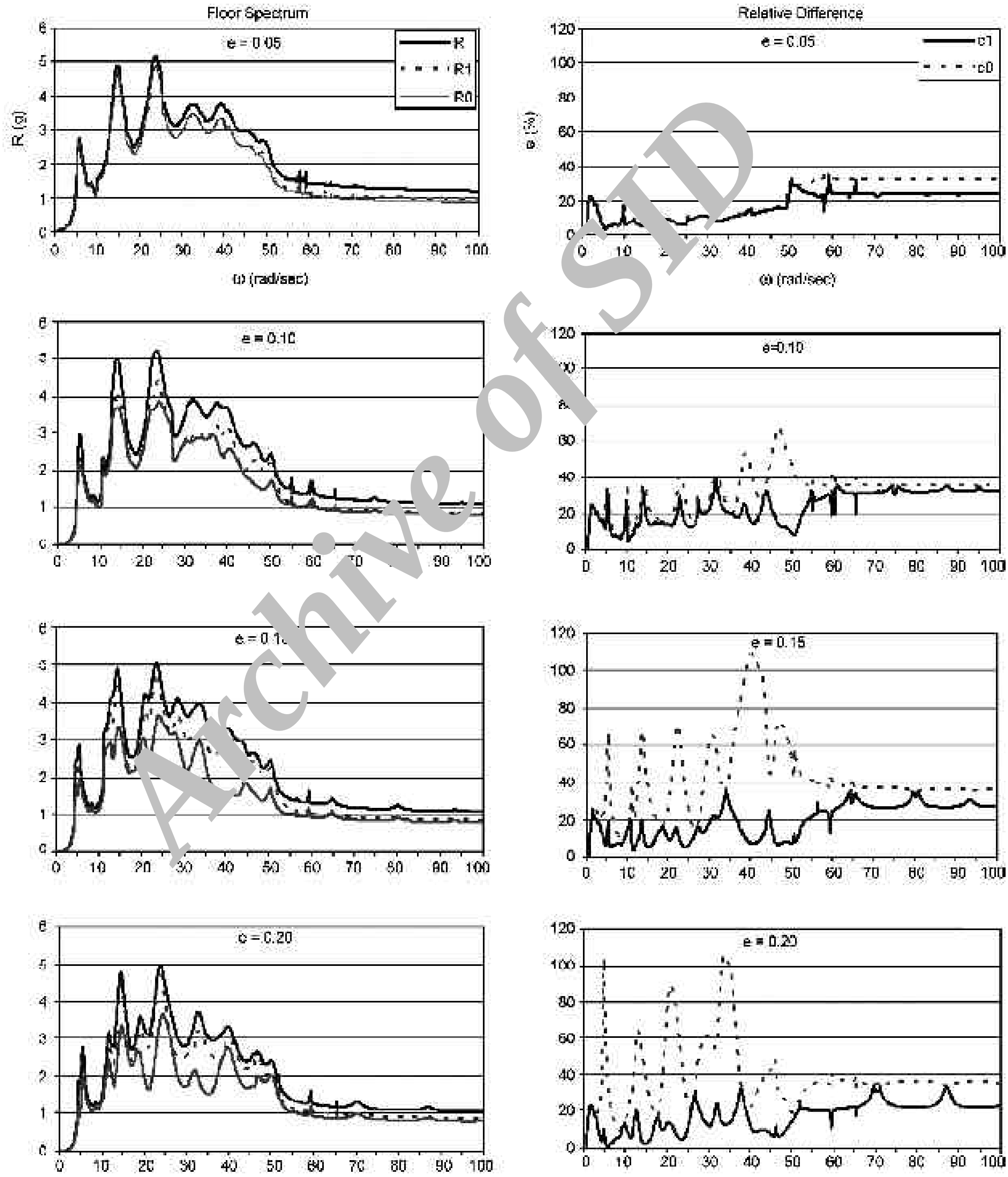

Figure 3. Effect of eccentricity.

5 have been calculated and plotted in Figure (4). In this case the damping ratio of both the primary and the secondary system are 0.05, and the eccentricity is equal to 15%. From Figure (4), it can be seen that by increasing the mass ratio, the floor response spectrum will decrease, especially in the tuning frequencies. Therefore interaction will be important for heavy secondary system and tuning frequency. This conclusion is in agreement with other researchers, such as Igusa and Derkiuregian [9]. Also from this figure it can be seen that by increasing the mass ratio, the difference between R, $R1$, and $R0$ will decrease and the error of neglecting multi-componentness of earthquake decrease.

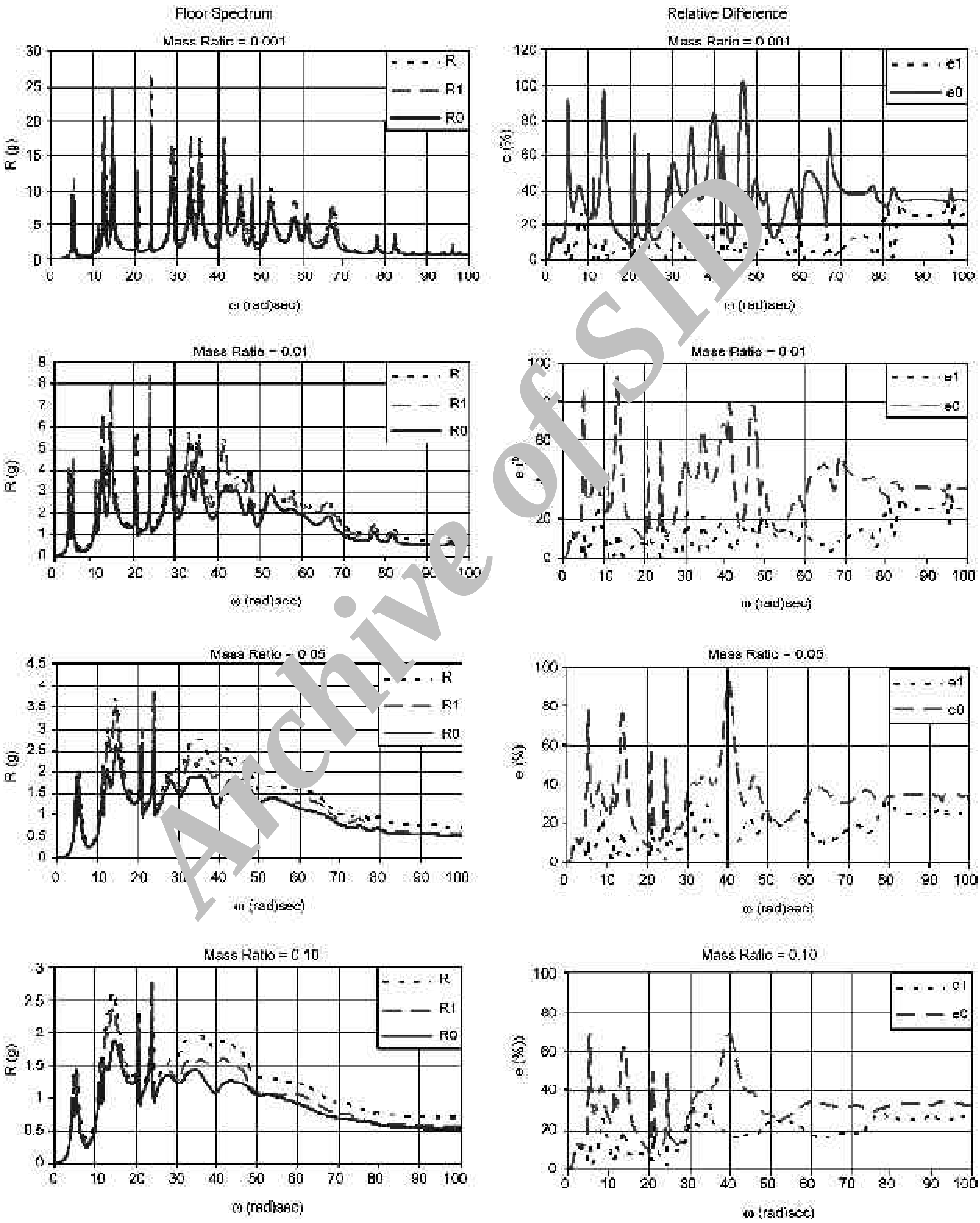

Figure 4. Effect of mass ratio.

3.3. Effect of Elevation of Secondary System from Ground

Figure (5) gives a comparison of the floor response spectrum and their respective differences in the first, fifth and tenth floors. For these cases the damping ratios of the both systems, the mass ratio, and the eccentricity ratio are taken 0.02, 0.05, and 0.05, respectively. It can be seen that for the higher elevation the absolute and the relative difference between R, $R1$, and $R0$ have been increased, especially in the case of the tuning frequencies. The average differences for "$e1$" are 6.3, 7.2, and 10.1% and for "$e0$" are 8.1, 10.6, and 13.3% for the floors number 1, 5 and 10, respectively. This means that although the effect of considering multicomponentness of earthquake is important for torsional buildings, but it is more important at the higher elevation from ground.

3.4. Effect of Nonproportional Damping

In order to study the effect of nonproportionality damping, the floor spectrum of the 10^{th} floor with mass ratio of 0.05 and eccentricity of 0.15 has been obtained. The proportional as well as nonproportional damping analysis have been performed and the results are shown in Figures (6) and (7). In Figure (6), the damping ratio of the primary system is considered to be constant and equal to 0.05, while the damping ratio of the secondary system varies as 0.02, 0.05, and 0.10. In Figure (7), the damping ratio of the secondary system is considered to be constant and equal to 0.05, while the damping ratio of the primary system varies as 0.02, 0.05, and 0.10. Figure (6) shows that the change in the damping ratio of the secondary system does not affect its response

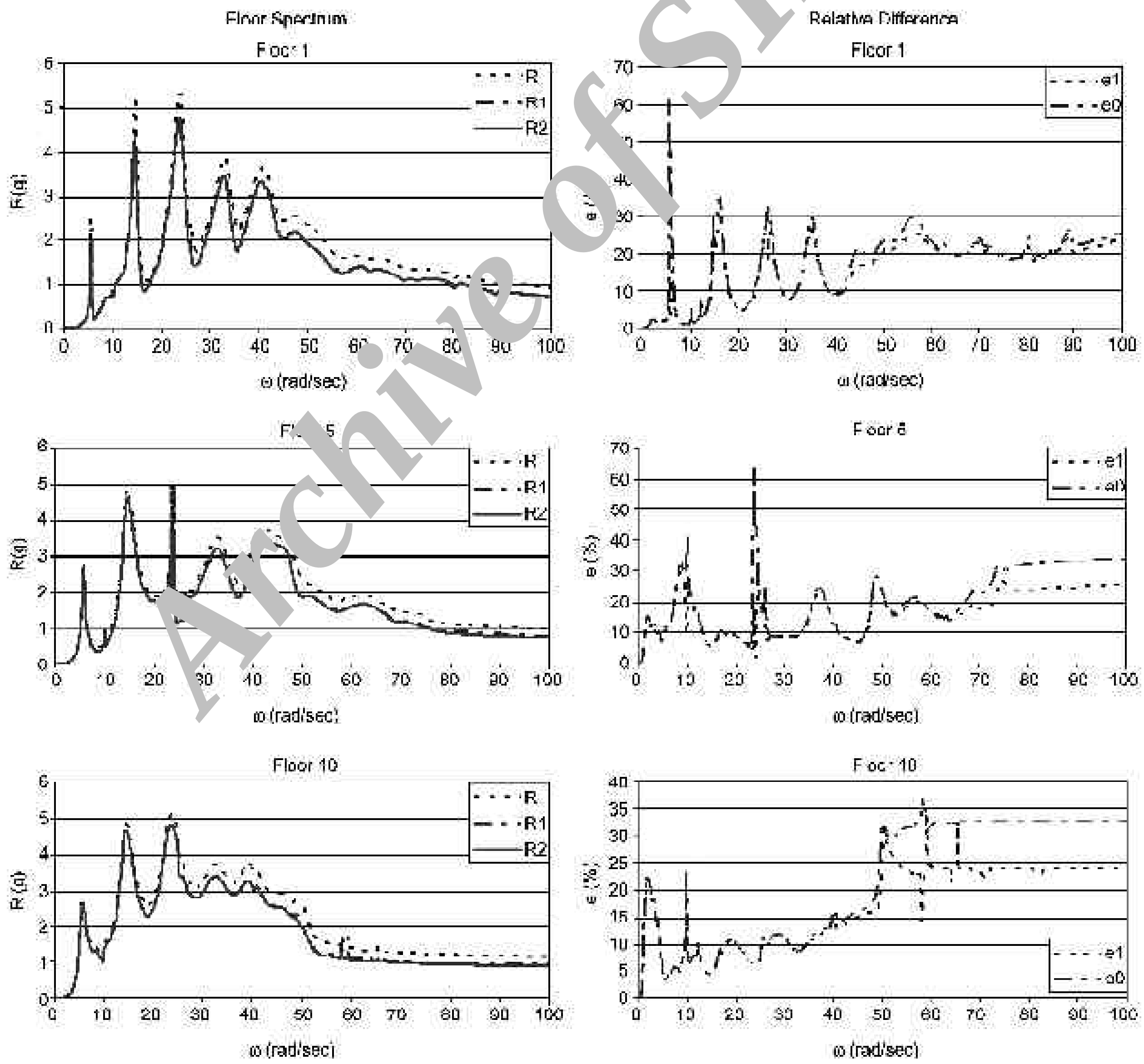

Figure 5. Effect of floor number.

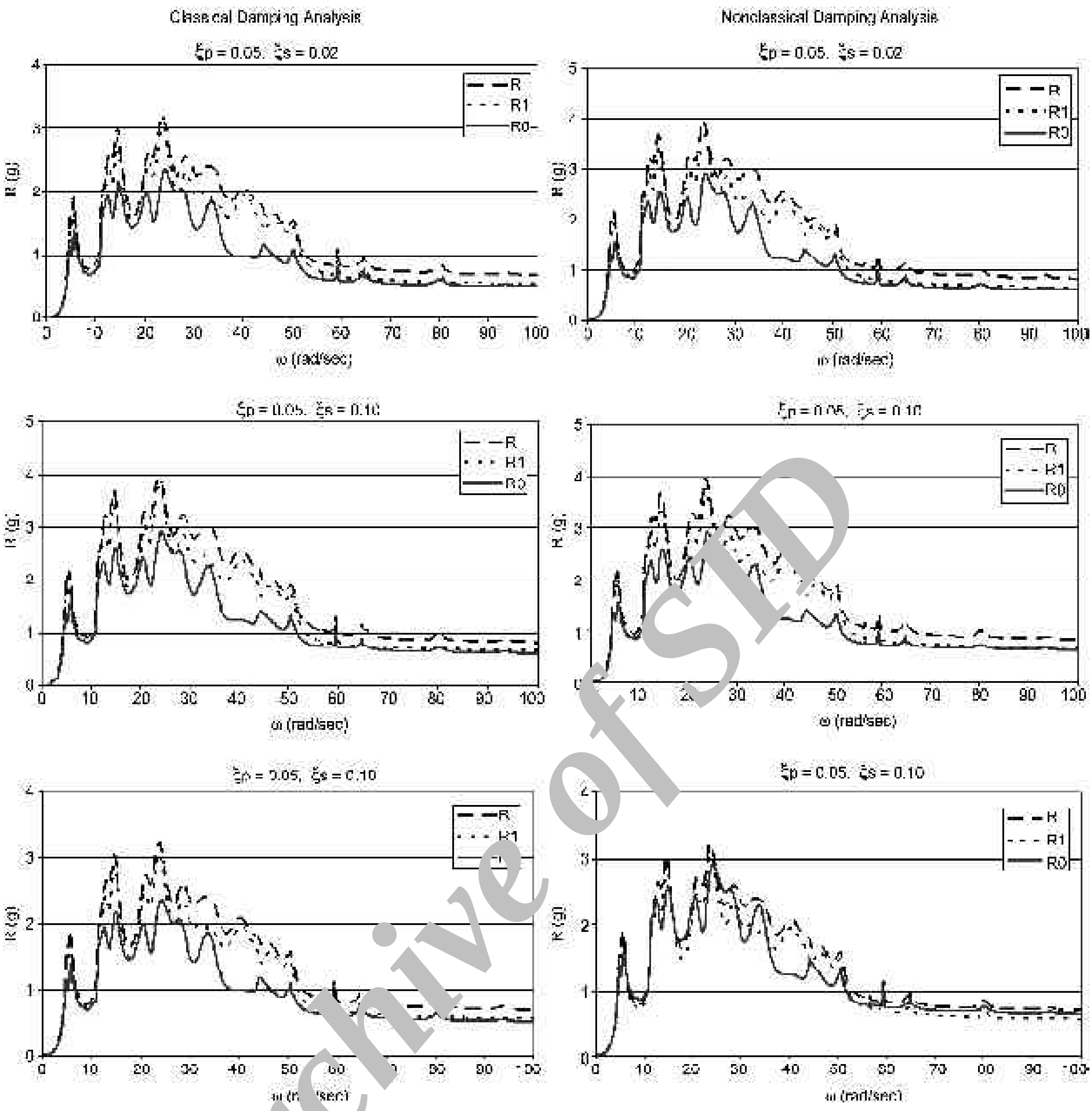

Figure 6. Effect of nonclassical damping, damping ratio of primary system is constant.

significantly. But Figure (7) shows that the increase in the damping ratio of the primary system cause the decrease in the response of the secondary system. This proves that the response of the secondary system is affected by the damping ratio of the primary structure rather than by the secondary system.

Figures (6) and (7) also show that the nonproportional damping analysis increase the response of the secondary system, especially in the case of tuning frequencies.

3.5. *Effect of Point of Connection of the Secondary System on the Floor*

The secondary system may be connected on a location other than the center of mass of the floor as it can be seen in Figure (8). In order to investigate the effect of variation of the location of the secondary system with respect to the center of mass of the floor, the response spectrum of a secondary system which installed on the 10^{th} floor with eccentricity of 0.01 has been obtained for various ratios. The results have been shown in Figure (9). A review of this figure shows that:

i) the response of the secondary system increases as the ratio increases.

ii) the effect of multicomponent earthquake becomes more important as the ratio between their spectra or spectral density function increases.

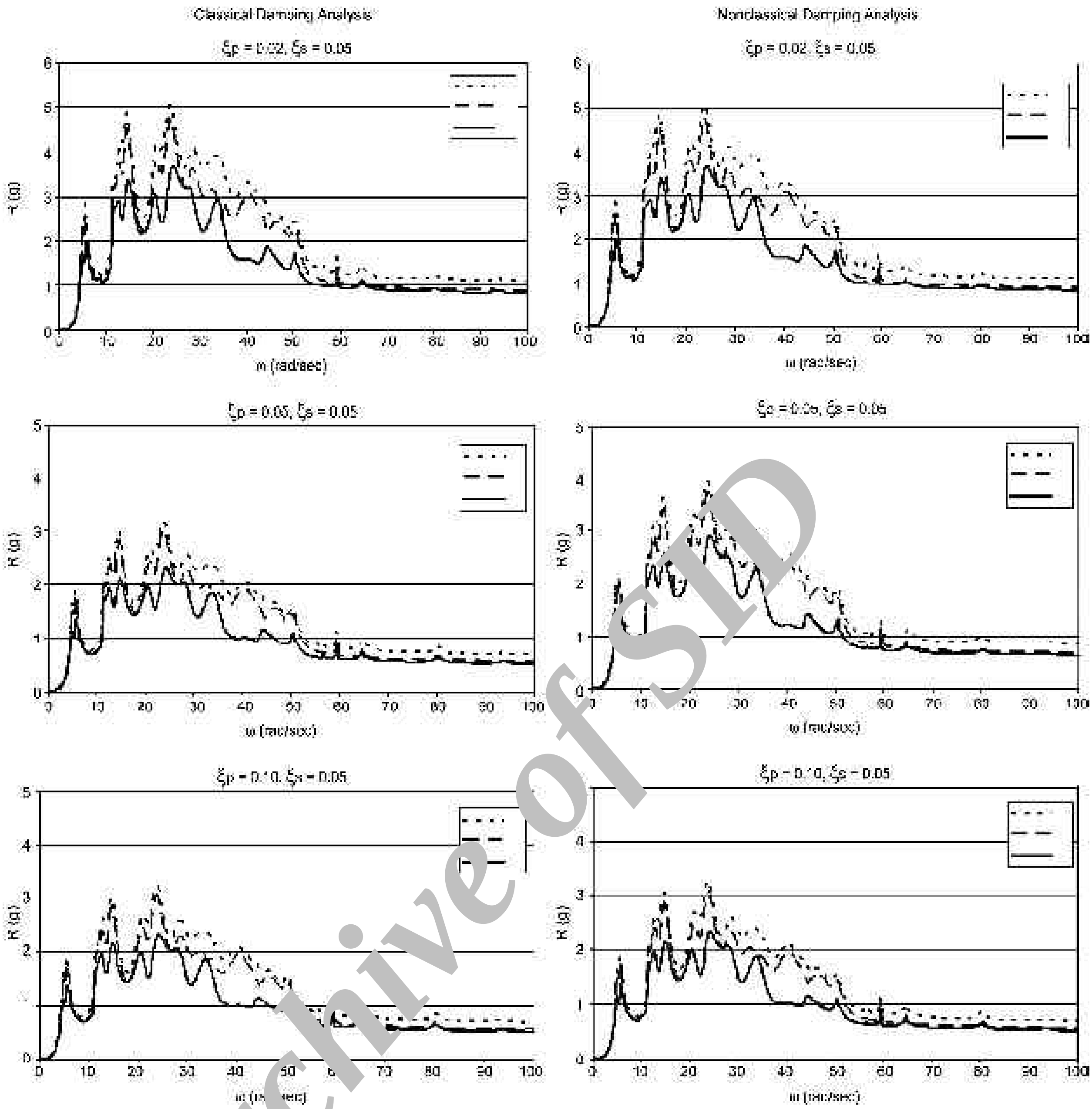

Figure Effect of nonclassical damping (damping ratio of secondary system is constant).

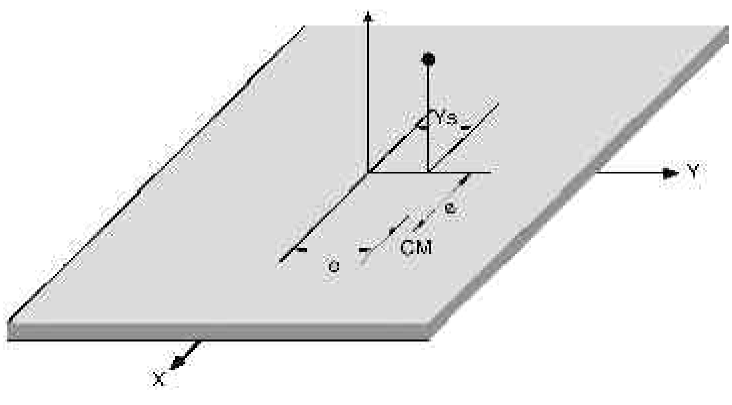

Figure 8. Point of connection of secondary system to the floor.

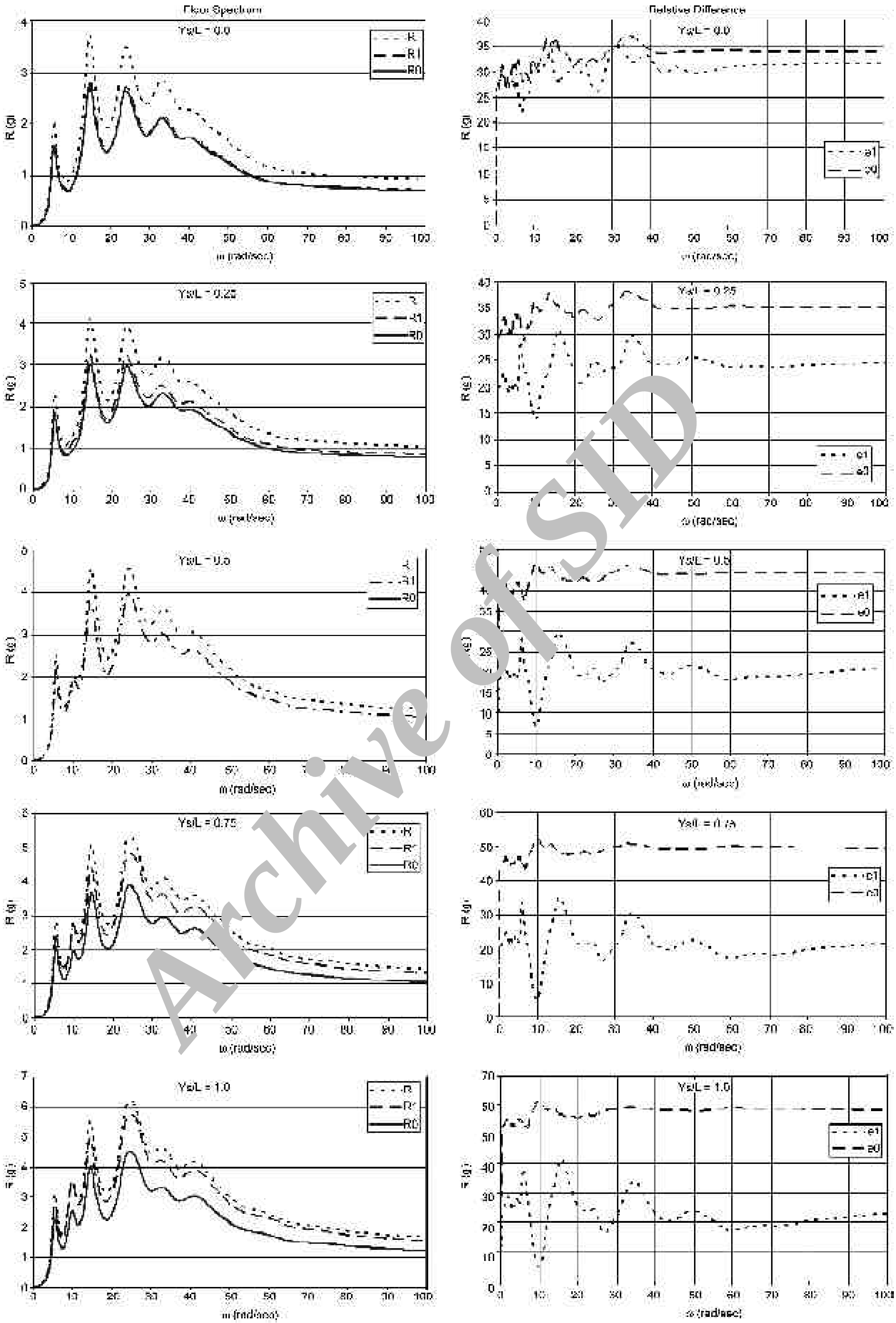

Figure 9. Effect of point of connection of secondary system on floor.

4. Conclusions

New formulation for the calculation of the floor response spectrum subjected to multicomponent earthquake is presented, which accounts for the interaction of the primary and the secondary systems and nonproportional damping which are inherent characteristics of combined systems. The autocorrelation, power spectral density functions, and mean square responses have been also derived. The two horizontal and one vertical components of earthquake is considered and the critical angle, which produce the maximum floor spectrum, has been obtained. The proposed method is efficient, since it generates floor response spectrum directly from the multicomponent ground response spectra. Numerical studies show that the effect of multicomponent earthquake input is important in the structures with large eccentricity, light secondary systems and in the case of tuned modes.

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Appendix I

Since the secondary system is the $N+1^{th}$ degree of freedom of the combined system, the parenthesis of the second term of Eq. (27) can be written as:

$$\sum_{l=1}^{3}\left[r_{N+1,l}+\sum_{j=1}^{N+1}\phi_{N+1,j}(\psi_{ja}F_{al})\right]\ddot{x}_{gl} \tag{I-1}$$

If $[Q]$ from Eq. (19) substitute in Eq. (23), $[F_a]$ becomes:

$$[F_a]=-\frac{\{\psi_a\}^T\begin{bmatrix}[0] & [0]\\ [0] & [I]\end{bmatrix}\begin{bmatrix}[0]\\ [\gamma]\end{bmatrix}}{\{\psi_a\}^T[A]\{\psi_a\}}=-\frac{\{\psi_a\}^T\begin{bmatrix}[0]\\ [\gamma]\end{bmatrix}}{\{\psi_a\}^T[A]\{\psi_a\}} \tag{I-2}$$

The l^{st} element of $[F_a]$ will be:

$$F_{al}=-\frac{\{\psi_a\}^T\begin{Bmatrix}\{0\}\\ \{\gamma_l\}\end{Bmatrix}}{\{\psi_a\}^T[A]\{\psi_a\}} \tag{I-3}$$

The size of $\{\psi_a\}$ is $2(N+1)$ by 1, which can be separated into the upper and lower parts:

$$\{\psi_a\}=\begin{Bmatrix}\{\psi_a^u\}\\ \{\psi_{la}^l\}\end{Bmatrix} \tag{I-4}$$

where $\{\psi_a^u\}$ and $\{\psi_{la}^l\}$ are the $(N+1)$ upper and lower element of $\{\psi_a\}$, respectively. Substituting this equation into Eq. (I-3), F_{al} becomes:

$$F_{al}=-\frac{\{\psi_a^l\}^T\{\gamma_l\}}{\{\psi_a\}^T[A]\{\psi_a\}} \tag{I-5}$$

which can be written as:

$$F_{al}=-\frac{\begin{Bmatrix}\{\psi_a^u\}\\ \{\psi_a^l\}\end{Bmatrix}^T\begin{bmatrix}[0] & [I]\\ [I] & [C]\end{bmatrix}\begin{Bmatrix}\{\gamma_l\}\\ \{0\}\end{Bmatrix}}{\{\psi_a\}^T[A]\{\psi_a\}} \tag{I-6}$$

The matrix in the middle part of the numerator of this equation is $[A]$, therefore,

$$F_{al}=-\frac{\{\psi_a\}^T[A]\begin{Bmatrix}\{\gamma_l\}\\ \{0\}\end{Bmatrix}}{\{\psi_a\}^T[A]\{\psi_a\}} \tag{I-7}$$

Since the imaginary mode shapes have the orthogonality and independency conditions, any vector can be expand in term of them. One can expand the vector $\begin{Bmatrix}\{\gamma_l\}\\ \{0\}\end{Bmatrix}$ as:

$$\begin{Bmatrix}\{\gamma_l\}\\ \{0\}\end{Bmatrix}=[\psi]\{\beta\}=\sum_{a=1}^{2(N+1)}\beta_a\{\psi_a\} \tag{I-8}$$

Where β's are the expansion coefficients which can be obtained by substituting Eq. (I-8) into Eq. (I-7):

$$F_{al}=-\frac{[\psi_a]^T[A][\psi]\{\beta\}}{\{\psi_a\}^T[A]\{\psi_a\}} \tag{I-9}$$

Since the imaginary mode shapes are orthogonal with respect to matrix [A], this equation becomes:

$$F_{al}=-\frac{[\psi_a]^T[A][\psi_a]}{[\psi_a]^T[A][\psi_a]}\beta_a=-\beta_a \tag{I-10}$$

The j^{th} row of the Eq. (I-8) is:

$$\gamma_{jl}=\sum_{a=1}^{2(N+1)}\beta_a\psi_{ja} \tag{I-11}$$

Substituting Eq. (I-10) into Eq. (I-11), and the results into Eq. (I-1), it becomes:

$$\sum_{l=1}^{3}\left(r_{N+1,l}-\sum_{j=1}^{N+1}\phi_{N+1,j}\gamma_{jl}\right)\ddot{x}_{gl} \tag{I-12}$$

Now it will be shown that the parenthesis in the Eq. (I-12) is zero. In the Eq. (16) the l^{st} element of $[\gamma_j]$ is:

$$\gamma_{jl}=\{\phi_j\}^T\begin{bmatrix}[M_p]\{r_{pl}\}\\ M_s r_{sl}\end{bmatrix}=$$

$$\{\phi_j\}^T\begin{bmatrix}[M_p] & \{0\}\\ [0] & M_s\end{bmatrix}\begin{Bmatrix}\{r_{pl}\}\\ r_{sl}\end{Bmatrix}=\{\phi_j\}^T[M]\{r_l\} \tag{I-13}$$

where $\{r_{pl}\}$ is the l^{th} column of $[r_p]$, r_{sl} is the l^{st} element of $[r_s]$, and where $\{r_l\}=\begin{Bmatrix}\{r_{pl}\}\\ r_{sl}\end{Bmatrix}$. Since the normal mode shapes have the orthogonality and independency conditions, any vector can be expand in term of them. One can expand the vector $\{r_l\}$ as:

$$\{r_l\}=[\phi]\{\alpha\}=\sum_{j=1}^{N+1}\alpha_j\{\phi_j\} \tag{I-14}$$

Where α's are the coefficients of the expansion. The $N+1^{th}$ element of $\{r_l\}$ is:

$$r_{N+1,l}=\sum_{j=1}^{N+1}\alpha_j\phi_{N+1,j} \tag{I-15}$$

Substituting Eq. (I-14) into Eq. (I-13) gives:

$$\gamma_{jl}=\{\phi_j\}^T[M]\{\phi\}\{\alpha\} \tag{I-16}$$

By the orthogonality of the mode shape, the Eq. (I-16) becomes:

$$\gamma_{jl}=\{\phi_j\}^T[M]\{\phi_j\}\alpha_j \tag{I-17}$$

Since the mode shapes normalized with respect to mass matrix ($\gamma_{jl}=\alpha_j$), Eq (I-15) becomes:

$$r_{N+1,l}=\sum_{j=1}^{N+1}\gamma_{jl}\phi_{N+1,j} \tag{I-18}$$

Therefore, the parenthesis in the Eq. (I-12) is zero, and consequently the Eq. (I-1) become zero.

Appendix II

The coefficients A_{jkaln} and B_{jkaln} in Eq. (41) are calculated as:

$$A_{jkaln}=4a_{jal}a_{kbn} \tag{II-1}$$

$$B_{jkaln}=4\Big[b_{jal}b_{kbn}+\xi_a^2(a_{jal}a_{kbn}-b_{jal}b_{kbn})+\xi_a^2\sqrt{1-\xi_a^2}(a_{jal}b_{kbn}+b_{jal}a_{kbn})\Big] \tag{II-2}$$

where:

$$a_{jal}=Re(\psi_{ja}F_{al}) \tag{II-3}$$

$$b_{jal}=Im(\psi_{ja}F_{al}) \tag{II-4}$$

Also the coefficients $A_{jkabln},\ldots,D_{jkabln}$ in Eq. (41) are obtained from the solution of following simultaneous equation:

$$[P]\{A\}=\{W\} \tag{II-5}$$

where

$$\{A\}^T=[A_{jkabln}\ B_{jkabln}\ C_{jkabln}\ D_{jkabln}] \tag{II-6}$$

$$[P]=\begin{bmatrix}0 & \omega_a^2 & 0 & \omega_b^2\\ \omega_a^4 & \omega_a^2\omega_b^2(4\xi_b^2-2) & \omega_b^4 & \omega_a^2\omega_b^2(4\xi_a^2-2)\\ \omega_a^2\omega_b^2(4\xi_b^2-2) & \omega_a^2\omega_b^4 & \omega_a^2\omega_b^4(4\xi_a^2-2) & \omega_a^4\omega_b^2\\ \omega_a^4\omega_b^4 & 0 & \omega_a^4\omega_b^4 & 0\end{bmatrix} \tag{II-7}$$

$$\{W\}=\begin{Bmatrix}B'_{jkabln}\\ A'_{jkabln}+B'_{jkabln}(-\omega_a^2-\omega_b^2+4\xi_a\omega_a\xi_b\omega_b)-2C'_{jkabln}(-\xi_a\omega_a-\xi_b\omega_b)\\ A'_{jkabln}(-\omega_a^2-\omega_b^2+4\xi_a\omega_a\xi_b\omega_b)+B'_{jkabln}\omega_a^2\omega_b^2-2C'_{jkabln}(-\xi_b\omega_b\omega_a^2+\xi_a\omega_a\omega_b^2)\\ A'_{jkabln}\omega_a^2\omega_b^2\end{Bmatrix} \tag{II-8}$$

in Eq. (I-8), the coefficients A'_{jkabln}, B'_{jkabln}, and C'_{jkabln} are given by:

$$A'_{jkabln}=4\omega_a^2\omega_b^2a_{jal}a_{kbn} \tag{II-9}$$

$$B'_{jkabln}=4\omega_a\omega_b\Big[\xi_a\xi_b a_{jal}a_{kbn}+\sqrt{(1-\xi_a^2)(1-\xi_b^2)}\times b_{jal}b_{kbn}+\xi_a\sqrt{(1-\xi_b^2)}a_{kbn}b_{jal}-\xi_b\sqrt{(1-\xi_a^2)}a_{jal}b_{kbn}\Big] \tag{II-10}$$

$$C'_{jkabln}=4\omega_a\omega_b\Big[a_{jal}a_{kbn}(\xi_a\omega_b-\xi_b\omega_a)+\omega_b\sqrt{(1-\xi_a^2)}a_{kbn}b_{jal}-\omega_a\sqrt{(1-\xi_b^2)}a_{jal}b_{kbn}\Big] \tag{II-11}$$

$$\xi_a=\frac{-Re(\lambda_a)}{|\lambda_a|^2} \tag{II-12}$$

$$\omega_a^2=|\lambda_a|^2. \tag{II-13}$$