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QUASI-PRIMITIVE RINGS AND DENSITY THEORY

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Abstract: In this paper, "quasi-primitive" rings and "almost semiprimitive" rings are introduced and is shown that a quasi-primitive ring R is a uniformly dense subring of a full linear ring $End(_DV)$, by which we mean that if $n \geq 1$, $\{v_1, v_2, \ldots, v_n\} \subseteq V$ is an independent set and $\{u_1, u_2, \ldots, u_n\} \subseteq V$, then there are $0 \neq d \in D$ and $f \in R$ such that $f(v_i) = du_i$ for all $1 \leq i \leq n$. Also, it is shown that nonsingular prime rings containing a uniform left ideal and a uniform right ideal are examples of quasi-primitive rings. Finally, we generalize a maim theorem of [4] as follows. If d is a derivation of an almost semiprimitive ring R so that the values of d on a left ideal L are nilpotent, then $Ld(L) = 0$.

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Introduction

Various versions of the density theorm have been studied and characterized by numerous authors. Some of the works are [1], [2], [3], [7], [8], [9], and [10].

Before we enter the main subject, we briefly build up a system to derive the main results. Then, it will be easily seen that almost all the works have been done in this regard can be deriven from this system. For the sake of simplicity and unified notations, we will mention some of the previous results.

Let R be a ring and L and K are modules. We call a submodule N of L, K-dense if for every $l \in L$ and $k \in K$, $(N : l)k = 0$ implies $k = 0$, where $(N : l) = \{r \in R \mid rl \in N\}$. An L-dense submodule of L is called dense submodule. It is easy to see that if K is faithfull (and elsewhere), then every K-dense submodule of L is also R-dense and if K is nonsingular, then every R-dense submodule of L is also K-dense. Furthermore, the R-dense left ideals of R are just the dense left ideals of R.

Now let R be a right-faithfull ring, M a nonzero cofaithfull (and elsewhere) module and $L, K \in \{M, R\}$. One may form the direct limit $\xrightarrow{lim}$ $Hom_R(N, K)$ over all M-R-dense submodules N of L: we denote this direct limit by $Q_m(L, K)$. Let $[f]$ denote the class of $f \in Hom_R(L_f, K)$ in $Q_m(L, K)$; we have $[f] = [g]$ if and only if $f = g$ on an M-R-dense (M-dense and also R-dense) sunmodule of L: addition of classes is defined by $[f] + [g] = [h]$ in $Q_m(L, K)$ where $h = f + g$ on an M-R-dense submodule of L. Now if $N \in \{M, R\}$, then we define the middle linear map $Q_m(L, K) \times Q_m(K, N) \longrightarrow Q_m(L, N)$ naturally. Set $D = Q_m(M, M)$, $V = Q_m(R, M), W = Q_m(M, R), S = Q_m(R, R)$ and $U = MD$. D, V, W and S are determind uniquely as follow.

A. S is an extension ring of R such that

1. For every $s \in S$ there exists an M-R-dense left ideal I of R such that $Is \subseteq R$.

2. For every $0 \neq s \in S$ and M-R-dense left ideal I of R, $Is \neq 0$.

3. For every M-R-dense left ideal I of R and $f \in Hom_R(I, R)$ there exists $s \in S$ such that $f(x) = xs$ for all $x \in I$.

It is worth to mention that S is a subring of the left maximal quotient ring of R, furthermore if M is faithfull and nonsingular, then S is the left maximal quotient ring of R.

B. V is an R-module containing M as an R-submodule such that

1. For every $v \in V$ there exists an M-R-dense left ideal I of R such that $Iv \subseteq M$.

2. For every $0 \neq v \in V$ and M-R-dense left ideal I of R, $Iv \neq 0$.

3. For every M-R-dense left ideal I of R and $f \in Hom_R(I, M)$ there exists $v \in V$ such that $f(x) = xv$ for all $x \in I$.

C. D is an extension ring of $Hom(M, M)$ and V is a R-D-bimodule such that

1. For every $d \in D$ there exists an M-R-dense submodule N of M such that $Nd \subseteq M$.

2. For every $0 \neq d \in D$ and M-R-dense submodule N of M, $Nd \neq 0$.

3. For every M-R-dense submodule N of M and $f \in Hom_R(N, M)$ there exists $d \in D$ such that $f(x) = xd$ for all $x \in N$.

D. W is an additive group with a middle linear map $M \times W \longrightarrow S$ such that

1. For every $w \in W$ there exists a M-R-dense submodule N of M such that $Nw \subseteq R$.
2. For every $0 \neq w \in W$ and M-R-dense submodule N of M, $Nw \neq 0$.
3. For every M-R-dense submodule N of M and $f \in Hom_R(N, R)$ there exists $w \in W$ such that $f(x) = xw$ for all $x \in N$.
4. For every $s \in S$, $v \in V$ and $w \in W$, $s(vw) = (sv)w$.

The above conditions determine S, V, D, W and U uniquely. Furthermore, (S, V, W, D) is a Morita context and if M is monoform, then U is the quasi-injective hull of M.
Let M be monoform [4]. If M is faithfull, then S and so R, can be considered as a subrings of $End(V_D)$ and if for every $0 \neq a \in R$, there exist $m \in M$ and $b \in R$ such that $bRa \neq 0$ and $a_l(m)b = 0$ (for example when R is prime and M is a left ideal), then S and so R, can be considered as a subrings of $End(_DW)$.
Our minimum expectation is that D can be a division ring and that can be fullfilled when M is monoform and either M is faithfull or R is nonsingular. In this case every nonzero submodule is M-R-dense. Now suppose that M is faithfull and monoform. All the works have been done so far can be explained and the results can be deriven within this system by one of these two approches. Amitsur in [1] and Zelmanowitz in [10] have considered R as a subring of $End(_DW)$. But in [7], in [9], in the Wedderburn density theorem and in the Goldie's theorems in [2]

R has been considered as a subring of $End(U_D)$.

The following can be proven easily.

E. Let R be a ring and M an R-module. If every essential left ideal is M-R-dense (as will happen if R is nonsingular and either M is nonsingular or M is a left ideal), then

1. For every R-submodule I of S and $f \in Hom_R(I, S)$ there exists $s \in S$ such that $f(x) = xs$ for all $x \in I$.
2. For every R-submodule I of S and $f \in Hom_R(I, V)$ there exists $v \in V$ such that $f(x) = xv$ for all $x \in I$.

F. Let R be a ring and M an R-module. If every essential submodule is M-R-dense (as will happen if either M is nonsingular and faithfull, or M is monoform and faithfull or M is monoform and R is nonsingular), then

1. For every R-submodule N of V and $f \in Hom_R(N, V)$ there exists $d \in D$ such that $f(x) = xd$ for all $x \in N$.
2. For every R-submodule N of V and $f \in Hom_R(N, S)$ there exists $w \in W$ such that $f(x) = xw$ for all $x \in N$.
3. If M is nonsingular, then V is a nonsigular R-module and a nonsigular S-module.

G. Let R be a ring and M a faithfull R-module. If every essential submodule is M-R-dense (it will happen if M is either nonsingular or monoform), then

1. Considering R as a subring of $End(U_D)$, for every left ideal J of R, finite dimensional subspace N of U, and $v \in U -$

$(\cap_{f\in J} Ker(f) + N)$, there exists $f \in J$ such that $f(N) = 0$ and $f(v) \neq 0$.

2. Considering S as a subring of $End(V_D)$, for every left ideal J of R, finite dimensional subspace N of V, and $v \in V - (\cap_{f\in J} Ker(f) + N)$, there exists $f \in J$ such that $f(N) = 0$ and $f(v) \neq 0$.

1. Preliminaries

We introduce the following notations and definitions for convenience.

1. An element s of a ring R is said to be primitive, if for every $x, y \in R$, $xsy = 0$ implies either $xs = 0$ or $sy = 0$.

2. Let S be a ring, R a subring of S and $\emptyset \neq X \subseteq S$. Then, we denote $\{a \in R \mid aX = 0\}$ by $a_l^R(X)$. We also use $a_r^R(X)$ similarly. Moreover, if there is no ambiguity, we use $a_l(X)$ instead of $a_l^S(X)$ and $a_r(X)$ instead of $a_r^S(X)$. If $a_r^R(R) = 0$, then R will be called right faithfull. Left faithfull is defined similarly.

3. Suppose that D is a division ring, V a left vector space over D and R a subring of $Hom_D(V, V)$.

1. R is called "uniformly (and essentially) dense" if $\{v_1, v_2, \dots, v_n\} \subseteq V$ is an independent set and $\{u_1, u_2, \dots, u_n\} \subseteq V$, then there are $0 \neq d \in D$ and $f \in R$ (with rank not greater than n) such that $f(v_i) = du_i$ for all $1 \leq i \leq n$.

2. R is called "semi-dense" if $\{v_1, v_2, \dots, v_n\} \subseteq V$ is an independent set and $\{u_1, u_2, \dots, u_n\} \subseteq V$, then there is $f \in R$ and $0 \neq d_i \in D$ such that $f(v_i) = d_iu_i$ for all $1 \leq i \leq n$.

3. R is called " critically dense" if for every independent set $\{v_1, v_2, \ldots, v_n\} \subseteq V$, there is a subspace L with $L\oplus < v_1, v_2, \ldots, v_n >= V$ such that for every $\{u_1, u_2, \ldots, u_n\} \subseteq V$, there exists $f \in R$ with $f(L) = 0$ and $f(v_i) = u_i$ for all $1 \leq i \leq n$.

4. R is called " uniformly compressible" if for every independent set $\{v_1, v_2, \ldots, v_n\} \subseteq V$, there is a subspace L with $L\oplus < v_1, v_2, \ldots, v_n >= V$ and an invertible $n \times n$ matrix $A = [a_{ij}]$ such that for every $m \geq 1$ and $w_{ki} \in V$ there exists $0 \neq d \in D$ depending only on w_{ki}'s and $f_k \in R$ with $f_k(L) = 0$ and $f_k(v_i) = \sum_{j=1}^{n} a_{ij}(dw_{ki})$ for all $1 \leq i \leq n$.

4. Let R be a ring. Then, $Q_{mr}(R)$ will represent the maximal right quotient ring of R and the maximal left quotient ring of R will be represented by $Q_m(R)$.

5. Let R be a ring and M a module.

1. M is said to be "total", if for $0 \neq m, n \in M$ that $a_l(m) = a_l(n)$, there exist $a \in R$ and $f, g \in Hom_R(M, M)$ such that $g \neq 0$, $am \neq 0$ and $f(am) = g(an)$. In this case, if R is right faithfull and M is monoform, then
   (a) $D_M = \{d \in D \mid Md \subseteq M\}$ is a right order in D.
   (b) For every $u_1, u_2, \ldots, u_n \in U$, there exists $0 \neq a \in D_M$ such that $u_i a \in M$ for all $1 \leq i \leq n$.
   (c) For every subspace L of V, $(L \cap M)D = L \cap U$.

2. M is called "quasi-compressible" if for every nonzero finitely generated submodule K and every nonzero submodule N,

$Hom_R(K, N) \neq 0$. In this case, if M is monoform and either M is faithfull or R is nonsingular, then for every $m_1, m_2, \ldots, m_n \in M$ and every nonzero submodule N, there exists $0 \neq d \in D$ such that $m_i d \in N$ for all $1 \leq i \leq n$.

3. M is called "almost compressible" if for every $0 \neq m, n \in M$, there exists $a \in R$ with $an \neq 0$ and $a_l(m)an = 0$. In this case, if M is monoform and either M is faithfull or R is nonsingular, then for every $0 \neq m \in M$ and every nonzero submodule N, there exists $0 \neq d \in D$ such that $md \in N$.

4. M is called "transitive" if there exists $m \in M$ and $s \in R$ such that $sa_r^R(a_l(m))$ is a uniform right ideal.

6. Let R be a ring.

1. If R has a faithfull, monoform, total and quasi-compressible module, then R is called "quasi-primitive".

2. If R has a faithfull, monoform, total and almost compressible module, then R is called "almost primitive".

3. An ideal P is said to be "almost primitive" if R/P is an almost primitive ring.

4. R is said to be "almost semiprimitive" if the intersection of all almost primitive ideals is zeroo.

Lemma 1.1. *Let R be a nonsingular ring and J a uniform left ideal. Then*

1. *Every element of J is primitive.*

2. *For every $a \in R$, either $Ja = 0$ or Ja is a uniform left ideal.*

3. *For every $0 \neq s \in J$, $\mathfrak{a}_r(s) = \mathfrak{a}_r(J)$.*

Proof. 1. We show that for every $\emptyset \neq X \subseteq R$, either $J \cap \mathfrak{a}_l(X) = 0$ or $J \subseteq \mathfrak{a}_l(X)$. Suppose $I = J \cap \mathfrak{a}_l(X) \neq 0$ and K is the closure of I. Since $I \subseteq \mathfrak{a}_l(X)$ and $\mathfrak{a}_l(X)$ is closed by [5, Lemma 1.1], $K \subseteq \mathfrak{a}_l(X)$. On the other hand, $J \subseteq K$.

2. Suppose $Ja \neq 0$. Then, $J \cap \mathfrak{a}_l(a) = 0$. Now let K and L be left ideals contained in Ja with $K \cap L = 0$. Set $K_0 = \{x \in J \mid xa \in K\}$. Then, $K_0 a = K$ and $L_0 a = L$. We claim that $K_0 \cap L_0 = 0$ because if $b \in K_0 \cap L_0$, then $ba \in K \cap L = 0$, hence $b \in J \cap \mathfrak{a}_l(a) = 0$. Thus, $K_0 = 0$ or $L_0 = 0$, therefore $K = 0$ or $L = 0$.

3. Suppose $b \in \mathfrak{a}_r(s)$, then $0 \neq s \in J \cap \mathfrak{a}_l(b)$. Thus, $J \subseteq \mathfrak{a}_l(b)$. ■

Lemma 1.2. *Let R be a ring and $s \in R$. If Rs is uniform then every left ideal $K \neq 0$ with $K \cap \mathfrak{a}_l(s) = 0$ is uniform.*

Proof. The natural map $K \longrightarrow Ks$ is an R-module isomorphism. ■

Lemma 1.3. *Let R be a nonsingular ring and $0 \neq s \in R$.*

1. *If every left ideal $K \neq 0$ with $K \cap \mathfrak{a}_l(s) = 0$ is uniform, then Rs is uniform.*
2. *If Rs is uniform then $\mathfrak{a}_l(s)$ is a maximal left annihilator ideal.*
3. *If Rs is uniform and $s^2 \neq 0$ then $Rs \oplus \mathfrak{a}_l(s)$ is an essential left ideal and $\mathfrak{a}_r(s) \cap \mathfrak{a}_r(\mathfrak{a}_l(s)) = 0$.*

Proof. 1. Let I and J be left ideals with $Is \neq 0$ and $Js \neq 0$. It is enough to show that $Is \cap Js \neq 0$. Since $I \nsubseteq \mathfrak{a}_l(s)$ and $\mathfrak{a}_l(s)$ is closed by [5, Lemma 1.1], there exists a left ideal $M \neq 0$ contained in I with $M \cap \mathfrak{a}_l(s) = 0$. Thus, we may assume that $I \cap \mathfrak{a}_l(s) = 0$ and $J \cap \mathfrak{a}_l(s) = 0$. If $(I + J) \cap \mathfrak{a}_l(s) = 0$, then $I \cap J \neq 0$, thus $0 \neq (I \cap J)s \subseteq Is \cap Js$. Now suppose that $(I + J) \cap \mathcal{A}_l(s) \neq 0$. There exist $x \in I$ and $y \in J$ such that $0 \neq x + y \in \mathcal{A}_l(s)$, then $xs = -ys \in Is \cap Js$. On the other hand $xs \neq 0$ because otherwise, $x \in I \cap \mathcal{A}_l(s) = 0$ and $y \in J \cap \mathcal{A}_l(s) = 0$, then $x + y = 0$ which is a contradiction.

2. Let $a \in R$ and $\mathcal{A}_l(s) \subset \mathcal{A}_l(a) \neq R$. Since $\mathcal{A}_l(s)$ and $\mathcal{A}_l(a)$ are closed, there exist nonzero left ideals K and L such that $K \cap \mathcal{A}_l(a) = 0$ and $L \subseteq \mathcal{A}_l(a)$ with $\mathcal{A}_l(s) \cap L = 0$, thus $(K \oplus L) \cap \mathcal{A}_l(s) = 0$ which is a contradiction by Lemma 1.2. ■

Lemma 1.4. *If R is a prime ring containing a nonzero primitive element, then R is left and right nonsingular.*

Proof. Let $a \in Sing(R)$ (singular ideal of R). We claim that $\mathcal{A}_l(a)$ contains all primitive elements of R. Suppose $x \neq 0$ is a primitive element of R. Since $\mathcal{A}_l(a)$ is essential, $\mathcal{A}_l(a) \cap Rx \neq 0$, so there is $r \in R$ such that $0 \neq rx \in \mathcal{A}_l(a)$, thus $x \in \mathcal{A}_l(a)$. Now by the hypothesis, there exists a nonzero primitive element s. Let $x, y \in R$. Since xsy is a primitive element, $xsy \in \mathcal{A}_l(a)$. Thus $RsR \subseteq \mathcal{A}_l(a)$. Therefore $a = 0$. ■

Corollary 1.5. *Let R be a prime ring containing either a maximal left annihilator ideal or a maximal right annihilator ideal. Then, R is left and right nonsingular.*

Lemma 1.6. *Let R be a prime ring containing a uniform left ideal. Then, for every primitive element $0 \neq s \in R$, Rs is uniform.*

Proof. Suppose $a, b \in R$ with $as \neq 0$ and $bs \neq 0$. It is enough to show that there are $x, y \in R$ with $xas = ybs \neq 0$. Suppose $I \neq 0$ is a uniform left ideal. Consider $c \in I$ with $sc \neq 0$, then $0 \neq asc, bsc \in I$. So, there are $x, y \in R$ with $xasc = ybsc \neq 0$, therefore $xas = ybs \neq 0$. ■

Lemma 1.7. *Let R be a prime ring containing a uniform left ideal and a uniform right ideal. Then, for every primitive element $s \in R$ with $s^2 \neq 0$, $\mathcal{A}_l(\mathcal{A}_r(s))$ and $\mathcal{A}_r(\mathcal{A}_l(s))$ are uniform.*

Proof. By Lemma 1.4, R is left and right nonsingular. sR is uniform by Lemma 1.6, so $a_l(s) \cap a_l(a_r(s)) = 0$ by Lemma 1.3, then $a_l(a_r(s))$ is uniform by Lemma 1.2. ■

Lemma 1.8. *Let R be a semiprime ring and I a nonzero left ideal of R. Then, I contains a nonzero square element.*

Proof. Suppose it is not. Then, for every $x, y \in I$,

$$0 = (x+y)^2 = x^2 + y^2 + xy + yx = xy + yx$$

hence $xyx = 0$. Thus, $xbzax = 0$ for all $x, a \in I$ and $b, z \in R$. Hence $axb = 0$ for all $a, x \in I$ and $b \in R$. Consequently $I^2R = 0$ which is a contradiction. ■

Lemma 1.9. *Let R be a nonsingular prime ring containing a uniform left ideal and a uniform right ideal. If $A = \{a_1, a_2, \ldots, a_n\}$ is a set of primitive element with $a_i^2 \neq 0$ for each i and $a_i a_j = 0$ for all $i < j$, then*

1. *$\sum_{i=1}^n Ra_i + a_l(A)$ is a direct sum and an essential left ideal.*
2. *$\sum_{i=1}^n a_i R + a_r(A)$ is a direct sum and an essential rigth ideal.*
3. *$a_l(a_r(A)) \cap a_l(A) = 0$ and RA is an essential submodule of $a_l(a_r(A))$.*
4. *$a_r(a_l(A)) \cap a_r(A) = 0$ and AR is an essential submodule of $a_r(a_l(A))$.*

Proof. 1. Obviously $\sum_{i=1}^n Ra_i + a_l(A)$ is a direct sum. We prove that $\sum_{i=1}^n Ra_i + a_l(A)$ is an essential left ideal by induction on n. It is clear for $n = 1$ by Lemma 1.3 and Lemma 1.6. Set $B = \{a_1, a_2, \ldots, a_{n-1}\}$ and $I = \sum_{i=1}^{n-1} Ra_i \oplus a_l(B)$. I is an essential left ideal by induction, so

$I \cap \mathfrak{a}_l(a_n)$ is an essential submodule of $\mathfrak{a}_l(a_n)$, thus $J = I \cap \mathfrak{a}_l(a_n) \oplus Ra_n$ is an essential left ideal by Lemma 1.3 and Lemma 1.6. Since $a_i \in \mathfrak{a}_l(a_n)$ for all $i < n$, $I \cap \mathfrak{a}_l(a_n) = \sum_{i=1}^{n-1} Ra_i \oplus \mathfrak{a}_l(A)$, then $J = \sum_{i=1}^{n} Ra_i \oplus \mathfrak{a}_l(A)$.
3. Since $\mathfrak{a}_r(\mathfrak{a}_l(A)) + \mathfrak{a}_r(A)$ is an essential left ideal and R is right nonsingular by Lemma 1.1 and Lemma 1.4, $\mathfrak{a}_l(A) \cap \mathfrak{a}_l(\mathfrak{a}_r(A)) = 0$. ■

Lemma 1.10. *Let R be a nonsingular prime ring containing a uniform left ideal and a uniform right ideal.*

1. *For every independent set $\{I_1, I_2, \dots, I_n\}$ of uniform left ideals, there exist $a_i \in I_i$ with $a_i^2 \neq 0$ such that $a_i a_j = 0$ for all $i < j$, furthermore, for every $1 \leq j \leq n$, there exists $r \in R$ with $I_j r \neq 0$ such that $I_i r = 0$ for all $i \neq j$.*

2. *if A is an independent finite set of uniform left ideals, then $\sum_{I \in A} I$ is an essential submodule of $\mathfrak{a}_l(\mathfrak{a}_r(\sum_{I \in A} I))$.*

Proof. 1. By induction on n. It is clear for $n = 1$ by Lemma 1.8. Set $B = \{I_1, I_2, \dots, I_{n-1}\}$. RB is an essential submodule of $\mathfrak{a}_l(\mathfrak{a}_r(B))$ by Lemma 1.9. On the other hand, $RB \cap I_n = 0$, so $I_n \nsubseteq \mathfrak{a}_l(\mathfrak{a}_r(B))$, hence
$I_n \mathfrak{a}_r(B) \neq 0$, thus $I_n \mathfrak{a}_r(B) I_n \neq 0$, implying $\mathfrak{a}_r(B) I_n \nsubseteq \mathfrak{a}_r(I_n)$. Considering $a_n \in \mathfrak{a}_r(B) I_n - \mathfrak{a}_r(I_n)$ completes the induction by Lemma 1.1.
Now suppose that there is no $r \in R$ with $I_j r \neq 0$ such that $I_i r = 0$ for all $i \neq j$. Then, $\mathfrak{a}_r(\sum_{i \neq j} I_i) \subseteq \mathfrak{a}_r(I_j)$. Thus, $I_j \subseteq \mathfrak{a}_l(\mathfrak{a}_r(\sum_{i \neq j} I_i))$ which is a contradiction by item (2).
2. Suppose $A = \{I_1, I_2, \dots, I_n\}$. There exist $a_i \in I_i$ with $a_i^2 \neq 0$ such that $a_i a_j = 0$ for all $i < j$. Set $B = \{a_1, a_2, \dots, a_n\}$. RB is an essential submodule of $\mathfrak{a}_l(\mathfrak{a}_r(B))$ by Lemma 1.9. On the other hand, $\mathfrak{a}_r(a_i) = \mathfrak{a}_r(I_i)$ by Lemma 1.1 , so $\mathfrak{a}_r(\sum_{I \in A}) = \mathfrak{a}_r(B)$. Thus, RB is an essential sumodule of $\mathfrak{a}_l(\mathfrak{a}_r(\sum_{I \in A}))$, moreover $RB \subseteq \sum_{I \in A} \subseteq \mathfrak{a}_l(\mathfrak{a}_r(\sum_{I \in A}))$. ■

Lemma 1.11. *Let R be a ring and M a uniform nonsingular module.*

1. *If either M is faithfull or R is nonsingular, then for every $0 \neq m \in M$, there exists $w \in W$ and left ideal $J \neq 0$ such that $wm = 1$ and $amw = a$ for all $a \in J$.*

2. *If M is faithfull, then W is faithfull.*

Proof. 1. Set $I = a_l^R(m)$. There exists a nonzero left ideal J of R with $I \cap J = 0$. The map $g : Jm \longrightarrow J$ given by $g(am) = a$ for $a \in J$ is a well defined R-module homomorphism. Thus, there exists $w \in W$ such that $a = amw$ for all $a \in J$. Then, $Jm(1 - wm) = 0$, implying $wm = 1$.
2. Let $0 \neq r \in R$. There exists $m \in M$ with $rm \neq 0$, then there exists $w \in W$ such that $w(rm) = 1$. Thus, $Wr \neq 0$. ■

Lemma 1.12. *Let R be a nonsingular irreducible ring and M a uniform nonsingular module. Then, W is a faithfull R-module.*

Proof. Set $T = MW \cap R$ and $Q = a_r^R(T)$. $T \neq 0$ by Lemma 1.11. First we show that for every left ideal $J \neq 0$ of R contained in T, there exist $m \in M$ and $w \in W$ with $0 \neq mw \in J$. Consider $0 \neq b = \sum_{i=1}^{n} m_i w_i \in J$ with n is as small as possible. We claim that $n = 1$. Set $m = m_1$. The map $p : Rm \longrightarrow Rb$, given by $p(rm) = rb$, is a well defined R-module homomorphism. Thus, there exists $w \in W$ such that $b = mw$.
Now suppose $Q \neq 0$, then $T \cap Q \neq 0$. Thus, there exists $m \in M$ and $w \in W$ such that $0 \neq mw \in T \cap Q$. On the other hand there exist $u \in W$ and a left ideal $J \neq 0$ such that $um = 1$ and $aum = a$ for all $a \in J$, by Lemma 1.11. Then, $amw = (amu)mw \in Tmw = 0$ for all $a \in J$, implying $Jm = 0$. Thus, $J = Jmu = 0$ which is a contradiction. (This proof has been adopted from [10, Theorem 2.1])

Lemma 1.13. *Let R be a ring and M a faithfull uniform nonsingular module. For every independent set $\{w_1, w_2, \ldots, w_n\} \subseteq W$, there exist $m_1, m_2, \ldots, m_n \in M$ such that the metrix $A = [w_i m_j]$ is invertible and for $f = \sum_{i=1}^n m_i w_i$, $Ker(f) \oplus < w_1, w_2, \ldots, w_n >= W$.*

Proof. Set $K = \{w_1, w_2, \ldots, w_n\}$ and $N =< K >$. Consider S^{op} as a subring of $End({}_D W)$ and choose $f \in MK$ such that $\dim(f(N))$ is maximal. Since $f(W) \subseteq N$ and N is finite dimensional, $f(N) = N$ if and only if $Ker(f) \cap N = 0$ and in this case $Ker(f) \oplus N = W$. We claim that $f(N) = N$. Suppose it is not so. Then, $Ker(f) \cap N \neq 0$ and $N \nsubseteq f(N)$. Consider $0 \neq w_0 \in Ker(f) \cap N$. There exists an M-R-dense submodule L of M such that $Lw_0 \subseteq R$, then there exists $m \in M$ such that $(Lw_0)m \neq 0$, impluing $d = w_0 m \neq 0$. Consider $w \in K - f(N)$ and set $g = f + mw$. For every $u \in N$, if $g(u) = 0$, then $f(u) + (um)w = 0$, so $um = 0$ and $f(u) = 0$. Thus, $Ker(g) \cap N \subseteq Ker(f) \cap N$. On the other hand, $w_0 \in Ker(f) \cap N$ and $g(w_0) = dw_0 \neq 0$, Thus, $\dim(g(N)) > \dim(f(N))$ which is a contradiction.
There exist $m_1, m_2, \ldots, m_n \in M$ such that $f = \sum_{i=1}^n m_i w_i$. Since $f(N) = N$, there exist $u_i \in N$ such that $f(u_i) = w_i$. On the other hand, there exist $a_{ij} \in D$ such that $u_k = \sum_{i=1}^n a_{ki} w_i$, then

$$(\sum_{i=1}^n a_{ki} w_i)(\sum_{j=1}^n m_j w_j) = w_k,$$

implying $\sum_{i=1}^n a_{ki}(w_i m_j) = \delta_{kj}$. ■

Lemma 1.14. *Let R be a nonsingular prime ring containing a uniform left ideal and a uniform right ideal and M be a uniform left ideal. Then, W is uniform.*

Proof. Suppose $0 \neq v, w \in W$. There exists a left ideal $0 \neq L \subseteq M$ such that $0 \neq Lv, Lw \subseteq R$. On the other hand, R contains a nonzero primitive element s by Lemma 1.1. Since $\{m \in L \mid smv = 0\} \cup \{m \in L \mid smw = 0\} \neq L$, there exists $m \in L$ such that $smv, smw \neq 0$. Since sR is uniform by Lemma 1.6, there exist $r, t \in R$ such that $smvr = smwt \neq 0$. Now there exists $u \in W$ such that $usm = 1$ by Lemma 1.11, implying $ur = wt \neq 0$. ■

2. Various Versions of the Density Theorem

Theorem 2.1. *Let R be a quasi-primitive ring. There exist a division ring D and a vector space U such that R can be embbeded in $End(U_D)$ as a unifomly dense subring.*

Proof. Let M be a faithfull, monoform, total and quasi-compressible module. Suppose that $0 \neq v_1, v_2, \ldots, v_m \in V$ and $u_1, u_2, \ldots, u_n \in U$. We claim that there exist $0 \neq d \in D$ and $f_{ij} \in R$ such that $f_{ij}(v_j) = u_i d$. There exists $0 \neq a \in D$ such thay $u_i a \in M$ for all $1 \leq i \leq n$. On the other hand, there exists a nonzero left ideal I of R such that $0 \neq Iv_j \subseteq M$ for all $1 \leq j \leq m$, then there exists $0 \neq b \in D$ such that $(u_i a)b \in \cap_{j=1}^{m} Iv_j$ for all $1 \leq i \leq n$. Thus, there exists $f_{ij} \in I$ such that $u_i ab = f_{ij}(v_j)$.
Now let $v_1, v_2, \ldots, v_n \in U$ be an independent set and $u_1, u_2, \ldots, u_n \in U$. Set $V_i = < v_1, v_2, \cdots, v_{i-1}, v_{i+1}, \cdots v_n >$. There exists $g_i \in R$ such that $g_i(V_i) = 0$ and $g_i(v_i) \neq 0$, then ther exist $0 \neq d \in D$ and $f_i \in R$ such that $f_i(g_i(v_i)) = u_i d$. Set $f = \sum_{i=1}^{n} f_i g_i$. ■

Theorem 2.2. *Let R be an almost primitive ring. There exist a division ring D and a vector space U such that R can be embbedded in $End(U_D)$ as a semi-dense subring.*

Theorem 2.3. *Let R be a ring having a faithfull nonsingular uniform module. Then, there exist a division ring D and a vector space V such that $Q_m(R)$ can be embbeded in $End(V_D)$ as an essentially dense subring.*

Proof. First we show that V is a simple S-module. For that, let $0 \neq u, v \in V$, we show that there exist $f \in S$ such that $f(v) = u$. is maximal respect to $I \cap J = 0$. Set $K = (Jv : u)^R = \{r \in R \mid ru \in Jv\}$. K Set $I = a_l^R(v)$. There exists a nonzero left ideal J of R such that is a nonzero left ideal of R, moreover for every $k \in K$ there is a unique $h \in J$ with $ku = hv$. Then, the map $g : K \longrightarrow J$ given by $g(k) = h$ is an R-module homomorphism. Thus, there exists $f \in S$ such that $g(k) = kf$ for all $k \in K$, implying $K(fv - u) = 0$. It is enough to show that K is an essential left ideal of R. Suppose L is a left ideal of R such that $K \cap L = 0$. Then, $Ku \cap Lu = 0$. Set $N = (Lu : v)^R$. Then, $N \cap J = 0$. On the other hand, I is closed by [1]. Thus, $I = N$, because of the maximality of J, then $Lu \cap Rv = 0$, so Jv is an essential R-submodule of V). Therefore, S is a dense subring of $Lu = 0$ which implies $L = 0$ (Or we coud say that K is essential because $End(V_D)$. Now it is enough to show that S contains a rank 1 element.
Consider $0 \neq m \in M$. There exists $w \in W$ with $wm = 1$. Set $f = mw$, then $f(V) = mD$. ■

Theorem 2.4. *Let R be a nonsingular irreducible ring containing a uniform left ideal. Then, there exist a division ring D and a vector space W and R^{op} can be embbeded in $End(_DW)$ such that for every $0 \neq w \in W$ and $h_1, h_2, \ldots, h_n \in End(_DW)$ that $h_1(w) \neq 0$, there exists $f \in R$ with $f(W) = Dw$ and $h_i f \in R$ for all $1 \leq i \leq n$.*
(See also [10, Theorem 2.1])

Proof. Let M be a uniform left ideal. There exists an M-R-dense submodule L such that $Lw, Lh_k(w) \subseteq R$. On the other hand, there exists $m \in M$ such that $mh_1(w) \neq 0$. Set $f = mw$ and $g_k = mh_k(w)$. Then $f, g_k \in R$, $h_k f = g_k$, $g_1 \neq 0$ and $f(W) = Dw$ by Lemma 1.11. ■

Theorem 2.5. *Let R be a right nonsingular, right irreducible ring containing a uniform right ideal. Then, there exist a division ring D and a vector space W and R can be embbeded in $End({}_DW)$ such that $Q_{mr}(R) = End({}_DW)$.*
(See also [10, Theorem 2.1])

Corollary 2.6. *Let R be a nonsingular irreducible ring containing a uniform leftt ideal. Then, every nonzero left ideal contains a uniform left ideal.*
(See also [10, Theorem 2.1])

Theorem 2.7. *Let R be a prime ring with nonzero socle. Then, there exist a division ring D and a vector space W and R can be embbeded in $End({}_DW)$ as a critically dense subring.*

Proof. Let M be a minimal right ideal. Consider M as a left R^{op}-module and consider R as a subring of $End({}_DW)$. Let $\{v_1, v_2, \ldots, v_n\} \subseteq W$ be an independent set. By Lemma 1.13, there exist $m_1, m_2, \ldots, m_n \in M$ such that the matrix $A = [v_i m_j]$ is invertible and for $g = \sum_{i=1}^n m_i v_i$, $Ker(g) \oplus < v_1, v_2, \ldots, v_n >= W$. Suppose $w_1, w_2, \ldots, w_n \in W$. There exist $x_1, x_2, \ldots, x_n \in W$ such that $w_i = \sum_{j=1}^n (v_i m_j) x_j$. Set $f = \sum_{i=1}^n m_i x_i$. Then, $f \in R$, moreover for every $1 \leq i \leq n$ and $w \in Ker(g)$,

$$f(w) = \sum_{i=1}^n w(m_i x_i) = \sum_{i=1}^n (w m_i) x_i = 0$$

$$f(v_i) = v_i \sum_{j=1}^n m_j x_j = \sum_{j=1}^n (v_i m_j) x_j = w_i. \blacksquare$$

Theorem 2.8. *Let R be a nonsingular prime ring containing a uniform right ideal. Then, there exist a division ring D and a vector space W and R can be embbeded in $End({}_DW)$ as a unifomly compressible*

subring.
(See also [10, Theorem A])

Proof. Let M be a uniform right ideal. Consider M as a left R^{op}-module and consider R as a subring of $End({}_DW)$. Let $\{v_1, v_2, \ldots, v_n\} \subseteq W$ be an independent set. By Lemma 1.13, there exist $m_1, m_2, \ldots, m_n \in M$ such that the matrix $A = [v_i m_j]$ is invertible and for $g = \sum_{i=1}^{n} m_i v_i$, $Ker(g) \oplus < v_1, v_2, \ldots, v_n >= W$. Let $w_{ki} \in W$. Since M is compressible by Lemma 3.5, there exists $0 \neq c \in D$ such that $cw_{ki} \in W_M$ for all $1 \leq i \leq n$ and $1 \leq k \leq m$. Set $f_k = \sum_{i=1}^{n} m_i c w_{k_i}$. Then, $f_k \in R$, moreover for every $1 \leq i \leq n$ and $w \in Ker(g)$,

$$f_k(w) = \sum_{i=1}^{n} w(m_i c w_{ki}) = \sum_{i=1}^{n} (wm_i) c w_{ki} = 0$$

$$f_k(v_i) = v_i \sum_{j=1}^{n} m_j c w_{kj} = \sum_{j=1}^{n} (v_i m_j) c w_{kj}. \blacksquare$$

Theorem 2.9. *Let R be a nonsingular prime ring containing a uniform right ideal and a uniform left ideal. Then, there exist a division ring D and a vector space W and R can be embbeded in $End({}_DW)$ as a unifomly and essentially dense subring.*
(See also [3, Theorem 3.1])

Proof. Let M be a uniform right ideal. Consider M as a left R^{op}-module and consider R as a subring of $End({}_DW)$. Let $\{v_1, v_2, \ldots, v_n\} \subseteq W$ be an independent set. There exist a nonzero submodule L_i such that $L_i v_i \subseteq R$. We show that $\{L_1 v_1, L_2 v_2, \cdots, L_n v_n\}$ is an independent set of uniform left ideals of R. Suppose $m_i \in L_i$ and $\sum_{i=1}^{n} m_i v_i = 0$. If $m_j \neq 0$, then there is $w \in W$ such that $wm_j \neq 0$ by Lemma 1.11, then $\sum_{i=1}^{n} (wm_i) v_i = 0$ which is a contradiction because $\{v_1, v_2, \ldots, v_n\}$ is independent. Obviously each $L_i v_i$ is uniform. Thus, by Lemma 1.10,

there exist $h_j \in R$ such that $(L_j v_j)h_j \neq 0$ while $(L_i v_i)h_j = 0$ implying $v_j h_j \neq 0$ and $v_i h_j = 0$ for all $i \neq j$.
By Lemma 1.13, for every $0 \neq v \in W$, there exist $m \in M$ such that $vm \neq 0$ and for $g = mv$, $Ker(g) \oplus < v >= W$. Let $0 \neq c \in D$ and $0 \neq u \in W$ and set $p_{vuc} = mcu$ and $d_v = vm$. Then, $vp_{vuc} = d_v cu$, moreover $wp_{vuc} = 0$ for all $w \in Ker(g)$, so p_{vuc} is of rank 1. Furthermore, if $Mcu \subseteq R$, then $p_{vuc} \in R$. Let $w_1, w_2, \ldots, w_n \in W$. Consider $0 \neq w \in W$. There exists $0 \neq c \in D$ such that $Mcw \subseteq R$ and $Mcw_i \subseteq R$ for all $1 \leq i \leq n$. On the other hand, there exist $g_i \in R$ such that $(v_1 h_1)g_1 = (v_2 h_2)g_2 = \cdots = (v_n h_n)g_n \neq 0$ by Lemma 1.14. Set $v = v_1 h_1 g_1$. Then, there exists a rank 1 element $p = p_{vwc} \in R$ and $0 \neq a = d_v \in D$ such that $vp = acw$. Furthermore, there exist $p_i = p_{ww_i c} \in R$ and $0 \neq b = d_w \in D$ such that $wp_i = bcw_i$. Set $f = \sum_{i=1}^{n} h_i g_i p p_i$ and $d = acbc$. Then $f \in R$, f is of rank at most n and $v_j f = v_j h_j g_j p p_j = vpp_j = acwp_j = dw_j$. ■
(It also can be deriven by applying Lemma 1.11 to drive that R contains a rank 1 element, then to apply Theorem 2.1 and Corollary 3.7)

Theorem 2.10. *Let R be a prime ring in which every left ideal is principal. Then, R is isomorphic to a total matrix ring over a unital Ore domain.*
(See also [2, Lemma 4.11])

Proof. In the following, E, N, σ and $l(L)$ are as those used in [3]. Since R is left Noetherian, R contains a uniform left ideal, moreover R is nonsingular. Thus, by Theorem 2.5, there exist a division ring D and a vector space W and R^{op} can be embbeded in $End(_D W)$ such that $Q_{mr}(R^{op}) = End(_D W)$. By [3, Theorem 2.10], there exists a subspace L with $Codim(L) = 1$ such that $R^{op} \leq_L Hom_D(W, W)$, so E is a unital left Ore domain by [3, Lemma 2.5]. We show that $R^{op} \cong Mat_n(E^{op})$, implying $R \cong Mat_n(E)$. Since every independent set of right ideals of

R^{op} is finite, every independent set of right ideals of $End(_DW)$ is also finite, thus, $dim(W) < \infty$. Set $n = dim(W)$. First, we show that $DN = W$. Suppose $v \in W$. Consider $u \in W - L$ and $0 \neq g \in R \cap l(L)$. Since $g(u) \neq 0$, there exist $0 \neq b \in D$ and $f \in R$ such that $f(g(u)) = bv$ by [3, Theorem 2.8]. Set $h = fg$ and $w = b^{-1}v$, then $h(w) = v$ and $h \in R \cap l(L)$. Thus, there exists a basis $\{u_1, u_2, \ldots, u_n\} \subseteq N$ for W. Consider $g_i \in R \cap l(L)$ such og R because otherwise there exist $0 \neq g \in R \cap l(L)$ with $gR \cap J = 0$, that $\sigma(g_i) = u_i$. $J = \sum_{i=1}^{n} g_i R$ is an essential right ideal implying that $\{u_1, u_2, \ldots, u_n\} \cup \{\sigma(g)\}$ is independent which is a contradiction. Hence, J contains a regular element g becaus R is Noetherian. $J = fR$ for some $f \in J$, implying $g = fh$ for some $h \in R$. Furthermore, there exist $p_i \in R$ such that $f = \sum_{i=1}^{n} g_i p_i$. Since g is regular, f is also regular. On the other hand, there exist $h_i \in R$ such that $g_i = fh_i$, then $f = \sum_{i=1}^{n} fh_i p_i$, implying $1 = \sum_{i=1}^{n} h_i p_i$, moreover $h_i \in R \cap l(L)$ because f is one to one. Set $v_i = \sigma(h_i)$ Then, $u_i = \sigma(g_i) = f(\sigma(h_i)) = f(v_i) \in N$. Thus, $v_i \in N$ and $\{v_1, v_2, \ldots, v_n\}$ is a basis for W. Now let $v \in N$. Then, $v = \sum_{i=1}^{n} h_i(p_i(v)) = \sum_{i=1}^{n} \mu(pi(v))v_i = \sum_{i=1}^{n} e_i v_i$. Thus, $\{v_1, v_2, \ldots, v_n\}$ is a basis for N as an E-module. There exist $e \in E$ such that $v_1 = ev_1$ implying $1 = e \in E$. Now suppose $q \in End(_DW)$ such that $q(N) \subseteq N$. Then, $q = \sum_{i=1}^{n} qh_i p_i \in R$ because it is easy to see that $qh_i \in R$. Thus, by [3, Lemma 2.7], $R^{op} = \{q \in End(_DW) \mid q(N) \subseteq N\} \cong End(_EN) \cong Mat_n(E^{op})$. ■

3. Some Quasi-primitive Rings and Derivations on Almost Semiprimitive Rings

Theorem 3.1. *If R be an almost primitive ring and μ is a derivatioin of R having only nilpotent values on the left ideal L of R, then $L\mu(L) = 0$ and $\mu(L)^2 = 0$.*

Proof. We may assume that R is a semi-dense subring of a full linear ring $End(_D V)$. Set $M = \cap_{f\in L} Ker(f)$. We show that $\mu(L)(V) \subseteq M$. Let $f \in L$. We show the following.

1. For every $v \in V$, if $f(v) = 0$, then $\mu(f)(v) = 0$.
Suppose it is not so. There exist $g \in L$ and $0 \neq d \in D$ such that $g\mu(f)(v) = dv$. Then, $\mu(gf)(v) = \mu(g)f(v) + g\mu(f)(v) = dv$. On the other hand, $\mu(gf)^n = 0$ for some $n \geq 1$, which is a contradiction.

2. For every $v \in V$, if $f(v) \in M$, then $\mu(f)(v) \in M$.
We have $\mu(L)f(v) = 0$, $\mu(Lf)(v) = 0$ and $\mu(Lf)(v) = \mu(L)f(v) + L\mu(f)(v)$. Thus, $L\mu(f)(v) = 0$. Therefore, $\mu(f)(v) \in M$.

3. For every $v \in V$, there exists $a_v \in D$ such that $\mu(f)(v) \in a_v f(v) + M$.
Suppose it is not so. There exist $v \in V$ such that $v \notin Df(v) + M$. Then, $\mu(f)(v) \notin M$, so $f(v) \notin M$. Thus, $\mu(f)(v)$ and $f(v)$ are independent, moreover $< \mu(f)(v), f(v) > \cap M = 0$. Hence, there exists $0 \neq d \in D$ and $g \in L$ such that $gf(v) = 0$ and $g\mu(f)(v) = dv$, implying $\mu(g)f(v) = 0$ and $\mu(gf)(v) = \mu(g)f(v) + g\mu(f)(v) = dv$. On the other hand, $\mu(gf)(v) = 0$ which is a contradiction.

4. For every $v, w \in V$, if $f(v) + M$ and $f(w) + M$ are independent, than $a_v = a_w$.
Since $\mu(f)(v) \in a_v f(v) + M$, $\mu(f)(w) \in a_w f(w) + M$ and $\mu(f)(v+w) \in a_{v+w} f(v+w) + M$, $a_v f(v) + a_w f(w) - a_{v+w} f(v+w) \in M$, implying $a_v = a_w$.

5. If $\dim(f(V)/M) \geq 2$ and $v, w \in V$ such that $f(v), f(w) \notin M$, then $a_v = a_w$.
It is clear for the case that $f(v) + M$ and $f(w) + M$ are independent by (4). Otherwise, there exists $u \in V$ such that $f(u) + M$ and $f(v) + M$ are independent, implying $a_v = a_u$. Since $f(u) + M$ and $f(w) + M$ are also independent, $a_w = a_u$.

6. If $\dim(f(V)/M) \geq 2$ and $\mu(f)^n = 0$, then for every $v \in V$, either

$f^n(v) \in M$ or $\mu(f)(v) \in M$.

Suppose $v \in V$ such that $f^n(v) \notin M$. There exist $m_i \in M$ and $a \in D$ such that $\mu(f)(f^i(v)) = af^{i+1}(v) + m_i$. Then, $\mu(f)^2(v) = \mu(f)(af(v) + m_0) \in a^2f^2(v) + M$. Continuing this process, we have $0 = \mu(f)^n(v) \in a^n f^n(v) \in M$, implying $a = 0$. Therefore, $\mu(f)(v) \in M$.

7. If $\dim(f(V)/M) = 1$ and $\mu(f)^n = 0$, then for every $v \in V$, either $f^n(v) \in M$ or $\mu(f)(v) \in M$.

It is clear for the case $n = 1$. So, let $n \geq 2$ and $v \in V$ such that $f^n(v) \notin M$. Set $w = f(v)$. Then, $w \notin M$, so $f(V) = Dw + M$, similarly $f(w) \notin M$ and $f(V) = Df(w) + M$. There exist $0 \neq a \in D$ such that $f(w) \in aw + M$ and $b \in D$ such that $\mu(f)(w) \in bf(w) + M = baw + M$. Set $c = ba$. Then, $\mu(f)^n(w) \in c^n w + M$. So, $c^n = 0$, implying $b = 0$. Thus, $\mu(f)(w) \in M$. Set $K = f^{-1}(M)$. There exists $d \in D$ such that $f(v) \in df(w) + M$, implying $v \in dw + K$. Since $f(K) \subseteq M$, $\mu(f)(K) \subseteq M$. Therefore, $\mu(f)(v) \in M$.

8. If $\mu(f)^n = 0$, then either $f^n(V) \subseteq M$ or $\mu(f)(V) \subseteq M$.

Set $A = \{v \in V \mid \mu(f)(v) \in M\}$ and $B = \{v \in V \mid f^n(v) \in M\}$. Since A and B are subgroups of V and $A \cup B = V$, either $A = V$ or $B = V$.

9. If there exists $v \in V$ such that $f^2(v) \in M$ and $f(v) \notin M$, then $\mu(f)(V) \subseteq M$.

It is clear if $f^n(V) \not\subseteq M$. Otherwise, set $w = f(v)$. We claim that $v + M$ and $w + M$ are independent. Suppose it is not so. There exists $a \in D$ such that $w + M = a(v + M)$. Then, ther exists $m \in M$ such that $f(v) = av + m$, implying $f^n(v) \in a^n v + M$. Thus, $a = 0$, implying $f(v) \in M$ which is a contradiction. Hencs, there exist $g \in L$ and $0 \neq a, b \in D$ such that $g(v) = av$ and $g(w) = bw$. Since for every $k \geq 1$, $g^k(w) \notin M$ and $(f + g)^k(w) \notin M$, $\mu(g)(V) \subseteq M$ and $\mu(f + g)(V) \subseteq M$. Therefore, $\mu(f)(V) \subseteq M$

10. If there exists no $v \in V$ such that $f^2(v) \in M$ and $f(v) \notin M$, then

$\mu(f)(V) \subseteq M$.
It is clear if there exists no $n \geq 1$ such that $f^n(V) \subseteq M$ by (8). Otherwise, $f(V) \subseteq M$, implying $\mu(f)(V) \subseteq M$ by (2). ■

Theorem 3.2. *If R be an almost semiprimitive ring and μ is a derivatioin of R having only nilpotent values on the left ideal L of R, then $L\mu(L) = 0$.*

Proof. The same proof of [4, Theorem 4] can be applied because, in an almost semiprimitive ring, every nil left (right) ideal is zero. ■

Lemma 3.3. *Let R be a nonsingular prime ring and M a module. If there exist a uniform left ideal K and a unifom right ideal L such that $LK \neq 0 \neq KM$, then M is transitive.*

Proof. $(a_l(M) \cap K) \cup (a_r(L) \cap K) \neq K$ because otherwise $a_l(M) \cap K = K$ or $a_r(L) \cap K = K$ which is impossible. So, there exists s is a primitive element by Lemma 1.1, sR is uniform by Lemma 1.4 and Lemma 1.6. $s \in K$ with $Ls \neq 0 \neq sM$. Consider $a \in M$ with $sa \neq 0$. Since On the other hand, $0 \neq sa_r^R(a_l(a)) \subseteq sR$. ■

Lemma 3.4. *Let R be a ring, K a left ideal and M a faithfull and cofaithfull module. If $u, v \in K$ such that $a_l(u) \subseteq a_l(v)$ and Ru is a transitive module, then there exist $f, g \in Hom_R(K, M)$ and $a \in R$ such that $f(au) = g(av) \neq 0$.*

Proof. Consider $a, s \in R$ such that $sa_r^R(a_l(au))$ is a uniform right ideal, then $sau \neq 0$. Since $sav \in sa_r^R(a_l(av)) \subseteq sa_r^R(a_l(au))$ and $sau \in sa_r^R(a_l(au))$, there exist $k, l \in R$ such that $sauk = savl \neq 0$, then there exists $m \in M$ with $saukm \neq 0$. Set $c = km$ and $d = lm$. Consider $f, g \in Hom_R(K, M)$ given by $f(x) = xc$ and $g(x) = xd$. Then, $f(xsau) = xsauc = xsavd = g(xsav)$ for all $x \in R$. On the other hand, there exists $b \in R$ such that $f(bsau) = g(bsav) \neq 0$. ■

Corollary 3.5. *Let R be a nonsingular prime ring containing a uniform left ideal and a uniform right ideal, then every left ideal is a total module.*

Lemma 3.6. *Let R be a nonsingular prime ring. Then, every uniform left ideal is compressible.*

Corollary 3.7. *Every nonsingular prime ring containing a uniform left ideal and a uniform right ideal, is quasi-primitive and almost primitive.*

Proof. Let M be a uniform left ideal and $0 \neq N \subseteq M$ a left ideal. There exists $n \in N$ with $Mn \neq 0$. Consider $f \in Hom_R(M, N)$ given by $f(x) = xn$. ■

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حلقه‌ها Quasi-primitive و ق ه چگالی

در ا ن مقاله حلقه‌های Quasi-primitive و almost-semiprimitive معرف و نشان داده م شود که ا ن حلقه‌های R ک ز ر حلقه به ور کنواخت چگال ک حلقه خودر خت‌ های $End(_DW)$ است که در ن D ک حلقه‌تقس م و V ک ف ای برداری روی D است عن برای هر مجموعه مستقل خ $\{v_1, v_2, ..., v_n\} \subseteq V$ و هر $\{u_1, u_2, ..., u_n\} \subseteq V$ $0 \neq d \in D$ و $f \in R$ وجود دارد به ور که $f(v_i) = du_i$ برای تمام مقاد ر $1 \leq i \leq n$ نگاه نشان م ده م حلقه‌های اول ناتک ن که شامل ک ا د ل چپ کنواخت و ک ا د ل راست کنواخت هستند نمونه‌ای از ا ن حلقه‌ها م باشند سرانجام ق ه اصل را تعم م م ده م به ا ن صورت که نشان م ده م هرگاه d ک مشتق روی ک حلقه almost-semiprimitive باشد و L ک ا د ل چپ به ور که برای هر l $d(l)$ پوچتوان باشد نگاه $Ld(L) = 0$