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TOPOLOGY ON COALGEBRAS

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Abstract: In this paper we define coprime subcoalgebra and we characterize finite dimensional coprime coalgebras. We then construct a topology on coprime subcoalgebras. Finally we discuss some properties of coprime subcoalgebras and the topology induced by this type of subcoalgebras.

Introduction and Preliminaries.

We assume the reader is familiar with topology [see, 3]. A coalgebra is a triple (C, Δ, ϵ),where C is a vector space over a field

$$K, \Delta : C \longrightarrow C \underset{K}{\otimes} C$$

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and $\epsilon : C \longrightarrow K$ are linear maps such that $(\Delta \otimes I)o\Delta = (I \otimes \Delta)o\Delta$ and $(\epsilon \otimes I)o\Delta = (I \otimes \epsilon)o\Delta = I$. A subcoalgebra D of a coalgebra C is simple if it has no non-trivial subcoalgebra. We denote the sum of all simple subcoalgebras of a coalgebra C by $corad(C)$. We say that a coalgebra C is semisimple if $corad(C) = C$, irreducible if it has a unique non-zero simple subcoalgebra and pointed if $dim(D) = 1$, for all simple subcoalgebras D.

Let V be any vector space, S a subset of V. By $S^{\perp} \subseteq V^*$ we mean $f \in V^*| < f, s >= 0$. If T is a subset of V^*, by $T^{\perp} \subseteq V$ we mean $\{v \in V| < f, v >= 0, \text{for all} f \in T\}$.

A subcoalgebra D of C is conilpotent if and only if $corad(C) \subseteq D$. For any subcoalgebras X and Y of a coalgebra C, we denote $X \wedge Y$ by $\Delta^{-1}(C \otimes Y + X \otimes C)$ or $(X^{\perp}Y^{\perp})^{\perp}$.

1. Coprime Subcoalgebras of a Coalgebra.

Definition. A non-zero subcoalgebra P of a coalgebra C is called coprime if $P \subseteq X \wedge Y$ then $P \subseteq X$ or $P \subseteq Y$, for any subcoalgebras X and Y of C.

Proposition 1.1. *Let C be a coalgebra and P be a prime ideal of C^* such that $P^{\perp\perp} = P$. Then $P^{\perp}$ is a coprime subcoalgebra of C.*

Proof. (C, Δ, ϵ) is a coalgebra, hence (C^*, M, U) is an algebra such that $M = \Delta^* o\rho$ where $\rho : C^* \otimes C^* \longrightarrow (C \otimes C)^*$ is canonical injection linear map [4, prop.1.1.1]. Let X and Y be subcoalgebras of C. We know that $X^{\perp}$ and $Y^{\perp}$ are two-sided ideals of C^* and if $P^{\perp} \subseteq X \wedge Y = \Delta^{-1}(X \otimes C + C \otimes Y)$, then

$$\begin{aligned} \Delta(P^{\perp}) &\subseteq X \otimes C + C \otimes Y \\ &= (X^{\perp})^{\perp} \otimes C + C \otimes (Y^{\perp})^{\perp} \\ &= \rho(X^{\perp} \otimes Y^{\perp})^{\perp} \end{aligned}$$

Hence $< \rho(X^\perp \otimes Y^\perp), \Delta(P^\perp) >= 0$ or $\Delta^* o\rho(X^\perp \otimes Y^\perp) \subseteq P$. We conclude that $X^\perp \subseteq P$ or $Y^\perp \subseteq P$, since P is a prime ideal of C^*. So $P^\perp \subseteq (X^\perp)^\perp = X$ or $P^\perp \subseteq Y$ and the proof is complete. ■

Note: If $dim(C^*) < \infty$ then the converse of Proposition 1.1 is true.

Proposition 1.2. *The subcoalgebra P of a coalgebra C is coprime if and only if $P^\perp$ is a prime ideal of C^*.*

Proof. Let P be a coprime subcoalgebra of C and let A,B be two-sided ideals of C^* such that $\Delta^* o\rho(A \otimes B) \subseteq P^\perp$. We must show that $A \subseteq P^\perp$ or $B \subseteq P^\perp$. We have $< \Delta^* o\rho(A \otimes B), P >= 0$, so $\Delta(P) \subseteq \rho(A \otimes B)^\perp = A^\perp \otimes C + C \otimes B^\perp$. Hence $P \subseteq A^\perp \wedge B^\perp$. P is coprime, so $P \subseteq A^\perp$ or $P \subseteq B^\perp$. Therefore $A \subseteq P^\perp$ or $B \subseteq P^\perp$. The converse is clear by Proposition 1.1. ■

Proposition 1.3. *Every simple subcoalgebra P of C is coprime.*

Proof. Suppose M and N are subcoalgebras of C and $P \subseteq N \wedge M$. Let $P \not\subseteq M$. Because P is a simple subcoalgebra, so $P \cap M = \{0\}$. Hence there exists $f \in C^*$ such that $f|_P = \epsilon$ and $f|_M = 0$. $P \subseteq N \wedge M$ so $\Delta(P) \subseteq N \otimes C + C \otimes M$. Let x be an arbitrary element of P, we have

$$
\begin{aligned}
x = (I \otimes \epsilon)(\Delta(x)) &= \sum_{(x)} x_{(1)} \epsilon(x_{(2)}) \\
&= \sum_{(x)} x_{(1)} < f, x_{(2)} > \\
&= (I \otimes f)(\Delta(x)).
\end{aligned}
$$

Since $\Delta(x) \in N \otimes C + C \otimes M$, so $x = (I \otimes f)(\Delta(x)) \in N < f, C > \subseteq N$. We conclude that $P \subseteq N$ and the proof is complete. ■

Example 1.1. Let C be a vector space with basis $\{C_i\}_{i=0}^{\infty}$. If $\Delta(C_i) = C_i \otimes C_i$ and $\epsilon(C_i) = 1$, $i = 1, 2, \ldots$, then (C, Δ, ϵ) is a coalgebra.

It is clear that the subcoalgebras generated by C_i $(i = 1, 2, \ldots)$ are simple, hence by Proposition 1.3, are coprimes. Let $T =< C_0, C_1 >$ be the subcoalgebra of C generated by C_0 and C_1. Since $< C_0, C_1 > \subseteq < C_0 > \wedge < C_1 >$ but $< C_0, C_1 > \not\subseteq < C_1 >$ and $< C_0, C_1 > \not\subseteq < C_2 >$, so T is not coprime. It is not difficult to show that the only coprime subcoalgebra of C has the form $< C_i >$ $(i = 1, 2, \ldots)$.

Example 1.2. Let C be a vector space with basis $\{C_i\}_{i=0}^{\infty}$. If $\Delta(C_i) = \sum_{j=0}^{i} C_j \otimes C_{i-j}$ and $\epsilon(C_i) = \delta_{i0}$, $i = 1, 2, \ldots$, then (C, Δ, ϵ) is a coalgebra. But $< C_0 >$ is a simple subcoalgebra of C then it is coprime.

We have $\Delta(< C_0, C_1, \ldots, C_i >) \subseteq < C_0, C_1, \ldots, C_{i-1} > \otimes C + C \otimes < C_0 >$, so $< C_0, C_1, \ldots, C_i > \subseteq < C_0, C_1, \ldots, C_{i-1} > \wedge < C_0 >$, but $< C_0, C_1, \ldots, C_i > \not\subseteq < C_0 >$ and $< C_0, C_1, \ldots, C_i > \not\subseteq < C_0, C_1, \ldots, C_{i-1} >$. Hence $< C_0, C_1, \ldots, C_{i-1}, C_i >$ is not coprime, (note that the subcoalgebra generated by C_i $(i = 1, 2, \ldots)$ is equal to the subspace generated by $\{C_0, C_1, \ldots, C_i\}$). However $\Delta(C_i) = \sum_{j=0}^{i} C_j \otimes C_{i-j}$, so the subcoalgebras generated by infinitly many of C_i's is equal to C and clearly C is coprime. We conclude that the only coprime subcoalgebras of is $< C_0 >$ and C.

Lemma 1.1. *Let $f : C \longrightarrow C$ be a coalgebra isomorphism. Then*

$$f\left(\sum_{P \text{ is coprime}} P\right) = \sum_{P \text{ is coprime}} P.$$

Proof. First we claim that $f(P)$ is a coprime subcoalgebra of C where P is a coprime subcoalgebra of C. Let X and Y be subcoalgebras of C such that $f(P) \subseteq X \wedge Y$, we have $\Delta(f(P)) \subseteq X \otimes C + C \otimes Y$. But f is coalgebra map, then $f \otimes f(\Delta(P)) \subseteq X \otimes C + C \otimes Y$. Hence $P \subseteq \Delta^{-1}(f^{-1}(X) \otimes C + C \otimes f^{-1}(Y)) = f^{-1}(X) \wedge f^{-1}(Y)$. Since P is coprime, so $f(P) \subseteq X$ or $f(P) \subseteq Y$. By a similar proof we have $f^{-1}(P)$ is coprime when P is a coprime and the proof is complete. ■

Lemma 1.2. *Let $\{P_\alpha\}_{\alpha\in I}$ be a family of coprime subcoalgebras of a coalgebra C such that for any $\alpha, \beta \in I$, $P_\alpha \subseteq P_\beta$ or $P_\beta \subseteq P_\alpha$. Then $\bigcup_{\alpha\in I} P_\alpha = \sum_{\alpha\in I} P_\alpha$ and it is a coprime subcoalgebra of C.*

Proof. By the assumption , we have $\bigcup_{\alpha\in I} P_\alpha = \sum_{\alpha\in I} P_\alpha$, so it is enough to show that $E = \bigcup_{\alpha\in I} P_\alpha$ is a coprime subcoalgebra. It is clear that $\bigcup_{\alpha\in I} P_\alpha$ is a subcoalgebra of C. Let C_1 and C_2 be subcoalgebras such that $E \subseteq C_1 \wedge C_2$, so for any $\beta \in I$, $P_\beta \subseteq C_1$ or $P_\beta \subseteq C_2$. If $P_\beta \subseteq C_1$ and $P_\beta \nsubseteq C_2$ then $P_\beta \subseteq P_\alpha$ or $P_\alpha \subseteq P_\beta$, for some $\alpha \in I$. Suppose that $P_\beta \subseteq P_\alpha$ since $P_\beta \nsubseteq C_2$, hence $P_\alpha \nsubseteq C_2$. Therefore $P_\alpha \subseteq C_1$ and

$E \subseteq C_1$. The proof is complete. ■

Lemma 1.3. *Let C be a cocommutative coalgebra and $M_1, \ldots, M_n$ are non-zero distinct simple subcoalgebras of C. Then $M_1 \wedge \cdots \wedge M_n = M_1 + \cdots + M_n$.*

Proof. It is clear that $M_1 + \cdots + M_n \subseteq M_1 \wedge \cdots \wedge M_n$. We must show that $M_1 \wedge \cdots \wedge M_n \subseteq M_1 + \cdots + M_n$. Since C^* is a commutative algebra, so $M_1^\perp \ldots M_n^\perp = M_1^\perp \cap \cdots \cap M_n^\perp$. Now we have

$$\begin{aligned} (M_1 \wedge \cdots \wedge M_n)^\perp &\supseteq M_1^\perp \ldots M_n^\perp \\ &= M_1^\perp \cap \cdots \cap M_n^\perp \\ &= (M_1 + \cdots + M_n)^\perp \end{aligned}$$

Hence

$$\begin{aligned} (M_1 + \cdots + M_n) &= (M_1 + \cdots + M_n)^{\perp\perp} \\ &\subseteq (M_1 \wedge \cdots \wedge M_n)^{\perp\perp} \\ &= M_1 \wedge \cdots \wedge M_n \end{aligned}$$

and the proof is complete. ■

Note: If P_1 and P_2 are coprime subcoalgebras then $P_1 \wedge P_2$ is not necessarily coprime. For example $< C_1 >$ and $< C_2 >$ are coprime (In Example 1.1) but $< C_1 > \wedge < C_2 >=< C_1 > + < C_2 >=< C_1, C_2 >$ is not coprime subcoalgebra.

In the following we will characterize the finite dimensional coprime coalgebras. A coalgebra C is coprime if for any subcoalgebras X and C Y, $C = X \wedge Y$ implies that $C = X$ or $C = Y$. By Proposition 1.2, a coalgebra is coprime if and only if $C^{\perp} = \{0\}$ is a prime ideal of C^*.

Theorem 1.1. *A finite dimensional coalgebra is coprime if and only if it is simple.*

Proof. Let C be a finite dimensional coalgebra, then C^* is a finite dimensional algebra. By [5, Example 2.3.7], C^* is Artinian and by [5, Theorem 2.3.9] every prime ideal of C^* is maximal. Since C is coprime, $C^{\perp} = \{0\}$ is a maximal ideal of C^* and $\{0\}^{\perp} = C$ is simple. The converse is true by Proposition 1.3. ■

Proposition 1.4. *Let C be a cocommutative coprime coalgebra. Then C has a unique simple subcoalgebra.*

Proof. Since C is cocommutative, $C = \bigoplus_{\alpha} C_\alpha$, where C_α is an irreducible component of C. We have $C = C_\alpha \oplus (\sum_{\beta \neq \alpha} C_\beta) \subseteq C_\alpha \wedge (\sum_{\beta \neq \alpha} C_\beta)$; since C is coprime, $C = C_\alpha$ or $C = \sum_{\beta \neq \alpha} C_\beta$. If $C = \sum_{\beta \neq \alpha} C_\beta$, Then $C_\alpha \subseteq \sum_{\beta \neq \alpha} C_\beta$. Hence $C_\alpha \cap (\sum_{\beta \neq \alpha} C_\beta) = C_\alpha \neq 0$, which is contradiction. We conclude that $C = C_\alpha$ and so C has a unique simple subcoalgebra.■

Note. An infinite dimensional cocommutative pointed coalgebra with at least two group-like elements is not neccesarily coprime. For example, let C be a coalgebra with basis $\{C_i\}_{i=0}^{\infty}$ with $\Delta(C_i) = C_i \otimes C_i$

and $\epsilon(C_i) = 1$, $(i = 0, 1, \ldots)$. We know that C is a cocommutative pointed coalgebra. However C is not coprime, because though $C = < C_0 > \wedge < C_1, C_2, \cdots >$, $C \neq < C_0 >$ and $C \neq < C_1, C_2, \cdots >$.

Conjecture. *Let C be an infinite dimensional (cocommutative) coalgebra. If C has a unique simple subcoalgebra then it is coprime.*

Proposition 1.5. *Let C be a non-zero coprime coalgebra and D be a coalgebra containing C as a subcoalgebra. Then C is a coprime subcoalgebra of D.*

Proof. Let X and Y be subcoalgebras of D and $C \subseteq X \wedge Y$. We know that $X \cap C$ and $Y \cap C$ are subcoalgebras of C. We will show that $C = (X \cap C) \wedge (Y \cap C)$. It is clear that $(X \cap C) \wedge (Y \cap C) \subseteq C$. Since $C^\perp$ is a two-sided ideal of D^* we have

$$\begin{aligned} (X \cap C) \wedge (Y \cap C) &= [(X \cap C)^\perp (Y \cap C)^\perp]^\perp \\ &= [(X^\perp + C^\perp)(Y^\perp + C^\perp)]^\perp \\ &\supseteq [X^\perp Y^\perp + C^\perp]^\perp \\ &= C^{\perp\perp} \cap (X^\perp Y^\perp)^\perp \\ &= C \cap (X \wedge Y) \\ &= C. \end{aligned}$$

Hence $C = X \cap C$ or $C = Y \cap C$. Therefore $C \subseteq X$ or $C \subseteq Y$. ■

2. Topology on Coprime Subcoalgebras.

Let C be a coalgebra and X be the set of coprime subcoalgebras on C. Suppose that E is an arbitrary subcoalgebra of C, $V(E) = \{P \in X | P \subseteq E\}$, $X_E = X - V(E)$ and $\tau = \{X_E | E \text{ is a subcoalgebra of } C\}$.

Proposition 2.1. *(X, τ) is a topological space with closed sets $V(E)$ (or open sets $X_E = X - V(E)$).*

Proof. Since $V(C) = X$ and $V(\{0\}) = \emptyset$, both X, $\emptyset$ belong to τ. Let D_1 and D_2 be subcoalgebras of C. If $P \in V(D_1) \cup V(D_2)$ then $P \subseteq D_1$ or $P \subseteq D_2$. Let $P \subseteq D_1$. Since $D_1 \subseteq D_1 + D_2 \subseteq D_1 \wedge D_2$, $P \in V(D_1 \wedge D_2)$. Conversely if $P \in V(D_1 \wedge D_2)$ then $P \subseteq D_1$ or $P \subseteq D_2$, since P is coprime. Hence $V(D_1 \wedge D_2) \subseteq V(D_1) \cup V(D_2)$. Therefore $V(D_1) \cup V(D_2) = V(D_1 \wedge D_2)$ and hence $X_{D_1} \cap X_{D_2} \in \tau$. It is clear that $\bigcap_\alpha V(D_\alpha) = V(\bigcap_\alpha D_\alpha)$ and hence $\bigcup_\alpha X_{D_\alpha} \in \tau$. The proof is complete. ■

Corollary 2.1. *Let $\{E_\alpha\}_{\alpha \in I}$ be a family of subcoalgebras of a coalgebra C. Then*
i) $X_{E_\alpha} \cap X_{E_\beta} = X_{E_\alpha \wedge E_\beta}$
ii) $X_{(\sum_{\alpha \in I} E_\alpha)} \subseteq \bigcup_{\alpha \in I} X_{E_\alpha}$.
The equality in (ii) does not necessarily hold.

Proof. The proofs of (i) and (ii) are easy. For the equality in (ii), let C be coalgebra in Example 1.1. Suuppose that $E_1 =< C_1 >$ and $E_2 =< C_2 >$.

Now we have

$X_{E_1+E_2} = \{< c_0 >, < c_3 >, < c_4 >, \dots\} = X_{E_1 \wedge E_2} \neq X = X_{E_1} \cup X_{E_2}$. ■

Proposition 2.2. *Let C be a coalgebra that is not coprime. Then $B = \{X_E | E$ is a finite dimension subcoalgebra of $C\}$ is a basis in topological space X.*

Proof. Let $P \in X$, there exists t, such that $P \not\subseteq < t >$ ($< t >$ is the subcoalgebra generated by t), for $P \neq \{0\}$. Now $< t >$ is finite dimensional, so $P \in X_{<t>}$,and therefore $X_{<t>} \in B$. Suppose that X_E and X_F belong to B and $P \in X_E \cap X_F$. Put $T =< c_1, \dots, c_k, d_1, \dots, d_n >$. Recall that E and F are finite dimensional, and set where $E =< c_1, \dots, c_k >$ and $F =< d_1, \dots, d_n >$. Since $T \subseteq E + F$, we have $dim\ T < \infty$. If $P \subseteq T$, then $P \subseteq E + F \subseteq E \wedge F$. Since P is coprime, hence $P \subseteq F$ or

$P \subseteq E$, which contradicts $P \in X_F \cap X_E$. We conclude that $P \not\subseteq T$, i.e. $P \in X_T$ and therefore $X_T \subseteq X_F \cap X_E$. The proof is complete. ■

Lemma 2.1. *Let P be a subcoalgebra of a coalgebra C. P is a simple subcoalgebra if and only if P is a coprime subcoalgebra and $V(P) = \{P\}$.*

Proof. Let P be a simple subcoalgebra. Then by Proposition 1.3, P is coprime and $V(P) = \{P\}$. Conversely, suppose that E is a non-zero subcoalgebra of C such that $E \subseteq P$, then there exists a simple subcolagebra $P' \subseteq E$. But $P' \in V(P)$, so $P' = P$. Hence $E = P$. ■

Corollary 2.2. *Let E be a subcoalgebra of a coalgebra C. Then $X_E = X$ if and only if $E = \{0\}$.*

Lemma 2.2. *Let P be a coprime subcoalgebra of a coalgebra C. Then $\{P\}$ closed in X if and only if P is a simple subcoalgebra.*

Proof. Let P be a simple subcoalgebra. By Lemma 2.1, $V(P) = \{P\}$ and so $\{P\}$ is closed in X. Conversely, suppose $S \subseteq P$ is a non-zero subcoalgebra. Hence there exists a non-zero simple subcoalgebra P' such that $P' \subseteq S$. But $V(E) = P$, for some subcolagebra E, so $P \subseteq E$. We conclude that $P' \in V(E)$ and so $P' = P = S$. The proof is complete. ■

Lemma 2.3. *Let P be a coprime subcoalgebra of C. Then $\overline{\{P\}} = V(P)$.*

Proof. Let $P_1 \in \overline{\{P\}}$ and $P_1 \not\subseteq P$, so that $P_1 \in X_P$. Now P_1 is a limit point of $\{P\}$, hence $X_P \cap \{P\} \neq \emptyset$, and $P \in X_P$, a contradiction. We conclude that $P_1 \in V(P)$ and $\overline{\{P\}} \subset V(P)$. Now suppose that $P' \in V(P)$ and X_E is a neighborhood of P'. Hence $P' \not\subseteq E$ and since $P' \subseteq P$, we have $P \in X_E$. Thus $P' \neq P \in X_E \cap \{P\}$ and we conclude that $P' \in \overline{\{P\}}$. ■

Lemma 2.4. *The topological space X is T_0.*

Proof. Suppose P_1 and P_2 are distinct points of X. If $P_1 \not\subset P_2$ then $P_1 \in X_{P_2}$ and $P_2 \notin X_{P_2}$. On the other hand, if $P_2 \not\subset P_1$ then $P_2 \in X_{P_1}$ and $P_1 \notin X_{P_1}$. ■

Lemma 2.5. *Let E be a subcoalgebra of a coalgebra C. If $X_E = \emptyset$. Then E is conilpotent subcoalgebra.*

Proof. Let $X_E = \emptyset$, so $V(E) = X$. Hence $P \subseteq E$, for any $P \in X$. But every simple subcoalgebra is coprime, so $corad(C) \subseteq E$. ■

Note: The converse of Lemma 2.5 is not true. In Example 1.2, we showed that $X = \{< C_o >, C\}$. Since the only simple subcoalgebra of C is $< C_0 >$ i.e. $corad(C) = < C_0 >$, and $< C_0 > \subseteq < C_0, C_1 >$, thus $E = < C_0, C_1 >$ is conilpotent, but $X_E = C$.

Lemma 2.6. *Let C be a coalgebra which is not coprime and C^* be a PID. If E is a conilpotent subcoalgebra then $X_E = \emptyset$.*

Proof. Let P be a coprime subcoalgebra of C, so $P^\perp$ is a prime ideal of C^*. But C^* is a PID, so $P^\perp$ is maximal. Since $0 \neq P = P^{\perp\perp}$, by [1, Thm. 2.3.4, p.80], P is a simple subcoalgebra. Therefore every coprime subcoalgebra is simple. But E is a conilpotent subcoalgebra, so E contains all coprime subcoalgebras of C. Hence $V(E) = X$ or $X_E = \emptyset$.■

Proposition 2.3. *Let C be an irreducible coalgebra. Then X is connected.*

Proof. Suppose that X is not connected; then there exist (non-zero) subcoalgebras E and F of C such that $X_E \cap X_F = \emptyset$ and $X = X_E \cup X_F$. Hence E and F contain a unique non-zero simple subcoalgebra P of C. Therefore $P \notin X_E \cup X_F$ but $P \in X$, a contradiction. We conclude that X is connected and the proof is complete. ■

Note: The irreduciblity condition in Proposition 2.3 is necessary. In Example 1.1, we showed that $X = \{< C_0 >, < C_1 >, \ldots\}$. We know that the coalgebra C in this example is not irreducible but $X = X_{<C_0>} \cup X_{<C_1, C_2, \cdots>}$ and $X_{<C_0>}, X_{<C_1, C_2, \cdots>}$ are non-empty open sets. Hence X is not connected.

Proposition 2.4. *Let C be a coalgebra. If every coprime subcoalgebra of C is simple and X is connected then C is irreducible.*

Proof. Suppose that P_1 and P_2 are distinct simple subcoalgebras of C and $T = \sum\{P | P$ is a coprime subcoalgebra and $P_1 \not\subset P\}$. Hence $X = X_{P_1} \cup X_T$ and since X_T contains the only subcoalgebra P_1, we have a contradiction. The proof is complete. ■

Theorem 2.1. *The topological space X is compact (Lindelof) if*
(i) C is irreducible or
(ii) The numbers of simple subcoalgebras of C is finite (countable).

Proof. An irreducible coalgebra has a unique simple subcoalgebra, so it is enough to show that part (ii) is true.

Suppose that $\{P_1, \ldots, P_n\}$ is the set of simple subcoalgebras of C and $X \subseteq \bigcup_\alpha X_{E_\alpha}$, where $\{X_{E_\alpha}\}_\alpha$ is a family of open sets. We claim that if $X \subseteq \bigcup_\alpha X_{E_\alpha} = X_{\bigcap_\alpha E_\alpha}$, then $\bigcap_\alpha E_\alpha = \{0\}$. If not, then $\bigcap_\alpha E_\alpha$ contains a non-zero simple subcoalgebra (coprime) which contradicts with $X \subseteq \bigcup_\alpha X_{E_\alpha}$. Hence, there exist indices α_i $(1 \leq i \leq n)$ such that $P_i \not\subset E_{\alpha_i}$. Therefore $\bigcap_{i=1}^n E_{\alpha_i} = \{0\}$ and so $\bigcup_{i=1}^n X_{E_{\alpha_i}} = X_{\bigcap_{i=1}^n E_{\alpha_i}} = X$. By a similar argument we can prove that if the number of simple subcoalgebras of C is countable then the topological space X is Lindelof and the proof is complete. ■

Note: If the set of simple subcoalgebras of a coalgebra C is infinite (countable) then the Theorem 2.1 is not true in general. In Example 1.1, we showed that $X = \{< C_0 >, < C_1 >, \ldots\}$. It is clear that $X \subseteq \bigcup_{i=1}^{\infty} X_{<C_i, C_{i+1}, \cdots>}$ which has no finite cover .

Theorem 2.2. *If the topological space X is Hausdorff then every coprime subcoalgebra of C is simple.*

Proof. Suppose that the coprime subcoalgebra P_1 of C is not simple, then there exists a non-zero simple subcoalgebra X_{E_1} and P_2 such that $P_2 \subset P_1$. Since X is Hausdorff, there exist two open sets X_{E_2} such that $P_1 \in X_{E_1}$, $P_2 \in X_{E_2}$ and $X_{E_1} \cap X_{E_2} = \emptyset$. Now $P_1 \in X_{E_1} \cap X_{E_2}$, for if $P_1 \notin X_{E_2}$ then $P_1 \subseteq E_2$. Hence $P_2 \subseteq X_{E_2}$ which contradicts $P_2 \in X_{E_2}$. We conclude that $X_{E_1} \cap X_{E_2} \neq \emptyset$, a contradiction; hence P_1 is a simple subcoalgebra and the proof is complete. ■

Proposition 2.5. *Let C be an irreducible coalgebra. Then the topological space X is not Hausdorff. (Assume that $|X| \geq 2$.)*

Proof. Every non-zero subcoalgebra of C contains the unique simple subcoalgebra P of C. So for every open set X_E, $P \notin X_E$, unless $E = \{0\}$. Hence $X_E = X$ and we conclude that open sets, containing P' and having no intersection with X, do not exist, for any $P \neq P' \in X$. Therefore X is not Hausdorff and the proof is complete. ■

Lemma 2.7. *If every coprime subcoalgebra of a coalgebra C is simple then the topology of X is discrete.*

Proof. Suppose that $P_1 \in E$ and

$$T = \sum \{P | P \text{ is a coprime subcoalgebra such that } P_1 \not\subset P\}$$

Put $F = X_T$. Since P_1 is the only of F, the open set F contains P_1 has an intersection with E only at point P_1. Hence P_1 is an isolated point of E and we conclude that the topology of X is discrete. ■

Corollary 2.3. *Let C be a coalgebra such that every coprime subcoalgebra of C is simple. Then the following conditions are satisfied:*
i) The topological space X is regular, normal, totally disconnected and locally connected.
ii) Urysohn's lemma and Tietze's extension theorem holds for C.

Proposition 2.7. *The sum of all coprime subcoalgebra of a coalgebra C is coprime if and only if X is an irreducible topological space.*

Proof. Let $P' = \sum\{P|P \text{ is a coprime subcoalgebra of } C\}$. Suppose that P' is coprime and X_E, X_F are two non-empty open sets. Let $P' \subseteq E \wedge F$, so that $P' \subseteq F$ or $P' \subseteq E$. If $P' \subseteq E$ or $P' \subseteq F$ then every coprime subcoalgebra is contained in E and hence $X_E = \emptyset$, a contradiction. Therefore $P' \not\subseteq E \wedge F$,and hence $P' \in X_{E\wedge F} = X_E \cap X_F$. We conclude that $X_E \cap X_F \neq \emptyset$ and so X is irreducible. Conversely, suppose that X is irreducible. We claim that P' is coprime. Let $P' \subseteq D_1 \wedge D_2$, for some subcoalgebras D_1 and D_2 of C. Suppose $P' \not\subseteq D_1$ and $P' \not\subseteq D_2$. Then there exist coprime subcoalgebras $P_1 \not\subseteq D_1$ and $P_2 \not\subseteq D_2$. Thus $X_{D_1} \neq \emptyset$ and $X_{D_2} \neq \emptyset$. If $X_{D_1} \cap X_{D_2} \neq \emptyset$, then there exists a coprime subcoalgebra P_0 such that $P_0 \in X_{D_1} \cap X_{D_2}$. Hence $P_0 \not\subseteq D_1 \wedge D_2$ and so $P' \not\subseteq D_1 \wedge D_2$, which contradicts to our assumption. Therefore we have $X_{D_1} \cap X_{D_2} = \emptyset$, we have a contradiction. The proof is complete. ■

Proposition 2.8. *Let C be a coalgebra. If C has no conilpotent subcoalgebra then $E = \{P|P$ is a simple subcoalgebra$\}$ is a dense subset of X.*

Proof. We claim that $\overline{E} = X$. Since $E \subseteq X$, so $\overline{E} \subseteq X$. Now we prove that $X \subseteq \overline{E}$. Let P be an arbitrary element of X. If P is simple then $P \in E \subseteq \overline{E}$. Now suppose that P is not simple. let X_F be an arbitrary open set containing P. Since F is not conilpotent, hence there exists a simple subcoalgebra $M \neq P$ such that $M \not\subseteq F$. Then $M \in X_F \cap E$ and so P is a limit point of E. Therefore $P \in E' \subseteq \overline{E}$. ■

Corollary 2.4. *Let C be a coalgebra. If C has no conilpotent subcoalgebra and the set of simple subcoalgebras of C is countable then the topological space X is separable.*

Proof. It is clear by Proposition 2.8. ■

Proposition 2.9. *Let C be a coalgebra and every coprime subcoalgebra of C be simple. Then*
i) The topological space X is not connected if $|X| \geq 2$.
ii) If $|X| = \infty$ then X is not compact.
iii) The principle T_1 is satisfied for X.

Proof. (i): Let E be a proper subset of X. By Lemma 2.7, E is both closed and open. Hence $X = E \cup (X \setminus E)$ and so X is not connected.

(ii): Let $\{P_\alpha\}_{\alpha \in I}$ be the family of all coprime subcoalgebras of C. Put $E_\beta = \sum_{\alpha \neq \beta} P_\alpha$. We claim that $P_\beta \in X_{E_\beta}$. If $P_\beta \notin X_{E_\beta}$ then $P_\beta \subseteq \sum_{\alpha \neq \beta} P_\alpha$. Since every coprime subcoalgebra is simple there exists a coprime subcoalgebra P_γ, $\gamma \neq \beta$ such that $P_\beta \subseteq P_\gamma$. Hence $P_\beta = P_\gamma$, a contradiction. It is clear that $X_{E_\beta} = \{P_\beta\}$ and $X_{E_\beta} \cap X_{E_\alpha} = \emptyset$ and so the cover $\bigcup_\beta X_{E_\beta}$ for X has no finite cover. Hence X is not compact.

(iii): Let P_1 and P_2 be two distinct elements of X. Since $X_{P_1}(X_{P_2})$ contains all coprime subcoalgebras except $P_1(P_2)$, so X_{P_1} and X_{P_2} are two disjoint open sets that contain P_2 and P_1 respectively. Therefore X satisfies T_1 and the proof is complete. ■

Note: If a coalgebra C has a coprime subcoalgebra that is not simple then the principle T_1 does not necessarily hold for X.

For example, in Example 1.2, we show that $X = \{< C_0 >, C\}$. Let X_E and X_F be open sets containing C and $< C_0 >$ respectively. Since C

is an irreducible coalgebra, so $F = \{0\}$. Hence $X_F = \{< C_0 >, C\} \supset X_E$ and the principle T_1 does not hold.

Proposition 2.10. *Let C be a coalgebra and $V_\alpha = \{M_\alpha\}$ such that M_α's are all simple subcoalgebras of C. If every coprime subcoalgebra of C contains a finite number of simple subcoalgebras then the family $B = \{V_\alpha\}_\alpha$ is locally finite.*

Proof. Let P be an arbitrary element of X and put $F = \sum\{M_\alpha | M_\alpha \not\subset P\}$. It is easy to show that $P \in X_F$. We claim that X_F has a finite intersection with B. Suppose that $\{M_{\alpha_1}, \ldots, M_{\alpha_k}\} \subseteq P$. First we show that $M_{\alpha_i} \in X_F$, for all i, $1 \leq i \leq n$. Suppose there exists $1 \leq j \leq n$, such that $M_{\alpha_j} \notin X_F$. Thus there exists M_γ such that $M_{\alpha_j} = M_\gamma$, which is in contradiction with $M_{\alpha_j} \subseteq P$. We conclude that $X_F \cap V_{\alpha_i} \neq \emptyset$, for all i, $1 \leq i \leq n$. Finally we show that $X_F \cap V_\alpha = \emptyset$, for any $\alpha \neq \alpha_i$ $(1 \leq i \leq n)$. Suppose that $M_\alpha \in X_F$, so $M_\alpha \subseteq P$. This contradicts with $\alpha \neq \alpha_i$ and the proof is complete. ■

Proposition 2.11. *The coalgebra C is irreducible if and only if every pair of non-empty closed sets in the topological space X have a non-empty intersection.*

Proof. Let C be an irreducible coalgebra and $V(E_1)$ and $V(E_2)$ be two non-empty closed sets in X. Hence $E_1 \cap E_2 \neq \{0\}$. Note that a coalgebra is irreducible if and only if the intersection of two non-zero subcoalgebras is non-zero, and so there exists a simple subcoalgebra $M \subseteq E_1 \cap E_2$. Hence $M \in V(E_1) \cap V(E_2)$. Conversely, suppose that E_1 and E_2 are non-zero two subcoalgebras of C. By Corollary 2.2, $V(E_1) \neq \emptyset, V(E_2) \neq \emptyset$, and by assumption $V(E_1) \cap V(E_2) \neq \emptyset$,so there exists a coprime subcoalgebra $P \in V(E_1) \cap V(E_2)$. Hence $P \subseteq E_1 \cap E_2$.■

Theorem 2.3. *Let C be a coalgebra. Then the following conditions hold.*

(i) If P is a coprime subcoalgebra of C then $Y = V(P)$ is an irreducible subspace of the topological space X.
(ii) If $Y = V(P)$ is an irreducible component then P is a maximal coprime subcoalgebra.

Proof. (i): Let U_1 and U_2 be non-empty open sets in Y. Then there exist open sets X_{E_1} and X_{E_2} of X such that $U_1 = Y \cap X_{E_1}$ and $U_2 = Y \cap X_{E_2}$. Therefore there exist two coprime subcoalgebras P_1 and P_2 such that $P_1 \in U_1$ and $P_2 \in U_2$. It is easy to show that $P \not\subseteq E_1$ and $P \not\subseteq E_2$. Hence $P \in U_1 \cap U_2$, so Y is an irreducible subspace of X.

(ii): Let P_1 be a coprime subcoalgebra of C such that $P \subseteq P_1$. $V(P) \subseteq V(P_1)$, also $V(P_1)$ is an irreducible subspace of X, so $V(P) = V(P_1)$. Hence $P = P_1$ and the proof is complete. ■

Lemma 2.8. *Let C be a coalgebra and $Y = \{P_i\}_{i=1}^n$ be an irreducible subspace of X. Then for any i, $1 \leq i \leq n$, there exists j, $1 \leq j \leq n$ such that $P_i \subseteq P_j$ or $P_j \subseteq P_i$.*

Proof. Suppose that there exists j, $1 \leq j \leq n$, such that for any i, $1 \leq i \leq n$, $P_i \not\subseteq P_j$ and $P_j \not\subseteq P_i$. Put $V_1 = X_{P_j} \cap Y$ and $V_2 = X_F \cap Y$ such that $F = \sum\{P_i \in Y | P_i \neq P_j\}$. We have $V_1 \cap V_2 = \emptyset$, $V_1 = Y \backslash \{P_j\}$ and $V_2 = \{P_j\}$ which is a contradiction. Hence $P_i \subseteq P_j$ or $P_j \subseteq P_i$ and the proof is complete. ■

Theorem 2.4. *Let $f : C \longrightarrow D$ be a coalgebra map and $X = \{P | P$ is a coprime subcoalgebra of $C\}$, $Y = \{P | P$ is a coprime subcoalgebra of $D\}$*

(i) If $P \in X$ then $f(P) \in Y$.

(ii) Define $\phi : X \longrightarrow Y$ by $\phi(P) = f(P)$, for any $P \in X$. Then ϕ is continuous.

(iii) If every coprime subcoalgebra of C is the inverse image of a subcoalgebra of D then ϕ is one-to-one.

(iv) If f is one-to-one so is ϕ.

(v) If ϕ is onto and f is one-to-one then ϕ is a closed and open map.

(vi) If f is one-to-one and onto so is ϕ and ϕ^{-1} is continuous.

Proof. (i) Since P is a coprime subcoalgebra of C and f is a coalgebra map, then $f(P)$ is a subcoalgebra of D and $P^{\perp}$ is a prime ideal of C^*. Now $(f^*)^{-1}(P^{\perp})$ is a prime ideal of D^*, since $f^* : D^* \longrightarrow C^*$ is an algebra map. Also $(f^*)^{-1}(P^{\perp}) = (f(P))^{\perp}$, so $(f(P))^{\perp}$ is a prime ideal of D^*. Hence by Proposition 1.2, $f(P)$ is a coprime subcoalgebra of D and the proof of part (i) is complete.

(ii) By (i), ϕ is well-defined. Suppose that E is a subcoalgebra of D. We claim that $\phi^{-1}(Y_E) = X_{f^{-1}(E)}$. $P \in X_{f^{-1}(E)}$ if and only if $f(P) \not\subseteq E$ which is equivalent to $P \in \phi^{-1}(Y_E)$.

E is a subcoalgebra of D and $f^{-1}(E)$ is a subcoalgebra of C, so $X_{f^{-1}(E)}$ is open in X. Hence ϕ is continuous.

(iii) Let $P_1, P_2 \in X$ and $\phi(P_1) = \phi(P_2)$. Hence $f(P_1) = f(P_2)$. By assumption there exist subcoalgebras of D, say D_1 and D_2 such that $f^{-1}(D_1) = P_1$ and $f^{-1}(D_2) = P_2$. We denote $f^{-1}(E) = (E)^c$ and $f(E') = (E')^e$. Then $D_1^{ce} = D_2^{ce}$ and therefore $D_1^c = D_1^{cec} = D_2^{cec} = D_2^c$. Thus $P_1 = D_1^c = D_2^c = P_2$

(iv) Clear.

(v) Suppose that $V(E)$ is a closed in X. It is easy to show that $\phi(V(E)) = V(f(E))$ and $Y_{f(E)} = \phi(X_E)$.

(vi) We must show that ϕ is onto. Let P' be a coprime subcoalgebra of D. Hence $f^{-1}(P')$ is a coprime subcoalgebra of C and $\phi(f^{-1}(P')) = P'$. Therefore ϕ is onto. Since ϕ is onto and f is one-to-one, so ϕ is an open map. Thus the inverse image of an open set under ϕ is also open, so ϕ^{-1} is continuous and the proof is complete. ■

Let D be a subcoalgebra of a coalgebra C and $rad(D)$ be the sum of all coprime subcoalgebras of C contained in D. It is clear that

$V(rad(D)) = V(D)$.

Theorem 2.5. *There is a one-to-one corespondence between the set of closed subsets of* X *and the set of subcoalgebras* D *of* C *such that* $rad(D) = D$.

Proof. Put $A = \{Y : Y \subseteq X\}$ and $T(Y) = \sum_{P \in Y} P$ and $T(\emptyset) = C$. Define a map $\varphi : A \to \{D | D \text{is a subcoalgebra of } C\}$ by $\varphi(Y) = T(Y)$, for any $Y \in A$. It is easy to show that
(i) φ is an increasing map
(ii) $T(V(E)) = rad(E)$,
(iii) $T(\bigcup_{l \in \mathrm{L}} Y_l) = \sum_{l \in \mathrm{L}} T(Y_l)$.

Now we show that $V(T(Y)) = \overline{Y}$. Since $Y \subseteq V(T(Y))$,hence $\overline{Y} \subseteq V(T(Y))$. Let $P \in V(T(Y))$ and $P \notin Y$. We claim that P is a limit point of Y. Let X_E be a neighborhood of P. So $P \not\subset E$ and there exists $P_1 \in Y$ such that $P_1 \not\subset E$, because if for every $P' \in Y$, $P' \subseteq E$, then $\sum_{P' \in Y} P' \subseteq E$, is contradiction. Hence $P_1 \in X_E \cap Y$ and so $P \in \overline{Y}$. Therefore if Y is a closed subset of X then $V(T(Y)) = Y$. Suppose that D is a subcoalgebra of C such that $rad(D) = D$, so $T(V(D)) = rad(D) = D$. ■

Conclusion

In this paper using the concepts of Zariski topology on rings and with the help of coprime subcoalgebras we have been able to construct a topology on coalgebras. So perhaps it seems that there is a one-to-one correspondence between the properties of coprime subcoalgebras C with the correspoding topology and the properties of the prime ideals of C^* with its topology (with duality). But the following statements reject the above.

i) In example 1.2, we proved that the only coprime subcoalgebras of C are $< C_0 >$ and C. But C is not a simple subcoalgebra. Recall that in a commutative ring with identity, every maximal ideal is prime.
ii) In proposition 1.3, we proved that every simple subcoalgebra is coprime. But the dual of this statement is not true in every ring.
iii) In lemma 2.7, we proved that if every coprime subcoalgebra of C is simple then every subset of X is closed and open. But in C^* the dual of this statement is not hold [2, page 14].

We have started to continue the use of this topology in non-commutative algebraic geometry and we hope to get more results .

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