

## ON REDUCING SEQUENCES AND AN APPLICATION TO LOCAL COHOMOLOGY MODULES

NONG QUOC CHINH

ABSTRACT. This paper is concerned with the reducing sequences introduced by Auslander-Buchsbaum [1]. A bound from above for the Krull dimension of the Koszul homology modules with respect to a reducing sequence is shown. Using reducing sequence, a result on the Artinianness and the finiteness of the support of the local cohomology modules is given.

### 1. Introduction

Throughout this paper, let  $(R, \mathfrak{m})$  be a Noetherian local ring and let  $M$  be a finitely generated  $R$ -module with  $\dim M = d$ . It is well known that for an ideal  $I$  of  $R$ , if  $x_1, \dots, x_n \in I$  is a regular sequence of  $M$  then the  $i$ -th local cohomology module  $H_I^i(M)$  is vanishing for all  $i < n$  and the  $s$ -th Koszul homology module  $H_s(x_1, \dots, x_n; M)$  is vanishing for all  $s \geq 1$ . Also the structure of Cohen-Macaulay modules (modules satisfying the condition that any system of parameters is a regular sequence) is fully known. Next, the notion of filter regular sequence was introduced by Cuong-Schenzel-Trung [4] as an extension of the notion of regular sequence. Lü-Tang [7] proved that if  $x_1, \dots, x_n \in I$  is a filter regular sequence of  $M$  then  $H_I^i(M)$  is Artinian for all  $i < n$ , and  $\ell(H_s(x_1, \dots, x_n; M)) < \infty$  for all  $s \geq 1$ . Moreover, the structure of generalized Cohen-Macaulay modules and  $\mathfrak{f}$ -modules (modules satisfying

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the condition that any system of parameters is a filter regular sequence) is described carefully in [4], [14]. Recently, the notion of generalized regular sequences as a generalization of that of filter regular sequences is introduced in [10]. It is shown that if  $x_1, \dots, x_n \in I$  is a generalized regular sequence of  $M$  then  $\text{Supp } H_I^i(M)$  is finite for all  $i < n$  and  $\dim H_s(x_1, \dots, x_n; M) \leq 1$  for all  $s \geq 1$ . Also in [11], the authors used generalized regular sequences in order to give some geometrical applications, specially in the study coordinate rings of algebraic varieties and Stanley-Reisner rings.

The notion of reducing sequences was introduced by Auslander-Buchsbaum [1]: A sequence  $x_1, \dots, x_n$  of elements in  $\mathfrak{m}$  is called a *reducing sequence* of  $M$  if for all  $i = 1, \dots, n$ ,  $x_i \notin \mathfrak{p}$  for all  $\mathfrak{p} \in \text{Ass } M/(x_1, \dots, x_{i-1})M$  satisfying  $\dim R/\mathfrak{p} \geq d - i$ . It is clear that any filter regular sequence and hence any regular sequence of  $M$  is a reducing sequence of  $M$ . Moreover, any generalized regular sequence of  $M$  of length at most  $d - 2$  is reducing sequence of  $M$ . Therefore, the notion of reducing sequences is in some sense an extension of all the above kinds of sequences.

The purpose of this paper is to study reducing sequences and its application to local cohomology modules. In the next section, we give some basic properties of reducing sequences. A bound from above for the Krull dimension of the  $s$ -th Koszul homology module  $H_s(x_1, \dots, x_k; M)$ , where  $x_1, \dots, x_k$  is a reducing sequence of  $M$ , is given (Proposition 2.5). In section 3, we present some applications of reducing sequences. We show a condition so that local cohomology modules  $H_I^i(M)$  are Artinian (resp.  $\text{Supp } H_I^i(M)$  is finite) for all but only one  $i$  (Theorem 3.3). We also consider the Krull dimension of the support of local cohomology modules (Theorem 3.5).

## 2. Reducing sequences and Koszul homology modules

We recall first the notion of reducing sequence which was introduced by Auslander-Buchsbaum [1].

**Definition 2.1.** Let  $k$  be a positive integer. A sequence  $x_1, \dots, x_k$  of elements in  $\mathfrak{m}$  is called a *reducing sequence* of  $M$  if for all  $i = 1, \dots, k$ ,  $x_i \notin \mathfrak{p}$  for all  $\mathfrak{p} \in \text{Ass}(M/(x_1, \dots, x_{i-1})M)$  satisfying  $\dim R/\mathfrak{p} \geq d - i$ .

Recall that a sequence  $x_1, \dots, x_k$  of elements in  $\mathfrak{m}$  is called a *filter regular sequence* (resp. *generalized regular sequence*) of  $M$  if for all  $i = 1, \dots, k$ ,  $x_i \notin \mathfrak{p}$  for all  $\mathfrak{p} \in \text{Ass}(M/(x_1, \dots, x_{i-1})M)$  satisfying  $\dim R/\mathfrak{p} > 0$ , cf. [4] (resp.  $x_i \notin \mathfrak{p}$  for all  $\mathfrak{p} \in \text{Ass}(M/(x_1, \dots, x_{i-1})M)$  satisfying  $\dim R/\mathfrak{p} > 1$ , cf. [10]). If  $I$  is an ideal of  $R$  such that  $\dim M/IM > 0$  (resp.  $\dim M/IM > 1$ ) then all maximal filter regular sequences (resp. maximal generalized regular sequences) of  $M$  in  $I$  have the same length and this common length is called *filter depth* (resp. *generalized depth*) of  $M$  in  $I$  and denoted by  $\text{f-depth}(I; M)$  (resp.  $\text{gdepth}(I; M)$ ), cf. [7], [10].

**Remark 2.2.** Reducing sequences are not necessarily filter regular sequences or generalized regular sequences. For example, let  $R = k[[x, y, z, t, v]]$  be the ring of formal series of 5 variables over a field  $k$ . Let  $M = R/(x) \cap (x^2, y) \cap (x^2, y^2, z)$ . Then  $z, t$  is a reducing sequence of  $M$ , but it is neither a filter regular sequence of  $M$  nor a generalized regular sequence of  $M$ .

For each ideal  $I$  of  $R$ , it should be noticed that if  $\dim M/IM > 0$  then any reducing sequence of  $M$  is a part of system of parameters of  $M$ . Therefore any reducing sequence of  $M$  in  $I$  has finite length. Of course, if  $\dim M/IM = 0$  then  $I$  contains a reducing sequence of  $M$  with infinite length. Therefore from now on, we always assume that  $I$  is an ideal of  $R$  such that  $\dim M/IM > 0$ .

**Definition 2.3.** The *reducing depth* of  $M$  in  $I$ , denoted by  $\text{rdepth}(I; M)$ , is the supremum of lengths of all reducing sequences of  $M$  in  $I$ .

**Proposition 2.4.** *If  $\dim M/IM = k > 0$ , then*

$$d - k - 1 \leq \text{rdepth}(I; M) \leq d - k.$$

**Proof.** Since any reducing sequence of  $M$  in  $I$  is a part of a system of parameters of  $M$ , we have  $\text{rdepth}(I; M) \leq d - k$ . Since  $\dim M/IM = k$ , the ideal  $I$  cannot be contained in any prime ideal  $\mathfrak{p} \in \text{Supp} M$  with  $\dim R/\mathfrak{p} > k$ . Therefore, by Prime Avoidance Theorem, there exist elements  $x_1, \dots, x_{d-k-1} \in I$  such that  $x_i \notin \mathfrak{p}$  for all  $\mathfrak{p} \in \text{Ass}(M/(x_1, \dots, x_{i-1})M)$  satisfying  $\dim R/\mathfrak{p} \geq d - i$  for all  $i = 1, \dots, d - k - 1$ . Thus,  $\text{rdepth}(I; M) \geq d - k - 1$ .  $\square$

Recall that  $M$  is called *f-module* (resp. *generalized f-module*) if any system of parameters of  $M$  is a filter regular sequence (resp. generalized regular sequence). The structure of f-modules and generalized f-modules are well-known by many properties of localizations, completions, multiplicities, local cohomology, and non-Cohen-Macaulay locus, cf. [4], [11]. Specially, when  $R$  is a quotient of a Cohen-Macaulay ring, f-modules are exactly generalized Cohen-Macaulay modules, i.e. modules with  $\ell(H_m^i(M)) < \infty$  for all  $i < d$ , cf. [4]. Therefore it is natural to ask about the structure of modules satisfying the condition that any system of parameters is a reducing sequence. The following result answers part of this question.

**Proposition 2.5.** *Assume that  $R$  is a quotient of a Cohen-Macaulay ring.*

- (i) *The following are equivalent:*
  - (a) *Any system of parameters of  $M$  is a reducing sequence.*
  - (b)  *$M$  is a generalized Cohen-Macaulay module.*
- (ii) *The following are equivalent:*
  - (a) *Any part of a system of parameters of  $M$  of  $d - 2$  elements is a reducing sequence.*
  - (b)  *$M$  is a generalized f-module.*

Before proving Proposition 2.5, we need the notion of polynomial type introduced by Cuong [3]. For each system of parameters  $\underline{x} = (x_1, \dots, x_d)$  of  $M$  and each  $d$ -tuple of positive integers  $\underline{n} = (n_1, \dots, n_d)$ , we set  $\underline{x}(\underline{n}) = (x_1^{n_1}, \dots, x_d^{n_d})$  and

$$I(\underline{x}(\underline{n}); M) = \ell(M/(\underline{x}(\underline{n})M)) - e(\underline{x}(\underline{n}); M).$$

In general  $I(\underline{x}(\underline{n}); M)$ , considered as a function in  $n_1, \dots, n_d$ , is not a polynomial when  $n_1, \dots, n_d \gg 0$ , but it always takes non negative values and is bounded above by polynomials. Specially, the least degree of all polynomials in  $\underline{n}$  bounding above the function  $I(\underline{x}(\underline{n}); M)$  does not depend on the choice of  $\underline{x}$ , cf. [3]. This least degree is denoted by  $p(M)$  and called *the polynomial type* of  $M$ .

**Proof of Proposition 2.5.** (i) is already proved in [4]. We prove (ii). It follows by [11, Theorem 3.2] that  $M$  is a generalized f-module if and only if  $p(M) \leq 1$ , where  $p(M)$  is the polynomial type of  $M$ . Moreover, we get by [3, Theorem 1.2] that  $p(M) \leq 1$  if and only if any part of a system of parameters of  $M$  of  $d - 2$  elements is a reducing sequence.  $\square$

For a sequence  $x_1, \dots, x_k$  of elements in  $\mathfrak{m}$ , we denote by  $H_s(x_1, \dots, x_k; M)$  for  $s \geq 0$ , the  $s$ -th Koszul homology module of  $M$  with respect to  $x_1, \dots, x_k$ .

**Proposition 2.6.** *Let  $x_1, \dots, x_k$  be a reducing sequence of  $M$ .*

- (i) *If  $k \geq d - 1$  then  $\ell(H_s(x_1, \dots, x_k; M)) < \infty$  for all  $s \geq 1$ .*
- (ii) *If  $k \leq d - 2$  then  $\dim(H_s(x_1, \dots, x_k; M)) \leq d - k - 1$  for all  $s \geq 1$ .*

**Proof.** The assertion (i) is a modification of a result of Dutta [5]. Note that any reducing sequence of  $M$  of length  $d$  is a system of parameters of  $M$ . Therefore if  $k \geq d$  then  $(x_1, \dots, x_k)$  is a multiplicative system of  $M$ . Hence  $\ell(H_s(x_1, \dots, x_k; M)) < \infty$  for all  $s \geq 0$ . Now let  $k = d - 1$ . Then there exists an element  $x_d \in \mathfrak{m}$  such that  $(x_1, \dots, x_d)$  is a system of parameters of  $M$ . Denote by  $e(x_1, \dots, x_d; M)$  the multiplicity of  $M$  with respect to  $(x_1, \dots, x_d)$ . It is clear that  $(x_1, \dots, x_d)$  is a reducing sequence of  $M$ . So, by [12, 7.9, Theorem 18], we have

$$\begin{aligned}
 e(x_1, \dots, x_d; M) &= e(x_2, \dots, x_d; M/x_1M) = \dots \\
 &= e(x_{d-1}, x_d; M/(x_1, \dots, x_{d-2})M) \\
 &= e(x_d; M/(x_1, \dots, x_{d-1})M) \\
 (2.1) \quad &= \sum_{i=1}^t \ell(M/(x_1, \dots, x_{d-1})M)_{\mathfrak{p}_i} e(\bar{x}_d, R/\mathfrak{p}_i),
 \end{aligned}$$

where  $\mathfrak{p}_1, \dots, \mathfrak{p}_t$  are all the minimal primes of  $\text{Ann } M + (x_1, \dots, x_{d-1})R$  and  $\bar{x}_d$  is the image of  $x_d$  in  $R/\mathfrak{p}_i$ . By the associativity law for multiplicities, cf. [12, 7.9, Theorem 18], we have

$$(2.2) \quad e(x_1, \dots, x_d; M) = \sum_{i=1}^t \left( e(x_1/1, \dots, x_{d-1}/1; M_{\mathfrak{p}_i}) e(\bar{x}_d; R/\mathfrak{p}_i) \right),$$

where  $x_j/1$  is the image of  $x_j$  in  $R_{\mathfrak{p}_i}$  for all  $j = 1, \dots, d - 1$ . We get by [12, 8.4, Theorem 5]

$$(2.3) \quad e(x_1/1, \dots, x_{d-1}/1; M_{\mathfrak{p}_i}) = \sum_{s=0}^{d-1} (-1)^s \ell(H_s(x_1/1, \dots, x_{d-1}/1; M_{\mathfrak{p}_i})).$$

From (2.1), (2.2) and (2.3) we have

$$\sum_{i=1}^t \left( e(\bar{x}_d, R/\mathfrak{p}_i) \sum_{s=1}^{d-1} (-1)^{s+1} \ell(H_s(x_1/1, \dots, x_{d-1}/1; M_{\mathfrak{p}_i})) \right) = 0.$$

Note that  $e(\bar{x}_d, R/\mathfrak{p}_i) > 0$ . Moreover, by Lichtenbaum [6],

$$\sum_{s=1}^{d-1} (-1)^{s+1} \ell(H_s(x_1/1, \dots, x_{d-1}/1; M_{\mathfrak{p}_i})) \geq 0$$

for all  $i$ . Therefore we obtain

$$\sum_{s=1}^{d-1} (-1)^{s+1} H_s(x_1/1, \dots, x_{d-1}/1; M_{\mathfrak{p}_i}) = 0$$

for all  $i = 1, \dots, t$ . So, by Lichtenbaum [6] again,  $H_s(x_1/1, \dots, x_{d-1}/1; M_{\mathfrak{p}_i}) = 0$  for all  $i = 1, \dots, r$  and all  $s \geq 1$ . Hence  $\ell(H_s(x_1, \dots, x_{d-1}; M)) < \infty$  for all  $s \geq 1$ .

(ii). Assume that  $k \leq d - 2$ . We set  $p = d - k$  and prove by induction on  $p \geq 2$  that

$$\dim(H_s(x_1, \dots, x_k; M)) \leq d - k - 1 = p - 1$$

for all  $s \geq 1$ . Let  $p = 2$ . We get the exact sequence

$$(2.4) \quad \begin{aligned} 0 &\longrightarrow H_s(x_1, \dots, x_{n-1}; M)/x_n H_s(x_1, \dots, x_{n-1}; M) \\ &\longrightarrow H_s(x_1, \dots, x_n; M), \end{aligned}$$

by [12, 8.3, Proposition 2], for all integers  $n \leq d$  and all  $s \geq 1$ . By the exact sequence (2.4) for  $n = d - 1$ ,  $\dim H_s(x_1, \dots, x_{d-2}; M) \leq 1 = 2 - 1$  for all  $s \geq 1$ , so the result is true when  $p = 2$ . Assume that the result is true for  $p$ , i.e.  $\dim H_s(x_1, \dots, x_{d-p}; M) \leq p - 1$  for all  $s \geq 1$ . Then by (2.4) for  $n = d - p$ , we have

$$\begin{aligned} \dim H_s(x_1, \dots, x_{d-(p+1)}; M) &\leq \dim H_s(x_1, \dots, x_{d-p}; M) + 1 \\ &\leq p - 1 + 1 = p \quad (s \geq 1). \end{aligned}$$

□

### 3. Local cohomology modules

**Lemma 3.1.**  $\text{Supp } H_I^{d-1}(M)$  is finite.

**Proof.** The proof is by induction on  $d$ . The case  $d = 1$  is clear. Let  $d \geq 2$ . and let  $\bar{M} = M/\Gamma_I(M)$ . Then  $H_I^i(M) \cong H_I^i(\bar{M})$  for all  $i \geq 1$ ,

and there exists  $a \in I$  which is  $\overline{M}$ -regular. So, the exact sequence  $0 \longrightarrow \overline{M} \xrightarrow{a} \overline{M} \longrightarrow \overline{M}/a\overline{M} \longrightarrow 0$  implies the exact sequence

$$H_I^{d-2}(\overline{M}/a\overline{M}) \longrightarrow 0 :_{H_I^{d-1}(\overline{M})} a \longrightarrow 0.$$

Since  $\dim \overline{M}/a\overline{M} \leq d-1$ ,  $\text{Supp } H_I^{d-2}(\overline{M}/a\overline{M})$  is finite by the induction hypothesis. So,  $\text{Supp } (0 :_{H_I^{d-1}(\overline{M})} a)$  is finite. Since  $\text{Supp } (0 :_{H_I^{d-1}(\overline{M})} a) = \text{Supp } H_I^{d-1}(M)$ , it follows that  $\text{Supp } H_I^{d-1}(M)$  is a finite set.  $\square$

**Remark.** Th. Marley [8, Corollary 2.5] proved the finiteness of  $\text{Supp } H_I^{d-1}(M)$  by using a result on the asymptotic behavior of associated primes. Here we give a simple inductive proof. The following result has been proved in [7] and [10].

**Lemma 3.2.**

(i) *If  $\dim M/IM > 0$  then  $\text{f-depth}(I; M)$  is finite and*

$$\text{f-depth}(I; M) = \inf\{i : H_I^i(M) \text{ is not Artinian}\}.$$

(ii) *If  $\dim M/IM > 1$  then  $\text{gdepth}(I; M)$  is finite and*

$$\text{gdepth}(I; M) = \inf\{i : \text{Supp } H_I^i(M) \text{ is not a finite set}\}.$$

*Moreover,  $\text{Ass } H_I^i(M)$  is a finite set for all  $i \leq \text{gdepth}(I; M)$ .*

It is clear that if  $I$  is an ideal of  $R$  such that  $\dim M/IM = 0$  (resp.  $\dim M/IM = 1$ ) then the local cohomology module  $H_I^i(M)$  is Artinian (resp.  $\text{Supp } H_I^i(M)$  is a finite set) for all integers  $i$ . In the case  $\dim M/IM > 0$ , the filter depth of  $M$  in  $I$  is finite. So, there is by Lemma 3.2 at least an integer  $i$  such that  $H_I^i(M)$  is not Artinian. Similarly, in the case  $\dim M/IM > 1$ , the generalized depth of  $M$  in  $I$  is finite. Therefore there is at least an integer  $i$  such that  $\text{Supp } H_I^i(M)$  is an infinite set. By using reducing sequences, we can show some cases in which  $H_I^i(M)$  are Artinian (resp.  $\text{Supp } H_I^i(M)$  is a finite) for all but only one index  $i$ .

**Theorem 3.3.** *Let  $I$  be an ideal of  $R$ .*

- (i) *Assume that  $\dim M/IM = 1$ . Then  $\text{rdepth}(I; M) = d-1$  if and only if  $H_I^i(M)$  is Artinian for all  $i \neq d-1$  and  $\text{Supp } H_I^{d-1}(M)$  is a finite set.*  
(ii) *Assume that  $\dim M/IM = 2$ . Then  $\text{rdepth}(I; M) = d-2$  if and*

only if  $\text{Supp } H_I^i(M)$  is a finite set for all  $i \neq d-2$  and  $\text{Ass } H_I^{d-2}(M)$  is a finite set. In this case,  $\text{gdepth}(I; M) = d-2$ .

**Proof.** (i). Assume that  $\text{rdepth}(I; M) = d-1$ . Then there exists, by the definition of reducing depth, a reducing sequence  $x_1, \dots, x_{d-1}$  of  $M$  in  $I$ . By Proposition 2.6, the Koszul homology module  $H_i(x_1, \dots, x_{d-1}; M)$  is of finite length for all  $i \geq 1$ . So,  $\ell(H^i(x_1, \dots, x_{d-1}; M)) < \infty$  for all  $i \leq d-2$ . Let  $J$  be the ideal of  $R$  generated by  $x_1, \dots, x_{d-1}$ . It follows by Melkersson [9, Theorem 5.5] that  $H_J^i(M)$  is Artinian for all  $i \leq d-2$ . By Lemma 3.2, the filter depth of  $M$  in  $J$  is the least integer  $i$  such that  $H_J^i(M)$  is not Artinian. So,  $\text{f-depth}(J; M) \geq d-1$ . Since  $J \subseteq I$ , it is easily seen that  $\text{f-depth}(I; M) \geq \text{f-depth}(J; M)$ . Therefore we have  $\text{f-depth}(I; M) \geq d-1$ . So, again by Lemma 3.2,  $H_I^i(M)$  is Artinian for all  $i \leq d-2$ . It should be mentioned that  $H_I^d(M)$  is always Artinian, cf. [13]. Therefore  $H_I^i(M)$  is Artinian for all  $i \neq d-1$ . Moreover,  $\text{Supp } H_I^{d-1}(M)$  is a finite set by Lemma 3.1. Conversely, assume that  $H_I^i(M)$  is Artinian for all  $i \neq d-1$ . Then we get by Lemma 3.2 that  $\text{f-depth}(I; M) \geq d-1$ . Hence,  $\text{rdepth}(I; M) \geq d-1$ . Since  $\dim M/IM = 1$ , we have  $\text{rdepth}(I; M) \leq d-1$  by Proposition 2.4. Thus  $\text{rdepth}(I; M) = d-1$ .

(ii). Assume that  $\text{rdepth}(I; M) = d-2$ . Then there exists a reducing sequence  $x_1, \dots, x_{d-2}$  of  $M$  in  $I$ . By Proposition 2.6,  $\dim H_i(x_1, \dots, x_{d-2}; M) \leq 1$  for all  $i \geq 1$ . Hence  $\ell\left(H_i(x_1, \dots, x_{d-2}; M)\right)_{\mathfrak{p}} < \infty$  for all  $i \geq 1$  and all prime ideal  $\mathfrak{p} \neq \mathfrak{m}$ . Let  $J = (x_1, \dots, x_{d-2})R$ . It follows by [9, Theorem 5.5] that  $H_{J_{R_{\mathfrak{p}}}}^i(M_{\mathfrak{p}})$  is Artinian for all  $i \leq d-3$ . Note that the support of any Artinian module is contained in  $\{\mathfrak{m}\}$ . It follows that

$$\min\{i : \dim \text{Supp } H_J^i(M) \geq 2\} \geq d-2.$$

Here,  $\dim \text{Supp } H_J^i(M)$  is the supremum of  $\dim R/\mathfrak{p}$  where  $\mathfrak{p}$  runs over the set  $\text{Supp } H_J^i(M)$ . Note that the generalized depth of  $M$  in  $J$  is the least integer  $i$  such that there exists  $\mathfrak{p} \in H_J^i(M)$  such that  $\dim R/\mathfrak{p} \geq 2$ , cf. [10, Proposition 4.5]. Hence  $\text{gdepth}(J; M) \geq d-2$ . Since  $J \subseteq I$ , we have  $\text{gdepth}(I; M) \geq d-2$ . Therefore we get by Lemma 3.2 that  $\text{Supp } H_I^i(M)$  is a finite set for all  $i \neq d-3$ . Moreover,  $\text{Supp } H_I^{d-1}(M)$  is a finite set by Lemma 3.1. Therefore  $\text{Supp } H_I^i(M)$  is a finite set for all  $i \neq d-2$ . Lastly,  $\text{Ass } H_I^{d-2}(M)$  is finite by Lemma 3.2. Conversely, since  $\text{Supp } H_I^i(M)$  is a finite set for all  $i \neq d-2$ , it follows by Lemma 3.2 that  $\text{gdepth}(I; M) \geq d-2$ . Hence  $\text{rdepth}(I; M) \geq d-2$ . Since



$\dim M/IM = 2$ , we get by Proposition 2.4 that  $\text{rdepth}(I; M) \leq d - 2$  and hence  $\text{rdepth}(I; M) = d - 2$ .  $\square$

For an ideal  $I$  generated by the elements  $x_1, \dots, x_n$  of  $R$  and an integer  $r \geq 0$ ,  $H_I^i(M)$  is Artinian for all  $i \leq r$  if and only if  $\ell(H^i(x_1, \dots, x_n; M)) < \infty$  for all  $i \leq r$ , cf. [9, Theorem 5.5]. Note that  $H_I^i(M)$  is Artinian for all  $i \leq r$  if and only if  $\text{Supp } H_I^i(M) \subseteq \{\mathfrak{m}\}$  for all  $i \leq r$ . Therefore by induction on  $k$ , with the standard arguments of localization, we can easily get the following result.

**Lemma 3.4.** *Let  $I = (x_1, \dots, x_n)R$  be an ideal of  $R$  and let  $r, k$  be positive integers. Then the following conditions are equivalent:*

- (i)  $\dim(H^i(x_1, \dots, x_n; M)) \leq k$  for all  $i \leq r$ .
- (ii)  $\dim \text{Supp } H_I^i(M) \leq k$  for all  $i \leq r$ .

Assume that  $\dim M/IM = t \geq 2$ . Then by Proposition 2.4 we get

$$d - t - 1 \leq \text{rdepth}(I; M) \leq d - t.$$

It is natural to ask when  $\text{rdepth}(I; M)$  is exactly  $d - t$ . The following result gives an answer to this question.

**Theorem 3.5** *If  $\dim M/IM = t$ , then the following conditions are equivalent:*

- (i)  $\text{rdepth}(I; M) = d - t$ .
- (ii)  $\dim \text{Supp } H_I^i(M) \leq t - 1$  for all  $i \leq d - t - 1$ .

Before proving Theorem 3.5, we need the notion of  $k$ -depth. Let  $k$  be an integer. A sequence  $x_1, \dots, x_n$  of elements in  $\mathfrak{m}$  is called a  $k$ -regular sequence if for all  $i = 1, \dots, n$ ,  $x_i \notin \mathfrak{p}$  for all  $\mathfrak{p} \in \text{Ass}(M/(x_1, \dots, x_{i-1})M)$  satisfying  $\dim R/\mathfrak{p} \geq k$ . It is clear that regular sequences (resp. filter regular sequences, generalized regular sequences) are 0-regular sequences (resp. 1-regular sequences, 2-regular sequences). Let  $I$  be an ideal of  $R$ . The  $k$ -depth of  $M$  in  $I$ , denoted by  $\text{depth}^k(I; M)$ , is the supremum of lengths of all  $k$ -regular sequences of  $M$  in  $I$ . It is easily checked that if  $\dim M/IM > k$  then any  $k$ -regular sequence of  $M$  in  $I$  is of finite length and all maximal  $k$ -regular sequences of  $M$  in  $I$  have the same length. In this case,

$$(3.1) \quad \begin{aligned} \text{depth}^k(I; M) &= \min\{\text{depth}(IR_{\mathfrak{p}}; M_{\mathfrak{p}}) : \mathfrak{p} \in \text{Supp } M/IM, \\ &\quad \dim R/\mathfrak{p} \geq k\} = \min\{i : \dim \text{Supp } H_I^i(M) \geq k\}. \end{aligned}$$

**Proof of Theorem 3.5.** Assume that  $\text{rdepth}(I; M) = d - t$ . Let  $x_1, \dots, x_{d-t}$  be a reducing sequence of  $M$  in  $I$ . By Proposition 2.6,

$$\dim H_i(x_1, \dots, x_{d-t}, M) \leq t - 1, \text{ for all } i \geq 1.$$

Let  $J = (x_1, \dots, x_{d-t})R$ . By Lemma 3.4 we get  $\dim \text{Supp } H_J^i(M) \leq t - 1$  for all  $i \leq d - t - 1$ . Therefore, it follows by (3.1), that  $\text{depth}^t(J; M) \geq d - t$ . Hence  $\text{depth}^t(I; M) \geq d - t$ . So we have again by (3.1) that  $\dim \text{Supp } H_I^i(M) \leq t - 1$  for all  $i \leq d - t - 1$ . Conversely, assume that  $\dim \text{Supp } H_I^i(M) \leq t - 1$  for all  $i \leq d - t - 1$ . Then  $\text{depth}^t(I; M) \geq d - t$ . Hence  $\text{rdepth}(I; M) \geq d - t$ . Note that  $\text{rdepth}(I; M) \leq d - t$  by Proposition 2.4. Thus,  $\text{rdepth}(I; M) = d - t$ .  $\square$

**Example 3.6.** Let  $S = k[x, y, z, t, v, w]$  be the polynomial ring of 6 variables over a field  $k$ . Let  $\mathfrak{m} = (x, y, z, t, v, w)S$  be the unique homogeneous maximal ideal of  $S$ . Set  $R = S_{\mathfrak{m}}$ , the localization of  $S$  with respect to  $\mathfrak{m}$ . Let  $I = (x, y, z, t)R$  and  $J = (x, y, z)R$ . Set

$$M = R/vR \cap (v^2)R \cap (x, y, z, t, v^3, w)R.$$

$$M' = R/vR \cap (z, t, v^2)R \cap (x, y, z, t, v^3, w)R.$$

Then we have  $\dim M = \dim M' = 5$  and

- (i)  $\dim M/IM = \dim M'/IM' = 1$ ;  $\text{rdepth}(I; M) = 4$ ;  $\text{rdepth}(I; M') = 3$ ;  $H_I^i(M)$  is Artinian for all  $i \neq 4$ .
- (ii)  $\dim M/JM = \dim M'/JM' = 2$ ;  $\text{rdepth}(I; M) = 3$ ;  $\text{rdepth}(I; M') = 2$ ;  $\text{Supp } H_I^i(M)$  is a finite set for all  $i \neq 3$ .

It is clear that  $\dim M = \dim M' = 5$ . For each element  $a \in S$ , we denote by  $\bar{a}$  the image of  $a$  in  $R$ . Then we can easily check that  $\bar{x}, \bar{y}, \bar{z}, \bar{t}$  is a reducing sequence of  $M$ . Therefore we get by Theorem 3.3 that  $\text{rdepth}(I; M) = 4$  and  $H_I^i(M)$  is Artinian for all  $i \neq 4$ . We can also check that  $\bar{x}, \bar{y}, \bar{z}$  is a maximal filter regular sequence of  $M'$  in  $I$ . It means that  $\text{f-depth}(I; M') = 3$ . Therefore  $\text{rdepth}(I; M') \neq 4$  by Theorem 3.2. Hence  $\text{rdepth}(I; M') = 3$  by Proposition 2.4. Proof of (ii) is similar to that of (i).

## REFERENCES

- [1] M. Auslander and D. Buchsbaum, Codimension and multiplicity, *Ann. Math.* **68** (2) (1958), 625-657.
- [2] M. Brodmann and R. Y. Sharp, *Local Cohomology: An Algebraic Introduction with Geometric Applications*, Cambridge University Press, 1998.

- [3] N. T. Cuong, On the dimension of non Cohen-Macaulay locus of local rings admitting dualizing complexes, *Math. Proc. Camb. Soc.* **109** (1991), 479-488.
- [4] N. T. Cuong, P. Schenzel and N. V. Trung, Verallgemeinerte Cohen-Macaulay Moduln, *Math. Nachr.* **5** (1978), 57-73.
- [5] P. Dutta, A note on the monomial conjecture, *Trans. Amer. Math. Soc.* **350** (7) (1998), 2871-2878.
- [6] S. Lichtenbaum, On the vanishing of Tor in regular local rings, *Illinois J. Math.* **10** (1966), 220-226.
- [7] R. Lü and Z. Tang, The  $f$ -depth of an ideal on a module, *Proc. Amer. Math. Soc.* **130** (7) (2002), 1905-1912.
- [8] Th. Marley, Associated primes of local cohomology modules over rings of small dimension, *Manuscripta Math.* **104** (4) (2001), 519-525.
- [9] L. Melkersson, Modules cofinite with respect to an ideal, *J. Algebra* **285** (2) (2005), 649-668.
- [10] L. T. Nhan, On generalized regular sequences and the finiteness for associated primes of local cohomology modules, *Comm. Algebra* **33** (3) (2005), 793-806.
- [11] L. T. Nhan and M. Morales, Generalized  $f$ -modules and the associated primes of local cohomology modules, to appear in *Comm. Algebra*.
- [12] D. G. Northcott, *Lessons on Rings, Modules and Multiplicities*, Cambridge University Press, London, 1968.
- [13] R. Y. Sharp, On the attached prime ideals of certain Artinian local cohomology modules, *Proc. Edinburgh Math. Soc.* **24** (2) (1981), 9-14.
- [14] J. Strückerad and W. Vogel, *Buchsbaum rings and Applications*, Springer-Verlag, Berlin-Heidelberg-New York, 1986.

**Nong Quoc Chinh**

Thai Nguyen Natural Sciences College

Thai Nguyen University, Thai Nguyen, Vietnam

e-mail: nongquocchinh2002@hn.vnn.vn

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