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SEMI-θ-OPEN SETS AND NEW CLASSES OF MAPS

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ABSTRACT. This paper is dealing with the application of the notion of semi-θ-open sets in topological spaces in order to present and study new classes of maps called contra pre-semi-θ-open maps and contra pre-semi-θ-closed maps. In this connection, two classes of maps called semi-θc-homeomorphism and contra semi-θc-homeomorphism are introduced. It turns out that the union of the families of all semi-θc-homeomorphisms and contra semi-θc-homeomorphisms forms a group under the composition of maps.

1. Introduction

Generalized open sets are now well-known important notions in Topology and its applications. Many topologists are focusing their research on these topics and this amounted to many important and useful results. In this respect, the variously modified forms of continuity, separation axioms etc by utilizing generalized open sets, play a significant role in General Topology and Real Analysis. In 1963 Norman Levine [14] introduced the notion of semi-open sets. According to Douglas E. Cameron [5] this notion was Norman Levine's most important contribution to the field of Topology. The motivation behind the introduction of semi-open sets was a problem of Kelley which Norman Levine has considered in [15], i.e. to show that $\mathrm{Cl}(U) = \mathrm{Cl}(U \cap D)$ for all open sets U and dense

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sets D. He proved that U is semi-open if and only if $\mathrm{Cl}(U) = \mathrm{Cl}(U \cap D)$ for all dense sets D and D is dense if and only if $\mathrm{Cl}(U) = \mathrm{Cl}(U \cap D)$ for all semi-open sets U. In 1987 , Di Maio and Noiri[9] used this notion and the semi-closure [7] of a set to introduce the concepts of semi-θ-open and semi-θ-closed sets which provide a formulation of semi-θ-closure of a set in a topological space. Noiri [19] defined and studied the concept of θ-semicontinuous functions by involving these sets. Mukherjee and Basu [18] continued the work of Di Maio and Noiri and defined the concepts of semi-θ-connectedness, semi-θ-components and semi-θ-quasi-components. Dontchev and Noiri [13] obtained , among others, that a topological space is semi-Hausdorff if and only if each singleton is semi-θ-closed. Recently the authors [2, 3, 4] have also obtained several new and important results and notions related to these sets. In this direction we shall introduce the concept of contra pre-semi-θ-openness (resp. contra pre-semi-θ-closedness) and study some of their basic properties. In this connection, two classes of maps called semi-θc-homeomorphism and contra semi-θc-homeomorphism are introduced. It turns out that the union of the families of all semi-θc-homeomorphisms and contra semi-θc-homeomorphisms forms a group under the composition of maps.

2. **Preliminaries**

Since we shall require the following known definitions, notations and some properties, we recall them in this section.

Let (X, τ) be a topological space and S a subset of X. We denote the closure and the interior of S by $\mathrm{Cl}(S)$ and $\mathrm{Int}(S)$, respectively. A subset S is said to be *semi-open* [14], if there exists an open set U such that $U \subseteq S \subseteq \mathrm{Cl}(U)$, or equivalently if $S \subseteq \mathrm{Cl}(\mathrm{Int}(S))$. The complement of a semi-open set is said to be *semi-closed* [7]. The intersection of all semi-closed sets containing S is called the *semi-closure* [7] of S and is denoted by $\mathrm{sCl}(S)$. The *semi-interior* of S denoted by $\mathrm{sInt}(S)$, is defined by the union of all semi-open sets contained in S. A subset S is called *semiregular* [9], if it is both semi-open and semi-closed. The family of all semi-open sets (resp. semiregular sets) of (X, τ) is denoted by $\mathrm{SO}(X, \tau)$ (resp. $\mathrm{SR}(X, \tau)$). The *semi θ-closure* of S [9], denoted by $\mathrm{sCl}_\theta(S)$, is defined to be the set of all $x \in X$ such that $\mathrm{sCl}(O) \cap S \neq \emptyset$ for every $O \in \mathrm{SO}(X, \tau)$ with $x \in O$. A subset S is called *semi-θ-closed* [10] if

S=sCl$_\theta(S)$. The complement of a semi-θ-closed set is called *semi-θ-open.* The family of all semi-θ-open (resp. semi-θ-closed) subsets of X is denoted by S$_\theta$O(X,τ) (resp. S$_\theta$C(X,τ)).

Definition 2.1. A map $f:(X,\tau)\to(Y,\sigma)$ is called *pre-semi-θ-closed* (resp. *pre-semi-θ-open)* [1], if for every semi-θ-closed (resp. semi-θ-open) set B in X, $f(B)$ is semi-θ-closed (resp. semi-θ-open) set in (Y,σ).

We recall the following results which are obtained by Di Maio and Noiri [9, 11] and Mukherjee and Basu [18].

Lemma 2.2. *Let A be a subset of a topological space (X,τ).*
(i) (Di Maio and Noiri [9, Proposition 2.3(a)]) *If $A\in SO(X,\tau)$, then $sCl(A)$ is semi-regular and $sCl(A)=sCl_\theta(A)$.*
(ii) (Di Maio and Noiri [9, Proposition 2.3(b)], [11, Proposition 2.2(2)]) *If A is semi-regular, then A is semi-θ-closed and semi-θ-open.*

Mukherjee and Basu [18, Corollary 2.5] note that the converse of (ii) above holds truly, because $S\subseteq sCl(S)\subseteq sCl_\theta(S)$ for any subset S of a topological space (X,τ).

Lemma 2.3. (Mukherjee and Basu [18, Corollary 2.5]) *$sCl_\theta(A)$ is semi-θ-closed for every $A\subset X$.*

Remark 2.4. (i) Any intersection of semi-θ-closed sets is semi-θ-closed (Di Maio [10, Lemma 3.3]). Hence, by complement, any union of semi-θ-open sets is semi-θ-open.
(ii) Intersection of semi-θ-open sets may fail to be semi-θ-open (see Example 3.9 below).

Recall that a mapping $f:(X,\tau)\to(Y,\sigma)$ is said to be *semi-continuous* [14] (resp. *quasi-irresolute* [11, Definition 3.1, Proposition 3.4], cf [20]) if the inverse image of each open set (resp. semi-θ-open set) of Y is semi-open (resp. semi-θ-open) in X.

Definition 2.5. A subset A of (X,τ) is said to be *quasi semi-θ-closed* (briefly *qst-closed*) in (X,τ) [1, 3] if, sCl$_\theta(A)\subseteq O$ whenever $A\subseteq O$ and O is semi-θ-open in (X,τ). A subset B is said to be *quasi semi-θ-open* (briefly *qst-open*) in (X,τ) [1] if its complement $B^c=X-B$ is qst-closed in (X,τ).

Remark 2.6. Every semi-θ-closed set is qst-closed , but not conversely. Let (X,τ) be a space such that $X=\{a,b,c,d\}$, $\tau=\{\emptyset,\{c,d\},X\}$ and $A=\{a,b,d\}$. Then $\mathrm{sCl}(A)=X$ and so A is not semi-closed. Hence A is not semi-θ-closed. Since X is the only semi-θ-open set containing A, then A is qst-closed.

Definition 2.7. A map $f:(X,\tau)\to(Y,\sigma)$ is said to be *approximately semi-θ-closed* (or *ap-semi-θ-closed*), if $f(B)\subseteq \mathrm{sInt}_\theta(A)$ whenever A is a qst-open subset of (Y,σ), B is a semi-θ-closed subset of (X,τ) and $f(B)\subseteq A$.

3. Contra pre-semi-θ-open maps and contra pre-semi-θ-closed maps.

Let $f:(X,\tau)\to(Y,\sigma)$ be a map from a topological space (X,τ) into a topological space (Y,σ).

Definition 3.1. A map $f:(X,\tau)\to(Y,\sigma)$ is said to be *contra pre-semi-θ-open* if $f(O)$ is semi-θ-closed in (Y,σ) for each $O\in \mathrm{S}_\theta\mathrm{O}(X,\tau)$.

Definition 3.2. A map $f:(X,\tau)\to(Y,\sigma)$ is said to be *contra pre-semi-θ-closed* if $f(B)\in \mathrm{S}_\theta\mathrm{O}(Y,\sigma)$ for each semi-θ-closed set B of (X,τ).

The following example shows that contra pre-semi-θ-closedness and contra pre-semi-θ-openness are independent.

Example 3.3. Let $X=Y=\{a,b,c\}$, $\tau=\{\emptyset,\{a\},\{b\},\{a,b\},\{b,c\},X\}$ and $\sigma=\{\emptyset,\{a\},\{b\},\{a,b\},Y\}$. Let $f:(X,\tau)\to(Y,\sigma)$ be defined by $f(a)=f(c)=c$, $f(b)=b$ and $g:(X,\tau)\to(Y,\sigma)$ by $g(a)=g(b)=a$, $g(c)=b$. Then we have: $\mathrm{S}_\theta\mathrm{O}(X,\tau)=\{\emptyset,\{a\},\{b,c\},X\}$ and $\mathrm{S}_\theta\mathrm{O}(Y,\sigma)=\{\emptyset,\{a\},\{b\},\{a,b\},\{b,c\},\{a,c\},Y\}$. Therefore f is contra pre-semi-θ-open but not contra pre-semi-θ-closed and g is contra pre-semi- θ-closed but not contra pre-semi-θ-open.

Remark 3.4. Contra pre-semi-θ-openness and contra pre-semi-θ-closedness are equivalent if the map is bijective.

Theorem 3.5. *For a map $f:(X,\tau)\to(Y,\sigma)$ the following properties are equivalent.*
(i) *f is contra pre-semi-θ-open.*

(ii) *For every subset B of Y and every semi-θ-closed subset F of X with $f^{-1}(B) \subseteq F$, there exists a semi-θ-open subset O of Y with $B \subseteq O$ and $f^{-1}(O) \subseteq F$.*
(iii) *For every point $y \in Y$ and every semi-θ-closed subset F of X with $f^{-1}(y) \subseteq F$, there exists a semi-θ-open subset O of Y with $y \in O$ and $f^{-1}(O) \subseteq F$.*

Proof. $(i) \Rightarrow (ii)$. Let B be a subset of Y and F be a semi-θ-closed subset of X with $f^{-1}(B) \subseteq F$. For the case where $B \neq \emptyset$, put $O = [f(F^c)]^c$. Then $f^{-1}(O) = [f^{-1}(f(F^c))]^c \subseteq F$ and O is a semi-θ-open subset of Y. We claim that $B \subseteq O$. There are two cases to be considered:
Case 1. $f^{-1}(B) \neq \emptyset$. Since $f^{-1}(B) \subseteq F$, we have $f(F^c) \subseteq B^c$ and $B \subseteq O$.
Case 2. $f^{-1}(B) = \emptyset$. Since $f^{-1}(B) = \emptyset \subseteq F$, we have $f(F^c) \subseteq f(X)$. We have that $B \bigcap f(X) = \emptyset$, because $f^{-1}(B) = \emptyset$ and $B \neq \emptyset$. Thus $B \subseteq [f(X)]^c \subseteq [f(F^c)]^c = O$. For the case where $B = \emptyset$, put $O = \emptyset$. Then, the set O is a required semi-θ-open set of Y.
$(ii) \Rightarrow (iii)$. It suffices to put $B = \{y\}$ in (ii).
$(iii) \Rightarrow (i)$. Let A be a semi-θ-open subset of X. Then, let $y \in [f(A)]^c$ and $F = A^c$. First we claim that $f^{-1}(y) \subseteq F$. For non-empty set $f^{-1}(y)$, let $z \in f^{-1}(y)$; $f(z) = y \notin f(A)$. Suppose that $z \notin F$. Then $z \in A$ and so $y = f(z) \in f(A)$. By contradiction, $z \in F$ for any $z \in f^{-1}(y)$, i.e., $f^{-1}(y) \subseteq F$. For the case where $f^{-1}(y) = \emptyset$, we have that $f^{-1}(y) = \emptyset \subseteq F$. For both cases, we can use (iii) and get the following: there exists a semi-θ-open set $O_y \subseteq Y$ such that $y \in O_y$ and $f^{-1}(O_y) \subseteq F = A^c$. Namely, $(*)$ $f^{-1}(O_y) \cap A = \emptyset$ holds for each $y \in [f(A)]^c$. Finally we claim that $[f(A)]^c = \bigcup\{O_y : y \in [f(A)]^c\}$. Obviously, we have that $[f(A)]^c \subseteq \bigcup\{O_y : y \in [f(A)]^c\}$. Conversely, let $z \in \bigcup\{O_y : y \in [f(A)]^c\}$. Then, there exists a point $w \in [f(A)]^c$ such that $z \in O_w$. Suppose that $z \notin [f(A)]^c$. Then $z \in f(A)$ and there exists a point $b \in A$ such that $f(b) = z$. Thus we have that $f(b) \in O_w$ and so $b \in f^{-1}(O_w)$. We have a contradiction to $(*)$ above, i.e., $b \in f^{-1}(O_w) \cap A$. Hence, we show that $\bigcup\{O_y : y \in [f(A)]^c\} \subseteq [f(A)]^c$ and so $[f(A)]^c = \bigcup\{O_y : y \in [f(A)]^c\}$. Consequently, by Remark 2.3(i), $f(A)$ is a semi-θ-closed subset in Y. $\square$

Theorem 3.6. *Let (1), (2) and (3) be properties of a map $f : (X, \tau) \to (Y, \sigma)$ as follows:*
(1) *f is contra pre-semi-θ-closed.*
(2) *For every subset B of Y and every semi-θ-open subset O of X with*

$f^{-1}(B) \subseteq O$, there exists a semi-θ-closed subset F of Y with $B \subseteq F$ and $f^{-1}(F) \subseteq O$.
(3) *For every point $y \in Y$ and every semi-θ-open subset O of X with $f^{-1}(y) \subseteq O$, there exists a semi-θ-closed subset F of Y with $y \in F$ and $f^{-1}(F) \subseteq O$.*
(i) *The implications* (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) *hold.*
(ii) *Suppose that $S_\theta C(Y,\sigma)$ is closed under arbitrary unions (i.e., the union of any collection of semi-θ-closed sets is semi-θ-closed). Then, the implication* (3) $\Rightarrow$ (1) *holds.*

Proof. (i) (1) $\Rightarrow$ (2). Let B be a subset of Y and O be a semi-θ-open subset of X with $f^{-1}(B) \subseteq O$. Put $F = [f(O^c)]^c$. Since f is contra pre-semi-θ-closed, then F is a semi-θ-closed set of Y. By $f^{-1}(B) \subseteq O$, we have $f(O^c) \subseteq B^c$. Moreover $f^{-1}(F) \subseteq O$.
(2) $\Rightarrow$ (1). Let E be a semi-θ-closed subset of X. Put $B = [f(E)]^c$ and $O = E^c$. Hence $f^{-1}(B) = f^{-1}(f(E))^c = [f^{-1}(f(E))]^c \subseteq E^c = O$. By assumption there exists a semi-θ-closed set $F \subseteq Y$ for which $B \subseteq F$ and $f^{-1}(F) \subseteq O$. It follows that $B = F$. If $y \in F$ and $y \notin B, y \in f(E)$. Therefore $y = f(x)$ for some $x \in E$ and we have that $x \in f^{-1}(F) \subseteq O = E^c$, which is a contradiction. Since $B = F$ (i.e., $[f(E)]^c = F$), $f(E)$ is semi-θ-open and hence f is contra pre-semi-θ-closed.
(2) $\Rightarrow$ (3). It suffices to put $B = \{y\}$ for $y \in Y$.
(ii) (3) $\Rightarrow$ (1). Let A be a semi-θ-closed subset of X. Let $y \in [f(A)]^c$. Then, we have that $f^{-1}(y) \subseteq A^c$. By (3) there exists a semi-θ-closed set $F_y \subseteq Y$ such that $y \in F_y$ and $f^{-1}(F_y) \subseteq A^c$. Namely, $f^{-1}(F_y) \cap A = \emptyset$ holds for each $y \in [f(A)]^c$. By an argument similar to that in the proof ($(iii) \Rightarrow (i)$) of Theorem 3.5, it is shown that $[f(A)]^c = \bigcup\{F_y : y \in [f(A)]^c\}$ holds. Consequently, using assumption in (ii), $f(A)$ is semi-θ-open in (Y,σ). □

Caldas and Jafari in [1] showed that pre-semi-θ-closed maps imply $\mathrm{sCl}_\theta(f(O)) \subseteq f(\mathrm{sCl}_\theta((O))$ for every subset O of X. The following theorem tests this result replacing the pre-semi-θ-closed requirement with contra pre-semi-θ-open.

Theorem 3.7. *Let $f : (X,\tau) \to (Y,\sigma)$ be a map.*
(i) *If f is contra pre-semi-θ-open, $\mathrm{sCl}_\theta(f(O)) \subseteq f(\mathrm{sCl}_\theta(O))$ for every semi-θ-open subset O of X.*

(ii) *If f is contra pre-semi-θ-closed, $f(O) \subseteq \mathrm{sInt}_\theta(f(\mathrm{sCl}_\theta(O)))$ for every subset O of X.* □

Our next two examples show that contra pre-semi-θ-closed (resp. contra pre-semi-θ-open) maps and pre-semi-θ-closed (resp. pre-semi-θ-open) are independent notions.

Example 3.8. Let (X,τ) and (Y,σ) be the space defined in Example 3.3. Let $f : (X,\tau) \to (Y,\sigma)$ be a map defined by $f(a) = f(c) = c$ and $f(b) = b$. Then f is contra pre-semi-θ-open. However $\{a\}$ is semi-θ-open in X, but $f(\{a\})$ is not semi-θ-open in Y. Therefore f is not pre-semi-θ-open.

Example 3.9. Let (X,τ) be a space such that $X = \{a,b,c,d\}$, $\tau = \{\emptyset, \{b\}, \{d\}, \{b,d\}, \{a,b,d\}, \{b,c,d\}, X\}$. Let $f : (X,\tau) \to (X,\tau)$ be the identity map. We obtain $S_\theta C(X,\tau) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{c,d\}, \{a,b,c\}, \{a,c,d\}, X\}$. Then f is a pre-semi-θ-closed map (resp. pre-semi-θ-open map), which is not contra pre-semi-θ-closed (resp. contra pre-semi-θ-open).

Definition 3.10. $f : (X,\tau) \to (Y,\sigma)$ is called a *perfectly contra pre-semi-θ-closed* map if the image of every semi-θ-closed set in X is semi-θ-clopen (i.e., semi-θ-open and semi-θ-closed) in Y.

Remark 3.11. Every perfectly contra pre-semi-θ-closed map is contra pre-semi-θ-closed and pre-semi-θ-closed, but not conversely.

Theorem 3.12. (i) *If $f : (X,\tau) \to (Y,\sigma)$ is a contra pre-semi-θ-closed map, then f is ap-semi-θ-closed.*
(ii) *If $S_\theta O(Y,\sigma) = S_\theta C(Y,\sigma)$, then the following properties are equivalent:*
(1) *f is ap-semi-θ-closed;*
(2) *f is contra pre-semi-θ-closed;*
(3) *f is pre-semi-θ-closed.*

Proof. (ii) We prove only (1) $\Leftrightarrow$ (2). Assume f is ap-semi-θ-closed. Let A be an arbitrary subset of (Y,σ) such that $A \subseteq Q$, where $Q \in S_\theta O(Y,\sigma)$. Then by hypothesis $\mathrm{sCl}_\theta(A) \subseteq \mathrm{sCl}_\theta(Q) = Q$. Therefore all subsets of (Y,σ) are qst-closed (and hence all are qst-open). Therefore for any semi-θ-closed subset B of (X,τ), $f(B)$ is qst-closed in (Y,σ). Since f is ap-semi-θ-closed $f(B)\subseteq \text{sInt}_\theta(f(B))$. Therefore $\text{sInt}_\theta(f(B)) = f(B)$, i.e., $f(B)$ is semi-θ-open in (Y,σ). Conversely, Let $f(B)\subseteq A$,where B is a semi-θ-closed subset of (X,τ) and A is a qst-open subset of (Y,σ). Therefore $\text{sInt}_\theta(f(B))\subseteq \text{sInt}_\theta(A)$. Then $f(B)\subseteq \text{sInt}_\theta(A)$. Thus f is ap-semi-θ-closed. □

The following diagram holds, (cf. Remark 3.11, Theorem 3.12, Examples 3.8, 3.9, Definitions 2.7 and 3.10).

$$\begin{array}{ccccc} & \not\swarrow\ \nearrow & \textit{contra pre-semi-}\theta\textit{-closed} & \not\nwarrow\ \searrow & \\ \textit{perf. contra pre-semi}\theta\textit{-closed} & & \not\uparrow\ \not\downarrow & & \textit{ap- semi-}\theta\textit{-closed} \\ & \not\nwarrow\ \searrow & \textit{pre-semi-}\theta\textit{-closed} & \not\swarrow\ \nearrow & \end{array}$$

Definition 3.13. A map $f:(X,\tau)\rightarrow(Y,\sigma)$ is said to be *semi-θ-pre-closed* (resp. *semi-θ-pre-open*) if $\text{sCl}_\theta(\text{sInt}_\theta(f(B)))\subseteq f(B)$ (resp. if $f(O)\subseteq \text{sInt}_\theta(\text{sCl}_\theta(f(O))))$ for every semi-θ-closed set B of X (resp. for every semi-θ-open set O of X).

Next we investigate conditions under which contra pre-semi-θ-closed (resp. contra pre-semi-θ-open) maps are semi-θ-closed (resp. semi-θ-open).

Theorem 3.14. *For a map $f:(X,\tau)\rightarrow(Y,\sigma)$, the following properties hold.*
(i) *f is pre-semi-θ-closed, whenever f is contra pre-semi-θ-closed and semi-θ-pre-closed.*
(ii) *f is pre-semi-θ-open, whenever f is contra pre-semi-θ-open and semi-θ-pre-open.* □

Theorem 3.15. *If a map $f:(X,\tau)\rightarrow(Y,\sigma)$ is quasi-irresolute and contra pre-semi-θ-closed, then the inverse image of every subset of Y is qst-closed in X. In fact it is even qst-open in X.*

Proof. Let A be a subset of (Y,σ). Suppose that $f^{-1}(A)\subseteq O$ where $O\in \text{S}_\theta\text{O}(X,\tau)$. Taking complements we obtain $O^c\subseteq f^{-1}(A^c)$ or $f(O^c)\subseteq A^c$. Since f is contra pre-semi-θ-closed $f(O^c)$ is semi-θ-open. Then $f(O^c) = \text{sInt}_\theta(f(O^c)) \subseteq \text{sInt}_\theta(A^c) = (\text{sCl}_\theta(A))^c$. Hence $f^{-1}(\text{sCl}_\theta(A))\subseteq O$. Since f is quasi-irresolute, $f^{-1}(\text{sCl}_\theta(A))$ is semi-θ-closed. Thus we have $\mathrm{sCl}_\theta(f^{-1}(A)) \subseteq \mathrm{sCl}_\theta(f^{-1}(\mathrm{sCl}_\theta(A))) = f^{-1}(\mathrm{sCl}_\theta(A)) \subseteq O$. This implies that $f^{-1}(A)$ is qst-closed in (X, τ). □

Theorem 3.16. *Suppose that $f : (X, \tau) \to (Y, \sigma)$ has the following property: $f(U) \subseteq sInt_\theta(f(\mathrm{Cl}(U)))$ for every $U \in \tau$. If $S_\theta C(Y, \sigma)$ is closed under arbitrary unions and if for each semi-θ-closed subset F of X and each fiber $f^{-1}(y) \subset F^c$ there exists an open subset U of X for which $F \subset U$ and $f^{-1}(y) \cap Cl(U) = \phi$, then f is contra pre-semi-θ-closed.*

Proof. Assume that F is a semi-θ-closed subset of X and let $y \in [f(F)]^c$. Thus $f^{-1}(y) \subset F^c$. Hence there exists an open subset U of X for which $F \subset U$ and $f^{-1}(y) \cap \mathrm{Cl}(U) = \phi$. Therefore $y \in [f(\mathrm{Cl}(U))]^c \subset [f(F)]^c$. By hypothesis $f(U) \subset \mathrm{sInt}_\theta(f(\mathrm{Cl}(U)))$. We obtain $y \in \mathrm{sCl}_\theta([f(\mathrm{Cl}(U))]^c) \subset [f(F)]^c$. Let $B_y = \mathrm{sCl}_\theta([f(\mathrm{Cl}(U))]^c)$. Then B_y is a semi-θ-closed subset of Y containing y. Hence $[f(F)]^c = \cup\{B_y : y \in [f(F)]^c\}$ is semi-θ-closed. Therefore $f(F)$ is semi-θ-open. □

Theorem 3.17. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \gamma)$ be two maps such that $g \circ f : (X, \tau) \to (Z, \gamma)$. Then, the following properties hold.*
(i) *If f is pre-semi-θ-open and g is contra pre-semi-θ-open, then $g \circ f$ is contra pre-semi- θ-open.*
(ii) *If f is contra pre-semi-θ-open and g is pre-semi-θ-closed, then $g \circ f$ is contra pre-semi-θ-open.*
(i*) *If f is pre-semi-θ-closed and g is contra pre-semi-θ-closed, then $g \circ f$ is contra pre-semi-θ-closed.*
(ii*) *If f is contra pre-semi-θ-closed and g is pre-semi-θ-open, then $g \circ f$ is contra pre-semi-θ-closed.* □

A map $f : (X, \tau) \to (Y, \sigma)$ is said to be a *contra semi-θc-homeomorphism* if f is bijective and contra pre-semi-θ-open and f^{-1} is contra pre-semi-θ-open. A map $f : (X, \tau) \to (Y, \sigma)$ is said to be a *semi-θc-homeomorphism* if f is bijective and pre-semi-θ-open and f^{-1} is pre-semi-θ-open. It is shown that a map f is a semi-θc-homeomorphism if and only if f is bijective and quasi-irresolute map and f^{-1} is quasi-irresolute (cf. [11, Proposition 3.4]). If $f : (X, \tau) \to (Y, \sigma)$ is a homeomorphism, then f and f^{-1} are irresolute bijections (cf. [14, Theorem 7] or [8, Theorem 1.2]) and so they are quasi-irresolute maps (cf. [11, Remark 3.1]). Thus, every homeomorphism is a semi-θc-homeomorphism. The family of all contra semi-θc-homeomorphisms (resp. semi-θc-homeomorphisms, homeomorphisms) from a topological space (X,τ) onto itself is denoted by $cs\theta ch(X,\tau)$ (resp. $s\theta ch(X,\tau)$, $h(X,\tau)$). We have the following property: $h(X,\tau) \subseteq s\theta ch(X,\tau)$ (cf. Corollary 3.14 (ii) and (iv) below).

Corollary 3.18. *Let (X,τ), (Y,σ) and (Z,γ) be topological spaces.*
(i) *The family $cs\theta ch(X,\tau) \cup s\theta ch(X,\tau)$ forms a group under the composition of maps.*
(ii) *The family $s\theta ch(X,\tau)$ is a subgroup of $cs\theta ch(X,\tau) \cup s\theta ch(X,\tau)$ and $h(X,\tau)$ is a subgroup of $s\theta ch(X,\tau)$.*
(iii) *A contra semi-θc-homeomorphism $f:(X,\tau)\to(Y,\sigma)$ induces an isomorphism $f_*: cs\theta ch(X,\tau) \cup s\theta ch(X,\tau) \to cs\theta ch(Y,\sigma) \cup s\theta ch(Y,\sigma)$ defined by $f_*(a) = f\circ a\circ f^{-1}$.*
Moreover, $(g\circ f)_* = g_*\circ f_* : cs\theta ch(X,\tau)\cup s\theta ch(X,\tau) \to cs\theta ch(Z,\gamma)\cup s\theta ch(Z,\gamma)$ holds whenever $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\gamma)$ are contra semi-θc-homeomorphisms; $(1_X)_*$ is the identity map on $cs\theta ch(X,\tau)\cup s\theta ch(X,\tau)$; and $f_*(s\theta ch(X,\tau)) = s\theta ch(X,\tau)$.
(iv) *A semi-θc-homeomorphism $f:(X,\tau)\to(Y,\sigma)$ induces an isomorphism $f_*: s\theta ch(X,\tau)\to s\theta ch(Y,\sigma)$ defined by $f_*(a) = f\circ a\circ f^{-1}$.*
Moreover, $(g\circ f)_* = g_*\circ f_* : s\theta ch(X,\tau)\to s\theta ch(Z,\gamma)$ holds whenever $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\gamma)$ are semi-θc-homeomorphisms; $(1_X)_*$ is the identity isomorphism on $s\theta ch(X,\tau)$, where 1_X is the identity map on X.
Specially, if $f:(X,\tau)\to(Y,\sigma)$ is a homeomorphism, then $f_*: s\theta ch(X,\tau)\to s\theta ch(Y,\sigma)$ is also an isomorphism of groups.

Proof. Throughout this proof, put $G_X = cs\theta ch(X,\tau)\cup s\theta ch(X,\tau)$ for a topological space (X,τ).
(i) A binary operation $\mu_X : G_X\times G_X\to G_X$ is defined by $\mu_X(a,b) = b\circ a$, where $a,b\in G_X$ and $b\circ a$ denotes the composition of maps a and b defined by $(b\circ a)(x) = b(a(x))$ for any $x\in X$. By using Remark 3.4 and Theorem 3.17, it is shown that $\mu_X : G_X\times G_X\to G_X$ is well defined, i.e., $\mu_X(a,b) = b\circ a\in G_X$ for any $a,b\in G_X$. It is evident to show that the axioms of group are satisfied.
(ii) Let a and b be any elements of $s\theta ch(X,\tau)$. Then, it is shown that $\mu_X(a,b^{-1}) = b^{-1}\circ a\in s\theta ch(X,\tau)$ and $s\theta ch(X,\tau)\neq\emptyset$, because 1_X is a semi-θc-homeomorphism on (X,τ). Thus $s\theta ch(X,\tau)$ is a subgroup of

G_X. It is obvious that $h(X,\tau)$ is a subgroup of $s\theta ch(X,\tau)$, because $h(X,\tau) \subseteq s\theta ch(X,\tau)$ holds.
(iii) By Remark 3.4 and Theorems 3.17, it is shown that f_* is well defined, i.e., $f_*(a) \in G_Y$ for any $a \in G_X$ and f_* is an isomorphism from G_X onto G_Y. The other properties follow immediately from definitions.
(iv) The first properties are proved by an argument similar to that in (iii) above.
The last property follows from a fact that $h(X,\tau) \subseteq s\theta ch(X,\tau)$ and the first properties. □

Example 3.19. In this example, we put also $G_X = cs\theta ch(X,\tau) \cup s\theta ch(X,\tau)$ for a topological space (X,τ).
(i) Let (X,τ) be a topological space similar to Example 3.9. We have that $cs\theta ch(X,\tau) = \emptyset$ and $G_X = s\theta ch(X,\tau) = h(X,\tau) = \{1_X, f_{a,c}, f_{b,d}, u\}$, where $f_{a,c}(x) = x$ for any $x \in \{a,c\}$, $f_{a,c}(b) = d$, $f_{a,c}(d) = b$; $f_{b,d}(x) = x$ for any $x \in \{b,d\}$, $f_{b,d}(a) = c$, $f_{b,d}(c) = a$; $u(a) = c$, $u(b) = d$, $u(c) = a$, $u(d) = b$.
(ii) Let $X = \{a,b,c\} = Y$, $\tau = \{\emptyset, \{a\}, \{a,b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c\}, Y\}$. For topological spaces (X,τ) and (Y,σ), it is shown that $S_\theta O(X,\tau) = \{\emptyset, X\}$ and $S_\theta O(Y,\sigma) = \{\emptyset, \{a\}, \{b,c\}, Y\}$. Then, we have the following group isomorphisms: $G_X = cs\theta ch(X,\tau) = s\theta ch(X,\tau) \cong S_3$ and $G_Y = cs\theta ch(Y,\sigma) = s\theta ch(Y,\sigma) = \{1_X, h_a\} \cong Z_2$, where S_3 is the symmetric group of degree 3 and $h_a : Y \to Y$ is a semi-θc-homeomorphism defined by $h_a(a) = a$, $h_a(b) = c$, $h_a(c) = b$. Thus, we have that $G_X \not\cong G_Y$ and $s\theta ch(X,\tau) \not\cong s\theta ch(Y,\sigma)$. By Corollary 3.18(iv), it is concluded that there does not exist any contra semi-θc-homeomorphism between (X,τ) and (Y,σ); there does not exist any semi-θc-homeomorphism between (X,τ) and (Y,σ); specially there does not exist any homeomorphism between (X,τ) and (Y,σ) in spite of $h(X,\tau) \cong h(Y,\sigma)$.

Remark 3.20. (i) Let $f : (X,\tau) \to (Y,\sigma)$ and $g : (Y,\sigma) \to (Z,\gamma)$ be two maps such that $g \circ f : (X,\tau) \to (Z,\gamma)$ is contra pre-semi-θ-closed (resp. contra pre-semi-θ-open).
(a) If f is a quasi-irresolute surjection, then g is contra pre-semi-θ-closed (resp. contra pre-semi-θ-open).
(b) If g is a quasi-irresolute injection, then f is contra pre-semi-θ-closed (resp. contra pre-semi-θ-open).
(ii) Let $G_X = cs\theta ch(X,\tau) \cup s\theta ch(X,\tau)$ be the group in Corollary 3.18

with the binary operation $\mu_X : G_X \times G_X \to G_X$ defined by $\mu_X(x, y) = y \circ x$ (composition of maps x and y). By using (i), we have the following properties: Suppose that $\mu_X(a, b) \in cs\theta ch(X, \tau)$, where a and $b \in G_X$. Then, if $a \in s\theta ch(X, \tau)$, then $b \in cs\theta ch(X, \tau)$; if $b \in s\theta ch(X, \tau)$, then $a \in cs\theta ch(X, \tau)$.

4. **Additional properties**

Recall that a space X is *regular* if whenever A is closed in X and $x \notin A$, then there are disjoint open sets U and V with $x \in U$ and $A \subseteq V$, equivalently if U is open in X and $x \in U$, then there is an open set V containing x such that $\mathrm{Cl}(V) \subseteq U$. Two non empty subsets A and B in X are *strongly separated* [21] if there exist open sets U and V in X with $A \subseteq U$ and $B \subseteq V$ and $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \phi$. If A and B are singleton sets we may speak of points being strongly separated . We will use the fact that in a normal space disjoint closed sets are strongly separated. Recall that a space X is said to be *semi-θ-Hausdorff* (briefly *semi-θ-T_2* [1]) (resp. *semi-Hausdorff*, sometimes called *semi-T_2* [17]) if for every distinct pair of points x and y there exist two semi-θ-open (resp. semi-open) sets U and V such that $x \in U$ and $y \in V$ and $U \cap V = \phi$.

Theorem 4.1. ([3]). *For a topological space (X, τ), the following are equivalent.*
(i) (X, τ) is semi-θ-T_2.
(ii) (X, τ) is semi-T_2.

Lemma 4.2. *For a regular space (X, τ), the following properties hold.*
(i) $\mathrm{sCl}_\theta(A) \subseteq \mathrm{Cl}_\theta(A) = \mathrm{Cl}(A)$.
(ii) *Every closed (resp.open) set of X is semi-θ-closed (resp. semi-θ-open).*

Proof. (i) It suffices to observe that X is a regular space [12] if and only if for each set $A \subseteq X$, $\mathrm{Cl}_\theta(A) = \mathrm{Cl}(A)$.
(ii) It is obvious from (i). □

Theorem 4.3. *Let X be a regular space. If $f : (X, \tau) \to (Y, \sigma)$ is contra pre-semi-θ-open surjection, then for each point y in Y and each open set U in X with $f^{-1}(y) \subseteq U$ there exists a semi-θ-open set A in Y containing y and $f^{-1}(A) \subseteq \mathrm{Cl}(U)$.*

Proof. Let $y \in Y$ and U be open in X with $f^{-1}(y) \subseteq U$. Then $f^{-1}(y) \cap \text{Cl}([\text{Cl}(U)]^c) = \phi$ and consequently, $\{y\} \cap f(\text{Cl}([\text{Cl}(U)]^c)) = \phi$. Since $[\text{Cl}(U)]^c$ is open, then by the regularity of X and by contra pre-semi- θ-openness $f([\text{Cl}(U)]^c)$ is semi-θ-closed. Hence $\text{sCl}_\theta(f([\text{Cl}(U)]^c))$ $= f([\text{Cl}(U)]^c) \subseteq f(\text{Cl}([\text{Cl}(U)]^c))$. Therefore $\{y\} \cap \text{sCl}_\theta(f([\text{Cl}(U)]^c)) = \phi$. Let $A = [\text{sCl}_\theta(f([\text{Cl}(U)]^c))]^c$. Then A is semi-θ-open with $y \in A$ and $f^{-1}(A) \subseteq [f^{-1}(\text{sCl}_\theta(f([\text{Cl}(U)]^c)))]^c \subseteq [f^{-1}f([\text{Cl}(U)]^c)]^c \subseteq \text{Cl}(U)$. □

Theorem 4.4. *Let X be a regular space. If $f : (X, \tau) \to (Y, \sigma)$ is contra pre-semi-θ-open surjection and all pairs of disjoint fibers are strongly separated, then Y is semi-T_2.*

Proof. Let y and z be two points in Y. Let U and V be open sets in X such that $f^{-1}(y) \in U$ and $f^{-1}(z) \in V$, respectively, with $\text{Cl}(U) \cap \text{Cl}(V) = \phi$. By contra pre-semi-$\theta$-openness of f (see Theorem 4.3) there are semi-θ-open sets F and B in Y such that $y \in F$ and $z \in B$, $f^{-1}(F) \subseteq \text{Cl}(U)$ and $f^{-1}(B) \subseteq \text{Cl}(V)$. Then $F \cap B = \phi$, since $\text{Cl}(U) \cap \text{Cl}(V) = \phi$ and f is a surjection. Therefore, Y is semi-T_2. □

Corollary 4.5. *Let X be a regular space. If $f : (X, \tau) \to (Y, \sigma)$ is contra pre-semi-θ-open surjection with all closed fibers and X is normal, then Y is semi-T_2.* □

Corollary 4.6. *Let X be a regular space. If $f : (X, \tau) \to (Y, \sigma)$ is continuous contra pre-semi-θ-open surjection with X compact T_2 space and Y a T_1 space, then Y is compact semi-T_2 space.*

Proof. Since f is a continuous surjection and Y is a T_1 space, Y is compact and all fibers are closed. Since X is normal, Y is also semi-T_2. □

Definition 4.7. A topological space X is said to be *quasi H-closed* [6] (resp. *N sθ-closed*), if every open (resp. semi-θ-closed) cover of X has a finite subfamily whose closures cover X. A subset A of a topological space X is *quasi H-closed relative to* X (resp. *N sθ-closed relative to* X) if every cover of A by open (resp. semi-θ-closed) sets of X has a finite subfamily whose closures cover A.

Lemma 4.8. *([16]). A function $f : (X, \tau) \to (Y, \sigma)$ is open if and only if for each $B \subseteq Y$, $f^{-1}(\text{Cl}(B)) \subseteq \text{Cl}(f^{-1}(B))$.*

Recall that a topological space (X,τ) is said to be *extremally disconnected* if the closure of each open set of X is open in X.

Theorem 4.9. *Let X be a regular extremally disconnected space and $S_\theta O(X,\tau)$ be closed under arbitrary intersections. Let $f:(X,\tau)\to(Y,\sigma)$ be an open one-to-one contra pre-semi-θ-open function such that $f^{-1}(y)$ is quasi H-closed relative to X for each y in Y. If G is $N\,s\theta$-closed relative to Y then $f^{-1}(G)$ is quasi H-closed relative to X.*

Proof. Let $\{V_\beta : \beta\in I\}$, I being the index set, be an open cover of $f^{-1}(G)$. Then for each $y\in G\cap f(X)$, $f^{-1}(y)\subseteq\cup\{\mathrm{Cl}(V_\beta):\beta\in I(y)\}=H_y$ for some finite subfamily $I(y)$ of I. Since X is extremally disconnected each $\mathrm{Cl}(V_\beta)$ is open, hence H_y is open in X. Let $U_y=f(\mathrm{Cl}(H_y))$. Then, the subset U_y is a semi-θ-closed set of Y. Indeed, the set $\mathrm{Cl}(H_y)$ is open, because (X,τ) is extremally disconnected. By Lemma 4.2(ii), $\mathrm{Cl}(H_y)$ is a semi-θ-open set. Since f is a contra pre-semi-θ-open, the subset U_y is a semi-θ-closed set of (Y,σ). It is shown that $y\in U_y$. Indeed, $y\in G\cap f(X)\subseteq f(H_y)\subseteq f(\mathrm{Cl}(H_y))=U_y$ and so $y\in U_y$. Since f is one-to-one, we have that $f^{-1}(U_y)=f^{-1}(f(\mathrm{Cl}(H_y)))=\mathrm{Cl}(H_y)$. Then, $\{U_y : y\in G\cap f(X)\}\cup\{\mathrm{Cl}([f(X)]^c)\}$ is a semi-θ-closed cover of G. By using an assumption of N-$s\theta$-closedness of G of relative to Y, $G\subseteq\cup\{\mathrm{Cl}(U_y):y\in K\}\cup\{\mathrm{Cl}([f(X)]^c)\}$ for some finite subset K of $G\cap f(X)$. By Lemma 4.8, it is shown that $f^{-1}(G)\subseteq\bigcup\{f^{-1}(\mathrm{Cl}(U_y)):y\in K\}\cup\{f^{-1}(\mathrm{Cl}([f(X)]^c))\}\subseteq\bigcup\{\mathrm{Cl}(f^{-1}(U_y)):y\in K\}=\bigcup\{\bigcup\{\mathrm{Cl}(V_\beta):\beta\in I(y)\}:y\in K\}$. Therefore, $f^{-1}(G)$ is quasi H-closed relative to X. □

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