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THE MAXIMUM NUMBER OF EDGES IN A STRONGLY MULTIPLICATIVE GRAPH

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ABSTRACT. We derive a formula for the maximum number of edges in a strongly multiplicative graph as a function of its order.

Recently, L.W. Beineke and S.M. Hegde [3] introduced the notion of a strongly multiplicative graph.

Definition (Beineke, Hegde [3]). A graph with n vertices is said to be *strongly multiplicative* if its vertices can be labeled $1, 2, ..., n$, so that the values on the edges, obtained as the product of labels of the end vertices, all are distinct.

An interesting problem is to obtain a formula for the maximum number of edges $\lambda(n)$ for a strongly multiplicative graph of order n. In [3], Beineke and Hegde gave an upper bound for $\lambda(n)$. In [2], C. Adiga et al. obtained a sharper upper bound for $\lambda(n)$. Then in [1], Adiga et al. established a formula for $\lambda(n)$ in terms of the divisor function. We quote their result now.

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Theorem (Adiga et al. [1]).

$$\lambda(n) = \sum_{k=1}^{n(n-1)} g(k),$$

where

$$g(k) = \min\{1, f(k)\},$$

$$f(k) = \begin{cases} [\frac{d(k)}{2}] & \text{if} \quad 1 \le k \le n, \\ [\frac{d(k)}{2}] - d_n(k) & \text{if} \quad n < k \le n(n-1) \end{cases}$$

and where $d(k)$ denotes the number of distinct divisors of k, $[x]$ denotes the largest integer less than or equal to x, and $d_n(k)$ denotes the number of divisors of k greater than n.

In this note we derive the following formula for $\lambda(n)$.

Theorem.

$$\lambda(n) = \frac{n(n-1)}{2} + \sum_{m=2}^{n} \sum_{k=1}^{m-1} [-\frac{\theta(m,k)}{[\sqrt{mk-1}] - k + 1}],$$

where

$$\theta(m,k) = \sum_{s=k+1}^{[\sqrt{mk-1}]} [\frac{[\frac{mk}{s}]}{\frac{mk}{s}}].$$

Proof. Let $\delta(n) = \lambda(n) - \lambda(n-1)$. Then

$$\lambda(n) = \sum_{m=2}^{n} \delta(m). \qquad (2.1)$$

Thus, in view of (2.1) it is enough to obtain a formula for $\delta(m)$. Consider the array of products

$$\begin{array}{cccccc}
1.2 & 1.3 & 1.4 & \dots & 1.(n-1) & 1.n \\
 & 2.3 & 2.4 & \dots & 2.(n-1) & 2.n \\
 & & 3.4 & \dots & 3.(n-1) & 3.n \\
 & & & & (n-2).(n-1) & (n-2).n \\
 & & & & & (n-1).n.
\end{array}$$

Let A_k denote the set of all elements of the k^{th} row. We count the number of terms in the last column which appear in other rows. If $k.n$ is divisible by s ($k \leq s-1$), then there exists an $m < n$ such that $k.n = s.m$, and hence $k.n$ repeats in the s^{th} row, i.e. $k.n \in A_s$. Observe that $k.n$ may belong to A_s, where s is the largest integer such that $k+1 < s < \sqrt{kn}$. Thus the number of repetition of $k.n$ in these rows is

$$\theta(n,k) = \sum_{s=k+1}^{[\sqrt{kn-1}]} [\frac{[\frac{nk}{s}]}{\frac{nk}{s}}].$$

By the definition of $\theta(n,k)$, it is clear that

$$0 \leq \theta(n,k) \leq [\sqrt{kn-1}] - k + 1.$$

Thus

$$[-\frac{\theta(n,k)}{[\sqrt{nk-1}]-k+1}] = \begin{cases} -1 & \text{if} \quad kn \in \bigcup_{s=k+1}^{\sqrt{kn-1}} A_s, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that

$$\delta(n) = (n-1) + \sum_{k=1}^{n-1} [-\frac{\theta(n,k)}{[\sqrt{nk-1}]-k+1}]. \qquad (2.2)$$

On using (2.2) in (2.1), we complete the proof. □

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