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SEPARATIVE IDEALS OF EXCHANGE RINGS

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Abstract. An ideal I of an exchange ring R is separative provided that for all $A, B \in FP(I), 2A \cong A \oplus B \cong 2B$ implies that $A \cong B$. We prove that I is separative if and only if so is the ideal of all (triangular) matrices over I. Further, we investigate diagonal reduction over such ideals. Comparability of modules over such ideals are studied as well.

1. Introduction

A ring R is said to be an exchange ring if for every right R-module A and any two decompositions $A = M \oplus N = \bigoplus_{i \in I} A_i$, where $M_R \cong R$ and the index set I is finite, there exist submodules $A_i' \subseteq A_i$ such that $A = M \oplus \big(\bigoplus_{i \in I} A_i'\big)$. The class of exchange rings is very large. It includes regular rings, π-regular rings, strongly π-regular rings, semiperfect rings, left or right continuous rings, clean rings, and unit C^*-algebras of real rank zero. For the general theory of exchange rings, refer to [10]. Following Ara et al. [3], an ideal I of an exchange ring R is separative provided that for all $A, B \in FP(I)$, $2A \cong A \oplus B \cong 2B \Rightarrow A \cong B$, where $FP(I)$ denotes the class of finitely generated projective right R-modules P such that $P = PI$. An exchange ring R is separative provided that R as an ideal of itself is separative. As is well known, an exchange ring R is separative if and only if so are I and R/I (cf. [10, Theorem 34.10]). Separativity plays a key role in the direct sum decomposition theory of

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exchange rings (cf. [2-3], [6] and [8-10]). We use $V(I)$ to stand for the monoid of isomorphism classes of objects from $FP(I)$. Applying [10, Lemma 34.5] to $V(I)$, one sees the following elementary result.

Theorem 1.1. *Let I be an ideal of an exchange ring R. Then, the followings are equivalent:*

(1) *I is separative.*
(2) *For all $A, B, C \in FP(I)$, $A \oplus 2C \cong B \oplus 2C \Rightarrow A \oplus C \cong B \oplus C$.*
(3) *For all $A, B, C \in FP(I)$, $A \oplus C \cong B \oplus C$ with $C \lesssim^{\oplus} A, B \Rightarrow A \cong B$.*
(4) *For all $A, B, C \in FP(I)$, $A \oplus C \cong B \oplus C$ with $C \propto A, B \Rightarrow A \cong B$.*
(5) *For all $A, B \in FP(I)$, $2A \cong 2B$ and $3A \cong 3B \Rightarrow A \cong B$.*
(6) *For all $A, B \in FP(I)$, $nA \cong nB$ and $(n+1)A \cong (n+1)B (n \in \mathbb{N}) \Rightarrow A \cong B$.*
(7) *For all $A, B, C \in FP(I)$, $A \oplus C \cong B \oplus C \lesssim^{\oplus} R$ with $C \lesssim^{\oplus} A, B \Longrightarrow A \cong B$.*

Here, we investigate new necessary and sufficient conditions under which an ideal of exchange rings is separative. For a regular ring R, we observe that the set $\{a \in R \mid End_R(aR) \text{ is separative}\}$ is a separative ideal. From this, we investigate diagonal reduction over such ideals. Furthermore, we show that such separativity can be characterized by comparability of modules.

Throughout, all rings are associative with identity and all modules are right modules. The notation $M \lesssim^{\oplus} N$ means that M is isomorphic to a direct summand of N. For any $A, B \in FP(I)$, we write $A \propto B$ if there exists a positive integer n such that $A \lesssim^{\oplus} nB$, where nB denotes the direct sum of n copies of a module B. We always use $\mathbb{N}$ to denote the set of all natural numbers.

2. Equivalent characterizations

The main purpose of this section is to give several equivalent characterizations for an ideal of exchange rings to be separative, which will be used in the sequel. We begin with a simple fact.

Lemma 2.1. *Let I be an ideal of an exchange ring R, and let $C \in FP(I)$. If A and B are any right R-modules such that $A \oplus C \cong B \oplus C$*

with $C \lesssim^{\oplus} A, B$*, then we have a refinement matrix,*

$$\begin{array}{c} \\ A \\ C \end{array} \begin{array}{c} \begin{array}{cc} B & C \end{array} \\ \left(\begin{array}{cc} D_1 & A_1 \\ B_1 & C_1 \end{array}\right) \end{array},$$

with $C_1 \lesssim^{\oplus} A_1, B_1$*.*

Proof. Suppose that $\psi : A \oplus C \cong B \oplus C$ with $C \lesssim^{\oplus} A, B$. Then, we have $A \oplus C = \psi^{-1}(B) \oplus \psi^{-1}(C)$. By [10, Proposition 28.6], C has the finite exchange property; hence, we have $B_1 \subseteq \psi^{-1}(B)$ and $C_1 \subseteq \psi^{-1}(C)$ such that $A \oplus C = A \oplus B_1 \oplus C_1$. So, $C \cong B_1 \oplus C_1$. It follows from $B_1 \subseteq \psi^{-1}(B) \subseteq B_1 \oplus A \oplus C_1$ that $\psi^{-1}(B) = \psi^{-1}(B) \cap (B_1 \oplus A \oplus C_1) = B_1 \oplus \psi^{-1}(B) \cap (A \oplus C_1)$. That is, B_1 is a direct summand of $\psi^{-1}(B)$. Likewise, C_1 is a direct summand of $\psi^{-1}(C)$. Assume now that $\psi^{-1}(B) = B_1 \oplus D_1$ and $\psi^{-1}(C) = C_1 \oplus A_1$. Then, $B \cong D_1 \oplus B_1, C \cong C_1 \oplus A_1$. As $B_1 \oplus D_1 \oplus C_1 \oplus A_1 = B_1 \oplus C_1 \oplus A$, we have $A \cong D_1 \oplus A_1$. Therefore, we get a refinement matrix,

$$\begin{array}{c} \\ A \\ C \end{array} \begin{array}{c} \begin{array}{cc} B & C \end{array} \\ \left(\begin{array}{cc} D_1 & A_1 \\ B_1 & C_1 \end{array}\right) \end{array},$$

as $C \lesssim^{\oplus} B$, $C_1 \lesssim^{\oplus} D_1 \oplus B_1$. Since C_1 as a direct summand of C, it has the finite exchange property. Similar to the consideration above, we have $C_1 = C_1' \oplus C_1''$ with $C_1' \lesssim^{\oplus} B_1$ and $C_1'' \lesssim^{\oplus} D_1$. Assume that $B_1 \cong C_1' \oplus B_1'$ and $D_1 \cong C_1'' \oplus D_1'$ for right R-modules B_1' and D_1'. Therefore, we get a refinement matrix,

$$\begin{array}{c} \\ A \\ C \end{array} \begin{array}{c} \begin{array}{cc} B & C \end{array} \\ \left(\begin{array}{cc} D_1' & A_1' \\ B_1' & C_1' \end{array}\right) \end{array},$$

where $A_1' = A_1 \oplus C_1''$ and $B_1' = B_1 \oplus C_1''$. Clearly, $C_1' \lesssim^{\oplus} B_1 \lesssim^{\oplus} B_1'$. Since $C_1' \lesssim^{\oplus} C \lesssim^{\oplus} A = A_1' \oplus D_1'$, analogous to the consideration above, we may also assume that $C_1' \lesssim^{\oplus} A_1'$. Therefore, we get the result. □

Theorem 2.2. *Let* I *be an ideal of an exchange ring* R*. Then, the followings are equivalent:*

(1) I *is separative.*
(2) *For all* $C \in FP(I)$*,* $A \oplus C \cong B \oplus C$ *with* $C \lesssim^{\oplus} A, B \Rightarrow A \cong B$*, for any right* R*-modules* A *and* B*.*

(3) *For all* $C \in FP(I)$, $A \oplus 2C \cong B \oplus 2C \Rightarrow A \oplus C \cong B \oplus C$, *for any right* R*-modules* A *and* B.
(4) *For all* $C \in FP(I)$, $A \oplus C \cong B \oplus C$ *with* $C \propto A, B \Rightarrow A \cong B$, *for any right* R*-modules* A *and* B.

Proof. (1) $\Rightarrow$ (2). Suppose that $A \oplus C \cong B \oplus C$ with $C \lesssim^{\oplus} A, B$ and $C \in FP(I)$. Clearly, C has the finite exchange property. Applying Lemma 2.1, we have a refinement matrix,

$$\begin{array}{c} \\ A \\ C \end{array} \begin{array}{c} \begin{array}{cc} B & C \end{array} \\ \begin{pmatrix} D_1 & A_1 \\ B_1 & C_1 \end{pmatrix} \end{array},$$

such that $C_1 \lesssim^{\oplus} A_1, B_1$. Thus, $C \cong A_1 \oplus C_1 \cong B_1 \oplus C_1$. Since $C \in FP(I)$, one easily checks that $C_1, A_1, B_1 \in FP(I)$. It follows from Theorem 1.1 that $A_1 \cong B_1$. Therefore, $A \cong D_1 \oplus A_1 \cong D_1 \oplus B_1 \cong B$, as desired.

(2) $\Rightarrow$ (3). This is clear.

(3) $\Rightarrow$ (4). Suppose that $C \in FP(I)$ and $A \oplus C \cong B \oplus C$ with $C \propto A, B$. Then, we have $k \in \mathbb{N}$ such that $C \lesssim^{\oplus} kA, kB$. By the finite exchange property of C, we have right R-module decomposition $C = C_1 \oplus \cdots \oplus C_k$ with all $C_i \lesssim^{\oplus} A$ $(1 \leq i \leq k)$. Clearly, all $C_i \lesssim^{\oplus} C \lesssim^{\oplus} kB$; hence, we have right R-module decompositions $C_i = C_{1i} \oplus \cdots \oplus C_{m_i i}$, for $1 \leq i \leq k$. Therefore, we get

$$\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m_i} C_{ji} \oplus A \cong \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m_i} C_{ji} \oplus B$$

with each $C_{ji} \lesssim^{\oplus} A, B$. Consequently, we have $A \cong B$, as required.

(4) $\Rightarrow$ (1). This is trivial by Theorem 1.1. □

Corollary 2.3. *Let* R *be an exchange ring. Then, the followings are equivalent:*

(1) R *is separative.*
(2) *For all* $C \in FP(R)$, $A \oplus C \cong B \oplus C$ *with* $C \lesssim^{\oplus} A, B \Rightarrow A \cong B$, *for any right* R*-modules* A *and* B.
(3) *For all* $C \in FP(R)$, $A \oplus 2C \cong B \oplus 2C \Rightarrow A \oplus C \cong B \oplus C$, *for any right* R*-modules* A *and* B.
(4) *For all* $C \in FP(R)$, $A \oplus C \cong B \oplus C$ *with* $C \propto A, B \Rightarrow A \cong B$, *for any right* R*-modules* A *and* B.

Proof. It is immediate from Theorem 2.2. □

Lemma 2.4. *Let I be a separative ideal of an exchange ring R, and let $e \in R$ be an idempotent. Then, eIe is a separative ideal of eRe.*

Proof. Given any idempotent $exe \in eIe$, we have $(exe)(eRe)(exe) = (exe)R(exe)$. Since $exe \in I$, by [10, Lemma 34.4], $(exe)R(exe)$ is a separative exchange ring. By [10, Lemma 34.4] again, we obtain the result. □

Theorem 2.5. *Let I be an ideal of an exchange ring R. Then, the followings are equivalent:*

(1) *I is separative.*
(2) *$M_n(I)$ is separative.*

Proof. (1) ⇒ (2). Suppose that $A \oplus B \cong A \oplus C$ with $A \lesssim^{\oplus} B, C$ and $A, B, C \in FP\big(M_n(I)\big)$. Then, $A \bigotimes\limits_{M_n(R)} R^{n\times 1} \oplus B \bigotimes\limits_{M_n(R)} R^{n\times 1} \cong A \bigotimes\limits_{M_n(R)} R^{n\times 1} \oplus C \bigotimes\limits_{M_n(R)} R^{n\times 1}$ with $A \bigotimes\limits_{M_n(R)} R^{n\times 1} \lesssim^{\oplus} B \bigotimes\limits_{M_n(R)} R^{n\times 1}, C \bigotimes\limits_{M_n(R)} R^{n\times 1}$. Clearly, $(A \bigotimes\limits_{M_n(R)} R^{n\times 1})I \subseteq A \bigotimes\limits_{M_n(R)} R^{n\times 1}$. Given any $\sum\limits_{i=1}^{m} a_i \bigotimes (x_{1i}, \cdots, x_{ni})^T \in A \bigotimes\limits_{M_n(R)} R^{n\times 1}$, we have $b_{ij} \in A, r_{ij} \in M_n(I)$ such that $\sum\limits_{i=1}^{m} a_i \bigotimes (x_{1i}, \cdots, x_{ni})^T = \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{k_i} (b_{ij} r_{ij}) \bigotimes (x_{1i}, \cdots, x_{ni})^T = \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{k_i} b_{ij} \bigotimes r_{ij}(x_{1i}, \cdots, x_{ni})^T$. Set $(c_1^{ij}, \cdots, c_n^{ij})^T = r_{ij}(x_{1i}, \cdots, x_{ni})^T$. Then, we see that $\sum\limits_{i=1}^{m} a_i \bigotimes (x_{1i}, \cdots, x_{ni})^T = \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{k_i} \sum\limits_{m=1}^{m_{ij}} \big(b_{ij} \bigotimes (0, \cdots, 1, \cdots, 0)^T\big) c_k^{ij} \subseteq (A \bigotimes\limits_{M_n(R)} R^{n\times 1})I$. That is, $A \bigotimes\limits_{M_n(R)} R^{n\times 1} \in FP(I)$. Likewise, $B \bigotimes\limits_{M_n(R)} R^{n\times 1}$, $C \bigotimes\limits_{M_n(R)} R^{n\times 1} \in FP(I)$. Since I is separative, we deduce that $B \bigotimes\limits_{M_n(R)} R^{n\times 1} \cong C \bigotimes\limits_{M_n(R)} R^{n\times 1}$. Therefore, we have $B \cong \big(B \bigotimes\limits_{M_n(R)} R^{n\times 1}\big) \bigotimes\limits_{R} R^{1\times n} \cong \big(C \bigotimes\limits_{M_n(R)} R^{n\times 1}\big) \bigotimes\limits_{R} R^{1\times n} \cong C$, as desired.

(2) $\Rightarrow$ (1). Choose $e = diag(1, 0, \cdots, 0) \in M_n(R)$. Then, $eM_n(I)e$ is a separative ideal of $eM_n(R)e$ from Lemma 2.4. Therefore, I is separative. $\square$

Corollary 2.6. *Let I be an ideal of an exchange ring R. Then, the followings are equivalent:*

(1) *I is separative.*
(2) *For all $P \in FP(I)$, $End_R(P)$ is separative.*

Proof. (1) $\Rightarrow$ (2). Since $P \in FP(I)$, by [10, Exercise 29.9], there exist idempotents $e_1, \cdots, e_n \in I$ such that $P \cong e_1R \oplus \cdots \oplus e_nR$. Hence,

$$End_R(P) \cong diag(e_1, \cdots, e_n)M_n(R)diag(e_1, \cdots, e_n).$$

In view of Theorem 2.4, $M_n(I)$ is separative. Thus, $End_R(P)$ is separative, by [10, Lemma 34.4].

(2) $\Rightarrow$ (1). Given any idempotent $e \in I$, one easily checks that $eR \in FP(I)$. Hence, $eRe \cong End_R(eR)$ is a separative ring. According to [10, Lemma 34.4], I is separative.

Recall that a rectangular matrix A admits diagonal reduction if there exist invertible P and Q such that PAQ is a diagonal matrix (cf. [2]). As in [10, Theorem 36.9], we can characterize separative ideals of exchange rings as follows.

Proposition 2.7. *Let I be an ideal of an exchange ring R. Then, the followings hold:*

(1) *I is a separative ideal.*
(2) *For all idempotents $e \in I$, every regular matrix in $M_2(eRe)$ admits a diagonal reduction.*

Proof. (1) $\Rightarrow$ (2). Let $e \in I$ be an idempotent. By virtue of [10, Lemma 34.4], eRe is a separative exchange ring. It follows from [2, Theorem 3.4] that every regular matrix in $M_2(eRe)$ admits a diagonal reduction.

(2) $\Rightarrow$ (1). Let $C \in FP(I)$. By [10, Exercise 29.9], there are idempotents $e_1, \cdots, e_n \in I$ such that $C \cong e_1R \oplus \cdots e_nR$. Suppose that $A \oplus C \cong B \oplus C$ with $C \lesssim^{\oplus} A, B$. Then, $e_1R \oplus (e_2R \oplus \cdots e_nR \oplus A) \cong e_1R \oplus (e_2R \oplus \cdots e_nR \oplus B)$. As $e_1R \lesssim^{\oplus} e_2R \oplus \cdots e_nR \oplus A, e_2R \oplus \cdots e_nR \oplus B$, we assume that $e_2R \oplus \cdots e_nR \oplus A \cong e_1R \oplus A'$ and $e_2R \oplus \cdots e_nR \oplus B \cong e_1R \oplus B'$. Then, $2(e_1R) \oplus A' \cong 2(e_2R) \oplus B'$. Clearly, e_1R has the finite

exchange property, and so does $2(e_1R)$. As in the proof of Lemma 2.1, we have a refinement matrix,

$$\begin{array}{cc} & \begin{array}{cc} A' & 2(e_1R) \end{array} \\ \begin{array}{c} B' \\ 2(e_1R) \end{array} & \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \end{array}.$$

Thus, we get $2(e_1R) \oplus C_{12} \cong (C_{21} \oplus C_{22}) \oplus C_{12} \cong C_{21} \oplus (C_{22} \oplus C_{12}) \cong 2(e_1R) \oplus C_{21}$. As a result, $2(e_1R) \bigotimes_R Re_1 \oplus C_{12} \bigotimes_R Re_1 \cong 2(e_1R) \bigotimes_R Re_1 \oplus C_{21} \bigotimes_R Re_1$. Since $e_1R \bigotimes_R Re_1 \cong e_1Re_1$, we have $2(e_1Re_1) \oplus C_{12} \bigotimes_R Re_1 \cong 2(e_1Re_1) \oplus C_{21} \bigotimes_R Re_1$. Clearly, e_1Re_1 is an exchange ring. According to [2, Proposition 3.3], $e_1Re_1 \oplus C_{12} \bigotimes_R Re_1 \cong e_1Re_1 \oplus C_{21} \bigotimes_R Re_1$; hence, $e_1Re_1 \bigotimes_{e_1Re_1} e_1R \oplus C_{12} \bigotimes_R Re_1 \bigotimes_{e_1Re_1} e_1R \cong e_1Re_1 \bigotimes_{e_1Re_1} e_1R \oplus C_{21} \bigotimes_R Re_1 \bigotimes_{e_1Re_1} e_1R$. Analogous to [2, Theorem 3.4], we get $e_1R \oplus C_{12} \cong e_1R \oplus C_{21}$. This proves that $e_1R \oplus A' \cong e_1R \oplus B'$; i.e., $e_2R \oplus \cdots e_nR \oplus A \cong e_2R \oplus \cdots e_nR \oplus B$. By repeating this process, we conclude that $A \cong B$. Therefore, I is a separative ideal, by Theorem 2.2. $\square$

3. Extensions

Let $P \in FP(R)$. We use $add_R(P)$ to denote the category whose objects are direct summands of finite copies of P.

Lemma 3.1. *Let R be a an exchange ring, $P \in FP(R)$, and $C \in add_R(P)$. If $End_R(P)$ is separative, then for any right R-modules A and B, $A \oplus C \cong B \oplus C$ with $C \lesssim^{\oplus} A, B$ implies that $A \cong B$.*

Proof. Given $A \oplus C \cong B \oplus C$ with $C \lesssim^{\oplus} A, B$, by hypothesis, $C \in FP(R)$. Hence, C has the finite exchange property. In view of Lemma 2.1, there exists a refinement matrix,

$$\begin{array}{cc} & \begin{array}{cc} A & C \end{array} \\ \begin{array}{c} B \\ C \end{array} & \begin{pmatrix} D_1 & B_1 \\ A_1 & C_1 \end{pmatrix} \end{array},$$

where $C_1 \lesssim^{\oplus} A_1, B_1$. Obviously, $C_1, A_1, B_1 \in add_R(C)$. According to [1, Lemma 12.3.19], there exist $\mathcal{F} : FP\big(End_R(C)\big) \to add_R(C)$ and

$\mathcal{G} : add_R(C) \to FP\big(End_R(C)\big)$ such that

$$\mathcal{F}\mathcal{G} = I_{add_R(C)} \text{ and } \mathcal{G}\mathcal{F} = I_{FP\left(End_R(C)\right)}.$$

Therefore, $\mathcal{G}(A_1)\oplus\mathcal{G}(C_1) \cong \mathcal{G}(B_1)\oplus\mathcal{G}(C_1)$ with $\mathcal{G}(C_1) \lesssim^{\oplus} \mathcal{G}(A_1), \mathcal{G}(B_1)$. As $End_R(C)$ is separative, we get $\mathcal{G}(A_1) \cong \mathcal{G}(B_1)$. Thus, $\mathcal{F}\mathcal{G}(A_1) \cong \mathcal{F}\mathcal{G}(B_1)$. By [1, Lemma 12.3.19] again, $A_1 \cong B_1$. Therefore, $A \cong D_1 \oplus A_1 \cong D_1 \oplus B_1 \cong B$, as required. □

Lemma 3.2. *Let R be an exchange ring, and let $x, y \in R$ be idempotents. If $End_R(xR)$ and $End_R(yR)$ are separative, then so is $End_R(xR\oplus yR)$.*

Proof. Suppose that $End_R(xR)$ and $End_R(yR)$ are separative. Given $A \oplus C \cong B \oplus C$ with $C \lesssim^{\oplus} A, B$, where $A, B, C \in FP\big(End_R(xR \oplus yR)\big)$, by [1, Lemma 12.3.19], there exist $\mathcal{F} : FP\big(End_R(xR \oplus yR)\big) \to add_R(xR \oplus yR)$ and $\mathcal{G} : add_R(xR \oplus yR) \to FP\big(End_R(xR \oplus yR)\big)$ such that

$$\mathcal{F}\mathcal{G} = I_{add_R(xR\oplus yR)} \text{ and } \mathcal{G}\mathcal{F} = I_{FP\left(End_R(xR\oplus yR)\right)}.$$

In addition, $\mathcal{F}$ and $\mathcal{G}$ preserve direct sums. Thus, $\mathcal{F}(A) \oplus \mathcal{F}(C) \cong \mathcal{F}(B) \oplus \mathcal{F}(C)$ with $\mathcal{F}(C) \lesssim^{\oplus} \mathcal{F}(A), \mathcal{F}(B)$. Clearly, $\mathcal{F}(C)$ has the finite exchange property. As in the proof of Lemma 2.1, we have some $C_1 \in add_R(xR), C_2 \in add_R(yR)$ such that $\mathcal{F}(C) = C_1 \oplus C_2$. Thus, $C_1 \oplus \big(C_2 \oplus \mathcal{F}(A)\big) \cong C_1 \oplus \big(C_2 \oplus \mathcal{F}(B)\big)$ with $C_1 \lesssim^{\oplus} C_2 \oplus \mathcal{F}(A), C_2 \oplus \mathcal{F}(B)$. In view of Lemma 3.1, $C_2 \oplus \mathcal{F}(A) \cong C_2 \oplus \mathcal{F}(B)$ with $C_2 \lesssim^{\oplus} \mathcal{F}(A), \mathcal{F}(B)$. By using Lemma 3.1 again, we get $\mathcal{F}(A) \cong \mathcal{F}(B)$. This implies that $\mathcal{G}\mathcal{F}(A) \cong \mathcal{G}\mathcal{F}(B)$. Therefore, $A \cong B$, and then we conclude that $End_R(xR \oplus yR)$ is a separative ring. □

A Morita context denoted by (A, B, M, N, ψ, ϕ) consists of two rings A, B, two bimodules ${}_AN_B, {}_BM_A$ and a pair of bimodule homomorphisms $\psi : N \bigotimes_B M \to A$ and $\phi : M \bigotimes_A N \to B$ satisfying the following conditions: $\psi(n \bigotimes m)n' = n\phi(m \bigotimes n'), \phi(m \bigotimes n)m' = m\psi(n \bigotimes m')$. These conditions insure that the set T of generalized matrices $\begin{pmatrix} a & n \\ m & b \end{pmatrix}$, $a \in A, b \in B, m \in M, n \in N$, forms a ring, called the ring of context.

Lemma 3.3. *Let T be the ring of a Morita context (A, B, M, N, ψ, ϕ). Then, T is separative if and only if so are A and B.*

Proof. Set $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in T$. Then, $A \cong eTe$ and $B \cong \big(\mathrm{diag}(1,1) - e\big)T\big(\mathrm{diag}(1,1) - e\big)$. Therefore, we get the result by [10, Lemma 34.4] and Lemma 3.2. $\square$

Let I be an ideal of an exchange ring R. Then, the set $LTM_n(I)$ of all $n \times n$ lower triangular matrices over I is an ideal of the exchange ring $LTM_n(R)$ of all $n \times n$ lower triangular matrices over R. Also, the set $UTM_n(I)$ of all $n \times n$ upper triangular matrices over I is an ideal of the exchange ring $UTM_n(R)$ of all $n \times n$ upper triangular matrices over R. Now, we extend Theorem 2.5 to triangular ideals.

Theorem 3.4. *Let I be an ideal of an exchange ring R. Then, the followings are equivalent:*

(1) *I is separative.*
(2) *$LTM_n(I)$ is separative.*
(3) *$UTM_n(I)$ is separative.*

Proof. (1) $\Rightarrow$ (2). It suffices to assume that $n = 2$. Let $\begin{pmatrix} e & 0 \\ * & f \end{pmatrix} \in LTM_2(I)$ be an idempotent. Then, $e, f \in I$ are idempotents. In view of [10, Lemma 34.4], eRe and fRf are both separative exchange rings. According to Lemma 3.3, $\begin{pmatrix} e & 0 \\ * & f \end{pmatrix} LTM_2(R) \begin{pmatrix} e & 0 \\ * & f \end{pmatrix}$ is a separative exchange ring. We infer that $LTM_2(I)$ is an exchange ideal of $LTM_2(R)$.

(2) $\Rightarrow$ (1). Choose $g = diag(1, 0, \cdots, 0)_n$. It follows from Lemma 2.4 that $gLTM_n(I)g$ is a separative ideal of $gLTM_n(R)g$; i.e., I is a separative ideal of R.

(1) $\Leftrightarrow$ (3). These are proved in the sam manner. $\square$

Lemma 3.5. *Let R be a regular ring. Then,*

$$\{a \in R \mid End_R(aR) \text{ is separative}\}$$

is a separative ideal of R.

Proof. Let $I = \{a \in R \mid End_R(aR) \text{ is separative}\}$. Let $x, y \in I$ and $z \in R$. Construct a map $\varphi : xR \to zxR$ given by $\varphi(xr) = zxr$, for any $r \in R$. Then, φ is a splitting R-epimorphism; hence, $zxR \oplus D \cong xR$, for some right R-module D. This implies that $zxR \lesssim^{\oplus} xR$. Write

$xR = eR, xzR = fR$, for some idempotents $e, f \in R$. It is easy to verify that $fR \oplus (1-f)eR = eR$, and so $xzR \subseteq^{\oplus} xR$. According to [10, Lemma 34.4], $End_R(xz)$ and $End_R(zx)$ are separative. Thus, $xz, zx \in I$. Write $(x + y)R = gR$ and $xR + yR = hR$, for some idempotents $g, h \in R$. Then, $gR \oplus (h - gh)R = hR$, and so $(x + y) \subseteq^{\oplus} xR + yR$. As R is regular, we have a splitting exact sequence,

$$0 \to xR \bigcap yR \to xR \oplus yR \to xR + yR \to 0,$$

and so $(xR+yR) \oplus (xR \bigcap yR) \cong xR \oplus yR$. This implies that $(x+y)R \lesssim^{\oplus} xR \oplus yR$. In view of Lemma 3.2, $End_R(xR \oplus yR)$ is separative, and so is $End_R\big((x + y)R\big)$. Therefore, $x + y \in I$. Consequently, I is an ideal of R. For any idempotent $e \in I$, eRe is separative. According to [10, Lemma 34.4], I is a separative ideal of R. □

Theorem 3.6. *Let R be a regular ring, and let $(a_{ij}) \in M_n(R)$. If each $End_R(a_{ij}R)$ are separative, then (a_{ij}) admits a diagonal reduction.*

Proof. Let $I = \{a \in R \mid End_R(aR) \text{ is separative}\}$. In view of Lemma 3.5, I is a separative ideal. Since each $End_R(a_{ij}R)$ is separative, we see that each $a_{ij} \in I$. As is well known, there exists an idempotent $e \in I$ such that all $a_{ij} \in eRe$. As $eRe \cong End_R(eR)$, eRe is separative. According to [10, Theorem 37.1], $(a_{ij}) \in M_n(eRe)$ admits a diagonal reduction; i.e., there exist some $U', V' \in GL_n(eRe)$ such that $U'AV' = diag(r_1, \cdots, r_n)$, for $r_1, \cdots, r_n \in eRe$. Let $E = diag(e, \cdots, e) \in M_n(R)$. Then, $U := U' + I_n - E, V = V' + I_n - E \in GL_n(R)$. Furthermore, $UAV = diag(r_1, \cdots, r_n)$, as asserted. □

4. Comparability of modules

In [8, Theorem 3.9], Pardo observed that every exchange rings satisfying general comparability is separative. The aim of this section is to investigate comparability of modules over separative ideals in a general case.

Lemma 4.1. *Let I be an ideal of an exchange ring R. Then, the followings are equivalent:*

(1) *I is separative.*
(2) *For any $A, B, C \in FP(I)$, $A \oplus C \cong B \oplus C$ with $C \lesssim^{\oplus} A, B \Rightarrow Ae \lesssim^{\oplus} Be$ and $B(1 - e) \lesssim^{\oplus} A(1 - e)$, for some $e \in B(R)$.*

(3) *For any* $A, B \in FP(I)$, $2A \cong A \oplus B \cong 2B \Rightarrow Ae \lesssim^{\oplus} Be$ *and* $B(1-e) \lesssim^{\oplus} A(1-e)$, *for some* $e \in B(R)$.
(4) *For any* $A, B \in FP(I)$, $2A \cong 2B$ *and* $3A \cong 3B \Rightarrow Ae \lesssim^{\oplus} Be$ *and* $B(1-e) \lesssim^{\oplus} A(1-e)$, *for some* $e \in B(R)$.

Proof. (1) $\Rightarrow$ (4). This is trivial using Theorem 1.1.

(4) $\Rightarrow$ (3). Given any $A, B \in FP(I)$ with $2A \cong A \oplus B \cong 2B$, we have $2A \cong 2B$ and $3A \cong 3B$, as desired.

(3) $\Rightarrow$ (2). This is obvious.

(2) $\Rightarrow$ (1). Suppose that $A \oplus C \cong B \oplus C$ with $C \lesssim^{\oplus} A, B$ for $A, B, C \in FP(I)$. Applying Lemma 2.1, we have a refinement matrix,

$$\begin{array}{c} \\ A \\ C \end{array} \begin{array}{c} \begin{array}{cc} B & C \end{array} \\ \left(\begin{array}{cc} D_1 & A_1 \\ B_1 & C_1 \end{array} \right) \end{array},$$

such that $C_1 \lesssim^{\oplus} A_1, B_1$. Since $A_1 \oplus C_1 \cong C \cong B_1 \oplus C_1$, we can find some $e \in B(R)$ such that $A_1e \lesssim^{\oplus} B_1e$ and $B_1(1-e) \lesssim^{\oplus} A_1(1-e)$. As $A_1e \lesssim^{\oplus} B_1e$, we have $B_1e \cong A_1e \oplus D$, for a right R-module D. We easily check that $Ce \cong C_1e \oplus B_1e \cong C_1e \oplus A_1e \oplus D \cong Ce \oplus D$. It follows that $Ae \cong Ae \oplus D$, because $C \lesssim^{\oplus} A$. Therefore, $Ae \cong Ae \oplus D \cong D_1e \oplus A_1e \oplus D \cong D_1e \oplus B_1e \cong Be$.

On the other hand, $B_1(1-e) \lesssim^{\oplus} A_1(1-e)$. Then, $A_1(1-e) \cong B_1(1-e) \oplus E$, for a right R-module E. So, $C(1-e) \cong C_1(1-e) \oplus A_1(1-e) \cong C_1(1-e) \oplus B_1(1-e) \oplus E \cong C(1-e) \oplus E$. It follows from $C \lesssim^{\oplus} B$ that $B(1-e) \cong B(1-e) \oplus E$. Consequently, $B(1-e) \cong B(1-e) \oplus E \cong D_1(1-e) \oplus B_1(1-e) \oplus E \cong D_1(1-e) \oplus A_1(1-e) \cong A(1-e)$. Hence, $A \cong Ae \oplus A(1-e) \cong Be \oplus B(1-e) \cong B$. Therefore, I is separative by Theorem 1.1. $\square$

Theorem 4.2. *Let* I *be an ideal of an exchange ring* R. *Then, the followings are equivalent:*

(1) I *is separative.*
(2) *For all* $C \in FP(I)$, $A \oplus C \cong B \oplus C$ *with* $C \lesssim^{\oplus} A, B \Rightarrow Ae \lesssim^{\oplus} Be$ *and* $B(1-e) \lesssim^{\oplus} A(1-e)$, *for some* $e \in B(R)$.
(3) *For all* $C \in FP(I)$, $A \oplus 2C \cong B \oplus 2C \Rightarrow (A \oplus C)e \lesssim^{\oplus} (B \oplus C)e$ *and* $(B \oplus C)(1-e) \lesssim^{\oplus} (A \oplus C)(1-e)$, *for some* $e \in B(R)$.
(4) *For all* $C \in FP(I)$, $A \oplus C \cong B \oplus C$ *with* $C \propto A, B \Rightarrow Ae \lesssim^{\oplus} Be$ *and* $B(1-e) \lesssim^{\oplus} C)(1-e)$, *for some* $e \in B(R)$.

Proof. As in the proof of Theorem 2.2, we obtain the proof by Theorem 1.1 and Lemma 4.1.

Corollary 4.3. *Let I be an ideal of an exchange ring R. Then, the followings are equivalent:*

(1) *I is separative.*
(2) *For all $C \in FP(I)$, $A \oplus C \cong B \oplus C \lesssim^{\oplus} R$ with $C \lesssim^{\oplus} A, B \Rightarrow Ae \lesssim^{\oplus} Be$ and $B(1-e) \lesssim^{\oplus} A(1-e)$, for some $e \in B(R)$.*

Proof. (1) $\Rightarrow$ (2). This is obvious using Theorem 4.2.

(2) $\Rightarrow$ (1). Suppose that $A \oplus C \cong B \oplus C \lesssim^{\oplus} R$ and $C \lesssim^{\oplus} A, B$, where $A, B, C \in FP(I)$. In view of Lemma 2.1, we have a refinement matrix,

$$\begin{array}{cc} & \begin{array}{cc} B & C \end{array} \\ \begin{array}{c} A \\ C \end{array} & \left(\begin{array}{cc} D_1 & A_1 \\ B_1 & C_1 \end{array} \right) \end{array},$$

with $C_1 \lesssim^{\oplus} A_1, B_1$. Clearly, $A_1 \oplus C_1 \cong A_2 \oplus C_1 \lesssim^{\oplus} R$. By hypothesis, we can find some $e \in B(R)$ such that $A_1e \lesssim^{\oplus} B_1e$ and $B_1(1-e) \lesssim^{\oplus} A_1(1-e)$. As in the proof of Lemma 4.1, we get $A \cong B$, and therefore the proof is complete by Theorem 1.1. □

Lemma 4.4. *Let I be an ideal of a regular ring R. Then, the followings are equivalent:*

(1) *I is separative.*
(2) *For any $a \in 1 + I$, $(a - a^2)R \lesssim^{\oplus} r(a), R/aR$ implies that there exists $e \in B(R)$ such that $r(a)e \lesssim^{\oplus} (R/aR)e$ and $(R/aR)(1-e) \lesssim^{\oplus} r(a)(1-e)$.*

Proof. (1) $\Rightarrow$ (2). Suppose that $a(1-a)R \lesssim^{\oplus} r(a), R/aR$ with $a \in 1+I$. Then, we can find a right R-module D such that $R = r(a) \oplus r(1-a) \oplus D$. So, $aR = ar(1-a) \oplus aD = r(1-a) \oplus aD$. As a result, $r(a) \oplus D \cong R/r(1-a) \cong R/aR \oplus aD$. Clearly, $D \cong aD \cong a(1-a)D$. This implies that $D \cong a(1-a)R$. Thus, we have $r(a) \oplus a(1-a)R \cong R/aR \oplus a(1-a)R$. Since $a \in 1+I$, we see that $a(1-a)R \in FP(I)$. In view of Theorem 4.2, we can find $e \in B(R)$ such that $r(a)e \lesssim^{\oplus} (R/aR)e$ and $(R/aR)(1-e) \lesssim^{\oplus} r(a)(1-e)$.

(2) $\Rightarrow$ (1). Given $A \oplus C \cong B \oplus C \lesssim^{\oplus} R$ and $C \lesssim^{\oplus} A, B$ with $A, B, C \in FP(I)$, we write $R = A_1 \oplus C_1 \oplus D = A_2 \oplus C_2 \oplus D$, where $A_1 \cong A$, $C_1 \cong C_2 \cong C$ and $A_2 \cong B$. Let $a \in R$ induce an endomorphism

of R_R, which is zero on A_1, an isomorphism from C_1 onto C_2, and the identity on D. One checks that $a(1-a)R \cong a(1-a)C_1 \lesssim^{\oplus} C_1 \lesssim^{\oplus} A_1 = r(a)$. In addition, we have $a(1-a)R \cong a(1-a)C_1 \lesssim^{\oplus} C_1 \cong C_2 \lesssim^{\oplus} A_2 \cong 2(R/aR)$. Thus, $a(1-a)R \propto r(a), R/aR$. As $(1-a)R \cong (1-a)\big(A_1 \oplus C_1\big)$, we see that $(1-a)R \bigotimes_R R/I \cong (1-a)\big(A_1 \oplus C_1\big) \bigotimes_R R/I = 0$, we deduce that $(1-a)R = (1-a)RI$, and then $a \in 1+I$. By hypothesis, there is $e \in B(R)$ such that $Ae \cong r(a)e \lesssim^{\oplus} (R/aR)e \cong Be$ and $B(1-e) \lesssim^{\oplus} A(1-e)$. In view of Corollary 4.3, I is separative. □

Theorem 4.5. *Let I be an ideal of a regular ring R. Then, the followings are equivalent:*

(1) *I is separative.*
(2) *For any $a \in 1+I$, $r(a) \oplus r(a) \cong r(a) \oplus R/aR \cong R/aR \oplus R/aR$ implies that there exists $e \in B(R)$ such that $r(a)e \lesssim^{\oplus} (R/aR)e$ and $(R/aR)(1-e) \lesssim^{\oplus} r(a)(1-e)$.*

Proof. (1) $\Rightarrow$ (2). For any $a \in 1+I$, $r(a), R/aR \in FP(I)$. It follows from Theorem 1.1 that there exists $e \in B(R)$ such that $r(a)e \lesssim^{\oplus} (R/aR)e$ and $(R/aR)(1-e) \lesssim^{\oplus} r(a)(1-e)$.

(2) $\Rightarrow$ (1). Suppose that $a \in 1+I$ and $(a-a^2)R \lesssim^{\oplus} r(a), R/aR$. Then, $r(a) \cong (a-a^2)R \oplus D$. As in the proof of Lemma 4.4, $r(a) \oplus a(1-a)R \cong R/aR \oplus a(1-a)R$. Hence, $r(a) \oplus r(a) \cong r(a) \oplus (a-a^2)R \oplus D \cong R/aR \oplus (a-a^2)R \oplus D \cong R/aR \oplus r(a)$. Likewise, $R/aR \oplus R/aR \cong R/aR \oplus r(a)$. Thus, $r(a) \oplus r(a) \cong r(a) \oplus R/aR \cong R/aR \oplus R/aR$. By hypothesis, there exists $e \in B(R)$ such that $r(a)e \lesssim^{\oplus} (R/aR)e$ and $(R/aR)(1-e) \lesssim^{\oplus} r(a)(1-e)$. According to Lemma 4.4, the proof is complete. □

Corollary 4.6. *Let I be an ideal of a regular ring R. Then, the followings are equivalent:*

(1) *I is separative.*
(2) *For any idempotents $e, f \in I$, $eR \oplus eR \cong eR \oplus fR \cong fR \oplus fR$ implies that there exists $u \in B(R)$ such that $ueR \lesssim^{\oplus} ufR$ and $(1-u)fR \lesssim^{\oplus} (1-u)eR$.*

Proof. (1) $\Rightarrow$ (2). For any idempotents $e, f \in I$, $eR, fR \in FP(I)$. Thus, there exists $u \in B(R)$ such that $ueR \lesssim^{\oplus} ufR$ and $(1-u)fR \lesssim^{\oplus} (1-u)eR$, by Theorem 1.1.

(2) $\Rightarrow$ (1). Suppose that $r(a) \oplus r(a) \cong r(a) \oplus R/aR \cong R/aR \oplus R/aR$, where $a \in 1 + I$. Write $a = axa$. Then, $r(a) = (1 - xa)R$ and $R/aR \cong (1 - ax)R$. Clearly, $x \in 1 + I$; hence, $1 - ax, 1 - xa \in I$ are both idempotents. By hypothesis, $(1 - ax)Re \lesssim^{\oplus} (1 - xa)Re$ and $(1 - xa)R(1 - e) \lesssim^{\oplus} (1 - ax)R(1 - e)$. According to Theorem 4.5, I is separative. □

Let I be an ideal of an exchange ring R. We say that I satisfies general comparability if for any regular $x, y \in I$, there exists $u \in B(R)$ such that $uxR \lesssim^{\oplus} uyR$ and $(1 - u)yR \lesssim^{\oplus} (1 - u)xR$. As is well known, every injective ideal of regular rings satisfies general comparability. Now, we extend [7, Proposition 8.8] to the ideals of exchange rings by means of a similar argument.

Lemma 4.7. *Let I be an ideal of an exchange ring R. Then, the followings are equivalent:*

(1) *I satisfies general comparability.*
(2) *For any $A, B \in FP(I)$, there exists $e \in B(R)$ such that $Ae \lesssim^{\oplus} Be$ and $B(1 - e) \lesssim^{\oplus} A(1 - e)$.*

Proof. (2) $\Rightarrow$ (1). Let $x, y \in I$ be regular. Then, $xR, yR \in FP(I)$. So, we have $e \in B(R)$ such that $e(xR) \lesssim^{\oplus} e(yR)$ and $(1 - e)(yR) \lesssim^{\oplus} (1 - e)(xR)$.

(1) $\Rightarrow$ (2). Let $A, B \in FP(I)$. Since R is an exchange ring, there exist idempotents $e'_1, \cdots, e'_n, e''_1, \cdots, e''_n \in I$ such that $A = e'_1R \oplus \cdots \oplus e'_nR$ and $B = e''_1R \oplus \cdots \oplus e''_nR$.

If $n = 1$, then the result follows. Assume that the result holds for $n-1$ ($n \geq 2$). Clearly, we have decompositions $A = A_1 \oplus A_2, B = B_1 \oplus B_2$ with $A_1, B_1 \lesssim^{\oplus} (n - 1)R, A_2, B_2 \lesssim^{\oplus} R$, where $A_1, A_2, B_1, B_2 \in FP(I)$. Hence, there exist $f_1, f_2 \in B(R)$ such that $A_1f_1 \lesssim^{\oplus} B_1f_1, B_1(1-f_1) \lesssim^{\oplus} A_1(1-f_1), A_2f_2 \lesssim^{\oplus} B_2f_2, B_2(1-f_2) \lesssim^{\oplus} A_2(1-f_2)$. Set $e_1 = f_1f_2, e_2 = (1 - f_1)(1 - f_2)$. It is easy to verify that $Ae_1 \lesssim^{\oplus} Be_1, Be_2 \lesssim^{\oplus} Ae_2$.

Set $g_1 = f_1(1-f_2)$ and $g_2 = f_2(1-f_1)$. We have $A_1g_1 \lesssim^{\oplus} B_1g_1, A_2g_2 \lesssim^{\oplus} B_2g_2, B_1g_2 \lesssim^{\oplus} A_1g_2$ and $B_2g_1 \lesssim^{\oplus} A_2g_1$. So $B_1g_1 \cong A_1g_1 \oplus D_1, B_2g_2 \cong A_2g_2 \oplus D_2, A_1g_2 \cong B_1g_2 \oplus C_1$ and $A_2g_1 \cong B_2g_1 \oplus C_2$, for some right R-modules C_1, C_2, D_1 and D_2. Clearly, $C_1 \oplus C_2, D_1 \oplus D_2 \lesssim^{\oplus} (n - 1)R$. In addition, $C_1 \oplus C_2, D_1 \oplus D_2 \in FP(I)$. Hence, there is $h \in B(R)$ such that $(C_1 \oplus C_2)h \lesssim^{\oplus} (D_1 \oplus D_2)h$ and $(D_1 \oplus D_2)(1 - h) \lesssim^{\oplus} (C_1 \oplus C_2)(1 - h)$.

Set $e_3 = gh, e_4 = g(1-h)$. Then, we see that

$$\begin{array}{rcl} Ae_3 & = & A_1g_1h \oplus A_1g_2h \oplus A_2g_1h \oplus A_2g_2h \\ & = & A_1g_1h \oplus B_1g_2h \oplus C_1h \oplus B_2g_1h \oplus C_2h \oplus A_2g_2h \\ & \lesssim^{\oplus} & B_1g_1h \oplus B_1g_2h \oplus B_2g_1h \oplus B_2g_2h \oplus (D_1 \oplus D_2)h \\ & \lesssim^{\oplus} & (B_1g_1 \oplus B_2g_2 \oplus B_1g_2 \oplus B_2g_1)h \\ & \cong & Be_3. \end{array}$$

Analogously, we have

$$\begin{array}{rcl} Be_4 & = & B_1g_1(1-h) \oplus B_1g_2(1-h) \oplus B_2g_1(1-h) \oplus B_2g_2(1-h) \\ & = & (A_1g_1 \oplus D_1 \oplus A_2g_2 \oplus D_2 \oplus B_1g_2 \oplus B_2g_1)(1-h) \\ & \lesssim^{\oplus} & (A_1g_1 \oplus B_1g_2 \oplus A_2g_2 \oplus B_2g_1 \oplus C_1 \oplus C_2)(1-h) \\ & \cong & (A_1g_1 \oplus A_1g_2 \oplus A_2g_2 \oplus A_2g_1)(1-h) \\ & \cong & Ae_5. \end{array}$$

Set $e = e_1 + e_3$. Then, $e \in B(R)$ with $1 - e = e_2 + e_4$. We conclude that $Ae \lesssim^{\oplus} Be$ and $B(1-e) \lesssim^{\oplus} A(1-e)$. By induction, the proof is complete. □

Proposition 4.8. *Let I be an ideal of an exchange ring R. If I satisfies general comparability, then it is separative.*

Proof. Suppose that $A, B \in FP(I)$ and $2A \cong A \oplus B \cong 2B$. Since I satisfies general comparability, by Lemma 4.7, we have $Ae \lesssim^{\oplus} Be$ and $B(1-e) \lesssim^{\oplus} A(1-e)$, for some $e \in B(R)$. Therefore, we get the result from Lemma 4.1. □

Example 4.9. *Let V be an infinite-dimensional vector space over a division ring D, and let R be a subring of $End_D(V)$ which contains $I = \{x \in End_D(V) \mid dim_D(xV) < \infty\}$. Then, I is a separative ideal of R.*

Proof. Let $S = End_D(V)$. Clearly, I is an ideal of S. Given any idempotents $x, y \in I$, we have $xS \lesssim^{\oplus} yS$ or $yS \lesssim^{\oplus} xS$, because S is a regular ring satisfying the comparability axiom. Observing that $xR = xI = xS$ and $yS = yI = yR$, we have either $xR \lesssim^{\oplus} yR$ or $yR \lesssim^{\oplus} xR$. Thus, I as an ideal of R satisfies general comparability. According to Proposition 4.8, I is separative of R, as asserted. □

References

[1] F. Anderson and K. Fuller, *Rings and Categories of Modules*, Springer, Berlin, 1973.

[2] P. Ara, K. R. Goodearl, K. C. O'Meara and E. Pardo, Diagonalization of matrices over regular rings, *Linear Algebra Appl.* **265** (1997) 147–163.

[3] P. Ara, K. R. Goodearl, K. C. O'Meara and E. Pardo, Separative cancellation for projective modules over exchange rings, *Israel J. Math.* **105** (1998) 105–137.

[4] H. Chen, Related comparability over exchange rings, *Comm. Algebra* **27** (1999) 4209–4216.

[5] H. Chen, Separativity of regular rings in which 2 is invertible, *Comm. Algebra* **35** (2007) 1661–1673.

[6] H. Chen, Cancellation of small projectives over exchange rings, *Comm. Algebra* **37** (2009) 2145–2158.

[7] K. R. Goodearl, *Von Neumann Regular Rings*, Pitman, London, San Francisco, Melbourne, 1979; second ed., Krieger, Malabar, Fl., 1991.

[8] E. Pardo, Comparability, separativity, and exchange rings, *Comm. Algebra* **24** (1996) 2915–2929.

[9] F. Perera, Lifting units modulo exchange ideals and C^*-algebras with real rank zero, *J. Reine. Math.* **522** (2000) 51–62.

[10] A. A. Tuganbaev, *Rings Close to Regular*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

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