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ITERATIVE METHOD FOR MIRROR-SYMMETRIC SOLUTION OF MATRIX EQUATION $AXB + CYD = E$

J.-F. LI*, X.-Y. HU, X.-F. DUAN AND L. ZHANG

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Abstract. Mirror-symmetric matrices have important applications in studying odd/even-mode decomposition of symmetric multiconductor transmission lines (MTL). In this paper, we propose an iterative algorithm to solve the mirror-symmetric solution of matrix equation $AXB + CYD = E$. With it, the solvability of the equation over mirror-symmetric X, Y can be determined automatically. When the equation is consistent, its solution can be obtained within finite iteration steps, and its least-norm mirror-symmetric solution can be obtained by choosing a special kind of initial iteration matrices. Furthermore, the related optimal approximation problem is also solved. Numerical examples are given to show the efficiency of the presented method.

1. Introduction

Real or complex matrices of rather high order are commonly encountered in real physical systems analysis. Usually, a physical system possesses certain geometrical symmetry. Mirror symmetry is the

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*Corresponding author

most common one. Interaction matrices of mirror-symmetric structures are centrosymmetric while one component or only one component is on the mirror plane. Centrosymmetric matrices play an important role in a number of areas such as pattern recognition, antenna theory, mechanical and electrical systems, quantum physics, and electrical packaging analysis [1]. While there is more than one component on the mirror plane, the mirror-symmetric matrix, which is originally proposed by Li and Feng [2], is the one and the only one that can reflect the structure properly. The background for introducing the concept of mirror-symmetric matrix is to study odd/even-mode decomposition of symmetric multiconductor transmission lines (MTL) [2, 3, 4].

Definition 1.1. An (r, p)-mirror matrix $W_{(r,p)}$ is defined by

$$W_{(r,p)} = \begin{pmatrix} & & J_r \\ & I_p & \\ J_r & & \end{pmatrix},$$

where I_p is the p-square identity matrix and J_r is the r-square backward identity matrix with ones along the secondary diagonal and zero elsewhere.

The dimension of the (r, p)-mirror matrix is $n = 2r + p$, where $r \geq 1, p \geq 0$. The (r, p)-mirror matrix $W_{(r,p)}$ is orthogonal and symmetric, i.e., $W^{-1} = W^T = W$. When $p = 0$ or 1, mirror matrix $W_{(r,p)}$ is backward identity matrix J_n.

Definition 1.2. A matrix $M \in R^{(2r+p)\times(2r+p)}$ is called the (r, p)-mirror symmetric matrix if and only if

$$M = W_{(r,p)} M W_{(r,p)}. \tag{1.1}$$

We denote the set of all (r, p)-mirror-symmetric matrices by $MS_{(r,p)}$.

From Definition 1.2, it is easy to see that the $(k, 1)$-mirror-symmetric matrices and the $(k, 0)$-mirrorsymmetric matrices are centrosymmetric matrices. That is to say, all centrosymmetric matrices are the special cases of mirror-symmetric matrices; i.e., when $p = 0$ or 1, mirror matrix $W_{(r,p)}$ is backward identity matrix J_n. Then, (1.1) becomes $M = J_n M J_n$, which is the definition of centrosymmetric matrices [5].

Throughout this paper, we denote the sets of all $m \times n$ real matrices by $R^{m\times n}$. For a matrix $A \in R^{m\times n}$, we denote its transpose by A^T. On $R^{m\times n}$ we define inner product: $\langle A, B\rangle = trace(B^T A)$, for all $A, B \in R^{m\times n}$. Then, $R^{m\times n}$ is a Hilbert inner product space and the norm of a matrix generate by this inner product is the Frobenious norm, denoted by $\|.\|$.

We study the following problems.

Problem I. For given matrices $A \in R^{s\times(2r+p)}$, $B \in R^{(2r+p)\times t}$, $C \in R^{s\times(2h+q)}$, $D \in R^{(2h+q)\times t}$, and $E \in R^{s\times t}$, find $X \in MS_{(r,p)}$ and $Y \in MS_{(h,q)}$ such that

$$AXB + CYD = E. \tag{1.2}$$

Problem II. When Problem I is consistent, let S_E be the set of mirror-symmetric solution of Problem I. For given matrices $\bar{X} \in MS_{(r,p)}$ and $\bar{Y} \in MS_{(h,q)}$, find $[\tilde{X}, \tilde{Y}] \in S_E$ such that

$$\|\tilde{X} - \bar{X}\|^2 + \|\tilde{Y} - \bar{Y}\|^2 = \min_{[X,Y]\in S_E} [\|X - \bar{X}\|^2 + \|Y - \bar{Y}\|^2]. \tag{1.3}$$

In fact, Problem II is to find the optimal approximation solutions to the given matrix pair $[X_0, Y_0]$. Problem II occurs frequently in experimental design. About Problem II, we refer the reader to references [13, 14, 16].

The well-known matrix equation (1.2) with arbitrary coefficient matrices A, B, C, D and the right-hand side E, has been studied actively for the past 40 or more years. For instance, Baksalary and Kala [6] presented a condition for the existence of a solution and derived a formula for the general solution of the matrix equation (1.2). Chu [7] gave the consistency conditions and the minimum-norm solution by making use of the generalized singular value decomposition (GSVD). Huang and Zeng [8] and Özgüler [9], respectively, gave the solvability conditions over a simple Artinian ring and a principal ideal domain by using the generalized inverse. Xu, et al.[10] gave the least-squares solution of the matrix equation (1.2) by making use of the canonical correlation decomposition (CDD). Shim and Chen [11] presented the least-squares solution with the minimum norm by using the singular value decomposition (SVD) and the GSVD. Liao, et al. [12] studied the best approximate solution of matrix equation (1.2) basing on the projection theorem and using the GSVD and the CCD simultaneously. In addition, Peng [13] presented an efficient iterative method for solving the matrix equation (1.2) by making use of the idea of the classical CG method. The methods cited in the above papers can find the solution X, Y of matrix equation in

$R^{n\times n}$. To our knowledge, however, no result exists about some certain structured solutions, such as mirror-symmetric solution, of the matrix equation (1.2). This paper represents a modest attempt to address this situation.

In this paper, using the idea of the classical conjugate gradient method, we present an iterative algorithm to solve Problem I. With it, the solvability of the systems of matrix equations can be determined automatically. When the matrix equation is consistent, for any initial mirror-symmetric matrix pair $[X_0, Y_0]$, we will show that the mirror-symmetric solution and the least norm mirror-symmetric solution of Problem I can be obtained within finite iteration steps in the absence of roundoff errors, (this will be done in Section 2). It is only required to compute a residual matrix and update the iterative solution and gradient matrices *linearly* in each iteration. In Section 3, the unique optimal approximation solution pair of Problem II to given matrix pair $[\bar{X}, \bar{Y}]$ in Frobenius norm will be obtained by finding the least norm mirror-symmetric solution of the new linear matrix equation $A\ddot{X}B + C\ddot{Y}D = \ddot{E}$, where $\ddot{E} = E - A\bar{X}B - C\bar{Y}D$. Finally, we present several numerical examples to illustrate the effectiveness of our algorithm.

2. An iterative method for solving Problem I

Here, we will construct an iterative method to solve Problem I Then, some basic properties of the introduced iterative method are described. Finally, we show that the method is convergent.

Lemma 2.1. *Suppose that* $X \in R^{(2r+p)\times(2r+p)}$. *Then,*

$$X + W_{(r,p)} X W_{(r,p)} \in MS_{(r,p)}.$$

Proof. The proof is easily at hand using the definition, and so we omit the details. □

Lemma 2.2. *Suppose that* $A \in R^{(2r+p)\times(2r+p)}$ *and* $X \in MS_{(r,p)}$. *Then,*

$$\langle A, X\rangle = \langle \frac{1}{2}[A + W_{(r,p)} A W_{(r,p)}],\ X\rangle.$$

Proof. $\langle \frac{1}{2}[A + W_{(r,p)} A W_{(r,p)}],\ X\rangle = \frac{1}{2}[\langle A, X\rangle + \langle W_{(r,p)} A W_{(r,p)},\ X\rangle] = \frac{1}{2}[\langle A, X\rangle + \langle A,\ W_{(r,p)} X W_{(r,p)}\rangle] = \langle A, X\rangle$. □

Algorithm 1. Input matrices $A \in R^{s\times(2r+p)}$, $B \in R^{(2r+p)\times t}$, $C \in R^{s\times(2h+q)}$, $D \in R^{(2h+q)\times t}$, $E \in R^{s\times t}$ and $X_0 \in MS_{(r,p)}$, $Y_0 \in MS_{(h,q)}$.

Step 1: Calculate

$$\begin{aligned} R_0 =& \ E - AX_0B - CY_0D; \\ P_{0,x} =& \ A^T R_0 B^T,\ P_{0,y} = C^T R_0 D^T; \\ Q_{0,x} =& \ \tfrac{1}{2}(P_{0,x} + W_{(r,p)}P_{0,x}W_{(r,p)}),\ Q_{0,y} = \tfrac{1}{2}(P_{0,y} + W_{(h,q)}P_{0,y}W_{(h,q)}) \\ k :=& \ 0; \end{aligned}$$

Step 2: If $R_k = 0$ then stop; else let $k := k + 1$.

Step 3: Compute

$$\begin{aligned} X_k =& \ X_{k-1} + \tfrac{\|R_{k-1}\|^2}{\|Q_{k-1,x}\|^2+\|Q_{k-1,y}\|^2} Q_{k-1,x}; \\ Y_k =& \ Y_{k-1} + \tfrac{\|R_{k-1}\|^2}{\|Q_{k-1,x}\|^2+\|Q_{k-1,y}\|^2} Q_{k-1,y}; \\ R_k =& \ E - AX_kB - CY_kD \\ =& \ R_{k-1} - \tfrac{\|R_{k-1}\|^2}{\|Q_{k-1,x}\|^2+\|Q_{k-1,y}\|^2}(AQ_{k-1,x}B + CQ_{k-1,y}D); \\ P_{k,x} =& \ A^T R_k B^T,\ P_{k,y} = C^T R_k D^T; \\ Q_{k,x} =& \ \tfrac{1}{2}(P_{k,x} + W_{(r,p)}P_{k,x}W_{(r,p)}) + \tfrac{\|R_k\|^2}{\|R_{k-1}\|^2} Q_{k-1,x}; \\ Q_{k,y} =& \ \tfrac{1}{2}(P_{k,y} + W_{(h,q)}P_{k,y}W_{(h,q)}) + \tfrac{\|R_k\|^2}{\|R_{k-1}\|^2} Q_{k-1,y}; \end{aligned}$$

Step 4: Go to step 2.

Remark 2.3. Obviously, $Q_{k,x} \in MS_{(r,p)}$, $Q_{k,y} \in MS_{(h,q)}$, $X_k \in MS_{(r,p)}$, and $Y_k \in MS_{(h,q)}$, for $k = 0, 1, \cdots$, from Algorithm 1.

In the next part, we will analyze properties of Algorithm 1, and then we will prove that the process will stop after a finite number of steps.

Lemma 2.4. *Suppose that the sequences R_i, $Q_{i,x}$, $Q_{i,y}(R_i \neq 0, i = 0, 1, 2, \cdots, k)$ are generated by Algorithm 1. We have*

(2.1)

$$\langle R_i, R_j\rangle = 0, \quad \langle Q_{i,x}, Q_{j,x}\rangle + \langle Q_{i,y}, Q_{j,y}\rangle = 0, (i, j = 0, 1, 2, \ldots, k, i \neq j).$$

Proof. We know that $\langle A,\ B\rangle = \langle B,\ A\rangle$ for arbitrary matrices $A,\ B \in R^{n\times n}$. Therefore, we only need to prove $\langle R_i,\ R_j\rangle = 0$ and $\langle Q_{i,x},\ Q_{j,x}\rangle + \langle Q_{i,y},\ Q_{j,y}\rangle = 0$, for $1 \leq i < j \leq k$. To this end, we use induction.

Step 1. Show that $\langle R_0,\ R_1\rangle = 0,\ \ \langle Q_{0,x},\ Q_{1,x}\rangle + \langle Q_{0,y},\ Q_{1,y}\rangle = 0$ when $k = 1$.

$$\begin{aligned}
&\langle R_0,\ R_1\rangle \\
&\quad = \langle R_0,\ R_0 - \tfrac{\|R_0\|^2}{\|Q_{0,x}\|^2+\|Q_{0,y}\|^2}(AQ_{0,x}B + CQ_{0,y}D)\rangle \\
&\quad = \|R_0\|^2 - \tfrac{\|R_0\|^2}{\|Q_{0,x}\|^2+\|Q_{0,y}\|^2}[\langle A^TR_0B^T,\ Q_{0,x}\rangle + \langle C^TR_0D^T,\ Q_{0,y}\rangle] \\
&\quad = \|R_0\|^2 - \tfrac{\|R_0\|^2}{\|Q_{0,x}\|^2+\|Q_{0,y}\|^2}[\langle P_{0,x},\ Q_{0,x}\rangle + \langle P_{0,y},\ Q_{0,y}\rangle] \\
&\quad = \|R_0\|^2 - \tfrac{\|R_0\|^2}{\|Q_{0,x}\|^2+\|Q_{0,y}\|^2}[\langle \tfrac{1}{2}(P_{0,x} + W_{(r,p)}P_{0,x}W_{(r,p)}),\ Q_{0,x}\rangle \\
&\qquad\qquad + \langle \tfrac{1}{2}(P_{0,y} + W_{(h,q)}P_{0,y}W_{(h,q)}),\ Q_{0,y}\rangle] \\
&\quad = \|R_0\|^2 - \tfrac{\|R_0\|^2}{\|Q_{0,x}\|^2+\|Q_{0,y}\|^2}[\langle Q_{0,x},\ Q_{0,x}\rangle + \langle Q_{0,y},\ Q_{0,y}\rangle] \\
&\quad = \|R_0\|^2 - \|R_0\|^2 \\
&\quad = 0.
\end{aligned}$$

$$\begin{aligned}
&\langle Q_{0,x},\ Q_{1,x}\rangle + \langle Q_{0,y},\ Q_{1,y}\rangle \\
&\quad = \langle Q_{0,x},\ \tfrac{1}{2}(P_{1,x} + W_{(r,p)}P_{1,x}W_{(r,p)}) + \tfrac{\|R_1\|^2}{\|R_0\|^2}Q_{0,x}\rangle \\
&\qquad + \langle Q_{0,y},\ \tfrac{1}{2}(P_{1,y} + W_{(h,q)}P_{1,y}W_{(h,q)}) + \tfrac{\|R_1\|^2}{\|R_0\|^2}Q_{0,y}\rangle \\
&\quad = \langle Q_{0,x},\ \tfrac{1}{2}[P_{1,x} + W_{(r,p)}P_{1,x}W_{(r,p)}]\rangle \\
&\qquad + \langle Q_{0,y},\ \tfrac{1}{2}[P_{1,y} + W_{(h,q)}P_{1,y}W_{(h,q)}]\rangle + \tfrac{\|R_1\|^2}{\|R_0\|^2}[\|Q_{0,x}\|^2 + \|Q_{0,y}\|^2] \\
&\quad = \langle Q_{0,x},\ P_{1,x}\rangle + \langle Q_{0,y},\ P_{1,y}\rangle + \tfrac{\|R_1\|^2}{\|R_0\|^2}[\|Q_{0,x}\|^2 + \|Q_{0,y}\|^2] \\
&\quad = \langle Q_{0,x},\ A^TR_1B^T\rangle + \langle Q_{0,y},\ C^TR_1D^T\rangle + \tfrac{\|R_1\|^2}{\|R_0\|^2}[\|Q_{0,x}\|^2 + \|Q_{0,y}\|^2] \\
&\quad = \langle AQ_{0,x}B,\ R_1\rangle + \langle CQ_{0,y}D,\ R_1\rangle + \tfrac{\|R_1\|^2}{\|R_0\|^2}[\|Q_{0,x}\|^2 + \|Q_{0,y}\|^2] \\
&\quad = \tfrac{\|Q_{0,x}\|^2+\|Q_{0,y}\|^2}{\|R_0\|^2}\langle R_0 - R_1,\ R_1\rangle + \tfrac{\|R_1\|^2}{\|R_0\|^2}[\|Q_{0,x}\|^2 + \|Q_{0,y}\|^2] \\
&\quad = -\tfrac{\|Q_{0,x}\|^2+\|Q_{0,y}\|^2}{\|R_0\|^2}\|R_1\|^2 + \tfrac{\|R_1\|^2}{\|R_0\|^2}[\|Q_{0,x}\|^2 + \|Q_{0,y}\|^2] \\
&\quad = 0.
\end{aligned}$$

Step 2. Suppose that (2.4) holds when $k = v$, and show for $k = v + 1$:

$$\begin{aligned}
&\langle R_v,\ R_{v+1}\rangle \\
&\quad = \langle R_v,\ R_v - \tfrac{\|R_v\|^2}{\|Q_{v,x}\|^2+\|Q_{v,y}\|^2}(AQ_{v,x}B + CQ_{v,y}D)\rangle \\
&\quad = \|R_v\|^2 - \tfrac{\|R_v\|^2}{\|Q_{v,x}\|^2+\|Q_{v,y}\|^2}[\langle A^TR_vB^T,\ Q_{v,x}\rangle + \langle C^TR_vD^T,\ Q_{v,y}\rangle] \\
&\quad = \|R_v\|^2 - \tfrac{\|R_v\|^2}{\|Q_{v,x}\|^2+\|Q_{v,y}\|^2}[\langle P_{v,x},\ Q_{v,x}\rangle + \langle P_{v,y},\ Q_{v,y}\rangle]
\end{aligned}$$

$$= \|R_v\|^2 - \frac{\|R_v\|^2}{\|Q_{v,x}\|^2+\|Q_{v,y}\|^2}[\langle \tfrac{1}{2}(P_{v,x} + W_{(r,p)}P_{v,x}W_{(r,p)}),\ Q_{v,x}\rangle + \langle \tfrac{1}{2}(P_{v,y} + W_{(h,q)}P_{v,y}W_{(h,q)}),\ Q_{v,y}\rangle]$$
$$= \|R_v\|^2 - \frac{\|R_v\|^2}{\|Q_{v,x}\|^2+\|Q_{v,y}\|^2}[\langle Q_{v,x} - \frac{\|R_v\|^2}{\|R_{v-1}\|^2}Q_{v-1,x},\ Q_{v,x}\rangle + \langle Q_{v,y} - \frac{\|R_v\|^2}{\|R_{v-1}\|^2}Q_{v-1,y},\ Q_{v,y}\rangle]$$
$$= \|R_v\|^2 - \|R_v\|^2 = 0.$$

$$\langle Q_{v,x},\ Q_{v+1,x}\rangle + \langle Q_{v,y},\ Q_{v+1,y}\rangle$$
$$= \langle Q_{v,x},\ \tfrac{1}{2}(P_{v+1,x} + W_{(r,p)}P_{v+1,x}W_{(r,p)}) + \frac{\|R_{v+1}\|^2}{\|R_v\|^2}Q_{v,x}\rangle + \langle Q_{v,y},\ \tfrac{1}{2}(P_{v+1,y} + W_{(h,q)}P_{v+1,y}W_{(h,q)}) + \frac{\|R_{v+1}\|^2}{\|R_v\|^2}Q_{v,y}\rangle$$
$$= \langle Q_{v,x},\ \tfrac{1}{2}(P_{v+1,x} + W_{(r,p)}P_{v+1,x}W_{(r,p)})\rangle + \langle Q_{v,y},\ \tfrac{1}{2}(P_{v+1,y} + W_{(h,q)}P_{v+1,y}W_{(h,q)})\rangle + \frac{\|R_{v+1}\|^2}{\|R_v\|^2}[\|Q_{v,x}\|^2 + \|Q_{v,y}\|^2]$$
$$= \langle Q_{v,x},\ P_{v+1,x}\rangle + \langle Q_{v,y},\ P_{v+1,y}\rangle + \frac{\|R_{v+1}\|^2}{\|R_v\|^2}[\|Q_{v,x}\|^2 + \|Q_{v,y}\|^2]$$
$$= \langle Q_{v,x},\ A^TR_{v+1}B^T\rangle + \langle Q_{v,y},\ C^TR_{v+1}D^T\rangle + \frac{\|R_{v+1}\|^2}{\|R_v\|^2}[\|Q_{v,x}\|^2 + \|Q_{v,y}\|^2]$$
$$= \langle AQ_{v,x}B,\ R_{v+1}\rangle + \langle CQ_{v,y}D,\ R_{v+1}\rangle + \frac{\|R_{v+1}\|^2}{\|R_v\|^2}[\|Q_{v,x}\|^2 + \|Q_{v,y}\|^2]$$
$$= \frac{\|Q_{v,x}\|^2+\|Q_{v,y}\|^2}{\|R_v\|^2}\langle R_v - R_{v+1},\ R_{v+1}\rangle + \frac{\|R_{v+1}\|^2}{\|R_v\|^2}[\|Q_{v,x}\|^2 + \|Q_{v,y}\|^2]$$
$$= -\frac{\|Q_{v,x}\|^2+\|Q_{v,y}\|^2}{\|R_v\|^2}\|R_{v+1}\|^2 + \frac{\|R_{v+1}\|^2}{\|R_v\|^2}[\|Q_{v,x}\|^2 + \|Q_{v,y}\|^2]$$
$$= 0.$$

For $j = 1, 2, \ldots, v-1$, we have

$$\langle R_j,\ R_{v+1}\rangle$$
$$= \langle R_j,\ R_v - \frac{\|R_v\|^2}{\|Q_{v,x}\|^2+\|Q_{v,y}\|^2}(AQ_{v,x}B + CQ_{v,y}D)\rangle$$
$$= -\frac{\|R_v\|^2}{\|Q_{v,x}\|^2+\|Q_{v,y}\|^2}[\langle A^TR_jB^T,\ Q_{v,x}\rangle + \langle C^TR_jD^T,\ Q_{v,y}\rangle]$$

$$
\begin{aligned}
&= -\tfrac{\|R_v\|^2}{\|Q_{v,x}\|^2+\|Q_{v,y}\|^2}[\langle P_{j,x},\ Q_{v,x}\rangle + \langle P_{j,y},\ Q_{v,y}\rangle]\\
&= -\tfrac{\|R_v\|^2}{\|Q_{v,x}\|^2+\|Q_{v,y}\|^2}[\langle \tfrac{1}{2}(P_{j,x} + W_{(r,p)}P_{j,x}W_{(r,p)}),\ Q_{v,x}\rangle\\
&\qquad + \langle \tfrac{1}{2}(P_{j,y} + W_{(h,q)}P_{j,y}W_{(h,q)}),\ Q_{v,y}\rangle]\\
&= -\tfrac{\|R_v\|^2}{\|Q_{v,x}\|^2+\|Q_{v,y}\|^2}[\langle Q_{j,x} - \tfrac{\|R_j\|^2}{\|R_{j-1}\|^2}Q_{j-1,x},\ Q_{v,x}\rangle\\
&\qquad + \langle Q_{j,y} - \tfrac{\|R_j\|^2}{\|R_{j-1}\|^2}Q_{j-1,y},\ Q_{v,y}\rangle] = 0.
\end{aligned}
$$

$$
\begin{aligned}
&\langle Q_{j,x},\ Q_{v+1,x}\rangle + \langle Q_{j,y},\ Q_{v+1,y}\rangle\\
&= \langle Q_{j,x},\ \tfrac{1}{2}(P_{v+1,x} + W_{(r,p)}P_{v+1,x}W_{(r,p)}) + \tfrac{\|R_{v+1}\|^2}{\|R_v\|^2}Q_{v,x}\rangle\\
&\quad + \langle Q_{j,y},\ \tfrac{1}{2}(P_{v+1,y} + W_{(h,q)}P_{v+1,y}W_{(h,q)}) + \tfrac{\|R_{v+1}\|^2}{\|R_v\|^2}Q_{v,y}\rangle\\
&= \langle Q_{j,x},\ \tfrac{1}{2}[P_{v+1,x} + W_{(r,p)}P_{v+1,x}W_{(r,p)}]\rangle\\
&\quad + \langle Q_{j,y},\ \tfrac{1}{2}[P_{v+1,y} + W_{(h,q)}P_{v+1,y}W_{(h,q)}]\rangle\\
&\quad + \tfrac{\|R_{v+1}\|^2}{\|R_v\|^2}[\langle Q_{j,x},\ Q_{v,x}\rangle + \langle Q_{j,y},\ Q_{v,y}\rangle]\\
&= \langle Q_{j,x},\ P_{v+1,x}\rangle + \langle Q_{j,y},\ P_{v+1,y}\rangle\\
&= \langle Q_{j,x},\ A^T R_{v+1} B^T\rangle + \langle Q_{j,y},\ C^T R_{v+1} D^T\rangle\\
&= \langle AQ_{j,x}B,\ R_{v+1}\rangle + \langle CQ_{j,y}D,\ R_{v+1}\rangle\\
&= \tfrac{\|Q_{j,x}\|^2+\|Q_{j,y}\|^2}{\|R_j\|^2}\langle R_j - R_{j+1},\ R_{v+1}\rangle\\
&= 0.
\end{aligned}
$$

From *step 1* and *step 2*, the conclusion $\langle R_i,\ R_j\rangle = 0$ and $\langle Q_{i,x},\ Q_{j,x}\rangle + \langle Q_{i,y},\ Q_{j,y}\rangle = 0$ hold, for all $i, j = 0, 1, 2, \ldots, k (i \neq j)$ by the principle of induction. □

Lemma 2.5. *Suppose that Problem I is consistent, and* $[X^*, Y^*]$ *is an arbitrary solution of Problem I. Then, for any initial mirror-symmetric matrix pair* $[X_0, Y_0]$*, the sequences* X_i*,* Y_i*,* R_i*,* $Q_{i,x}$ *and* $Q_{i,y}$ *generated by Algorithm 1 satisfy*

$$\langle Q_{i,x},\ X^* - X_i\rangle + \langle Q_{i,y}, Y^* - Y_i\rangle = \|R_i\|^2, \quad (i = 0, 1, 2. \ldots). \tag{2.2}$$

Proof. We prove this by induction. For $i = 0$,

$$\begin{aligned}
&\langle Q_{0,x},\ X^* - X_0\rangle + \langle Q_{0,y},\ Y^* - Y_0\rangle \\
&\quad= \langle \tfrac{1}{2}(P_{0,x} + W_{(r,p)}P_{0,x}W_{(r,p)}),\ X^* - X_0\rangle \\
&\qquad+ \langle \tfrac{1}{2}(P_{0,y} + W_{(h,q)}P_{0,y}W_{(h,q)}),\ Y^* - Y_0\rangle \\
&\quad= \langle P_{0,x},\ X^* - X_0\rangle + \langle P_{0,y},\ Y^* - Y_0\rangle \\
&\quad= \langle A^T R_0 B^T,\ X^* - X_0\rangle + \langle C^T R_0 D^T,\ Y^* - Y_0\rangle \\
&\quad= \langle R_0,\ A(X^* - X_0)B\rangle + \langle R_0,\ C(Y^* - Y_0)D\rangle \\
&\quad= \langle R_0,\ AX^*B + CY^*D - AX_0B - CY_0D\rangle \\
&\quad= \langle R_0,\ R_0\rangle \\
&\quad= \|R_0\|^2.
\end{aligned}$$

Suppose that the result holds for $i = v(v \geq 0)$, that is, $\langle Q_{v,x},\ X^* - X_v\rangle + \langle Q_{v,y}, Y^* - Y_v\rangle = \|R_v\|^2$. Then, for $i = v + 1$,

$$\begin{aligned}
&\langle Q_{v+1,x},\ X^* - X_{v+1}\rangle + \langle Q_{v+1,y},\ Y^* - Y_{v+1}\rangle \\
&\quad= \langle \tfrac{1}{2}(P_{v+1,x} + W_{(r,p)}P_{v+1,x}W_{(r,p)} + \tfrac{\|R_{v+1}\|^2}{\|R_v\|^2}Q_{v,x}),\ X^* - X_{v+1}\rangle \\
&\qquad+ \langle \tfrac{1}{2}(P_{v+1,y} + W_{(h,q)}P_{v+1,y}W_{(h,q)} + \tfrac{\|R_{v+1}\|^2}{\|R_v\|^2}Q_{v,y}),\ Y^* - Y_{v+1}\rangle \\
&\quad= \langle P_{v+1,x},\ X^* - X_{v+1}\rangle + \langle P_{v+1,y},\ Y^* - Y_{v+1}\rangle \\
&\qquad+ \tfrac{\|R_{v+1}\|^2}{\|R_v\|^2}[\langle Q_{v,x},\ X^* - X_{v+1}\rangle + \langle Q_{v,y},\ Y^* - Y_{v+1}\rangle] \\
&\quad= \langle A^T R_{v+1} B^T,\ X^* - X_{v+1}\rangle + \langle C^T R_{v+1} D^T,\ Y^* - Y_{v+1}\rangle \\
&\qquad+ \tfrac{\|R_{v+1}\|^2}{\|R_v\|^2}[\langle Q_{v,x},\ X^* - X_{v+1}\rangle + \langle Q_{v,y},\ Y^* - Y_{v+1}\rangle] \\
&\quad= \langle R_{v+1},\ A(X^* - X_{v+1})B\rangle + \langle R_{v+1},\ C(Y^* - Y_{v+1})D\rangle \\
&\qquad+ \tfrac{\|R_{v+1}\|^2}{\|R_v\|^2}[\langle Q_{v,x},\ X^* - X_{v+1}\rangle + \langle Q_{v,y},\ Y^* - Y_{v+1}\rangle] \\
&\quad= \langle R_{v+1},\ AX^*B + CY^*D - AX_{v+1}B - CY_{v+1}D\rangle \\
&\qquad+ \tfrac{\|R_{v+1}\|^2}{\|R_v\|^2}[\langle Q_{v,x},\ X^* - X_{v+1}\rangle + \langle Q_{v,y},\ Y^* - Y_{v+1}\rangle] \\
&\quad= \|R_{v+1}\|^2 + \tfrac{\|R_{v+1}\|^2}{\|R_v\|^2}[\langle Q_{v,x},\ X^* - X_v - \tfrac{\|R_v\|^2}{\|Q_{v,x}\|^2 + \|Q_{v,y}\|^2}Q_{v,x}\rangle \\
&\qquad\qquad+ \langle Q_{v,y},\ Y^* - Y_v - \tfrac{\|R_v\|^2}{\|Q_{v,x}\|^2 + \|Q_{v,y}\|^2}Q_{v,y}\rangle] \\
&\quad= \|R_{v+1}\|^2 + \tfrac{\|R_{v+1}\|^2}{\|R_v\|^2}[\langle Q_{v,x},\ X^* - X_v\rangle + \langle Q_{v,y},\ Y^* - Y_v\rangle - \|R_v\|^2] \\
&\quad= \|R_{v+1}\|^2.
\end{aligned}$$

By the principle of induction, the result (2.2) holds for all $i = 0, 1, 2, \ldots$. □

Theorem 2.6. *Suppose that Problem I is consistent. Then, for any initial mirror-symmetric matrix pair* $[X_0, Y_0]$*, a solution of Problem I can be obtained within at most st iteration steps by Algorithm 1.*

Proof. If $R_i \neq 0 (i = 0, 1, 2, \ldots, st-1)$, then $Q_{i,x} \neq 0$ and $Q_{i,y} \neq 0$, for $i = 0, 1, 2, \ldots, st-1$, from Lemma 2.5. Therefore $Q_{st,x}$, $Q_{st,y}$ and R_{st} can be calculated by Algorithm 1. Also, from Lemma 2.3, we can write

$$\langle R_i,\ R_{st} \rangle = 0, \quad i = 0, 1, 2, \ldots, st-1$$

and

$$\langle R_i,\ R_j \rangle = 0, \quad i, j = 0, 1, 2, \ldots, st-1, i \neq j.$$

So, the set composed of $R_0,\ R_1,\ \ldots, R_{st-1}$ is an orthogonal basis for the matrix space $R^{s \times t}$, which implies that $R_{st} = 0$; i.e., $[X_{st}, Y_{st}]$ is a mirror-symmetric solution of Problem I. □

Remark 2.7. Theoretically, for any initial mirror-symmetric matrix pair, a mirror-symmetric solution of Problem I can be obtained within at most $s \times t$ iteration steps by Algorithm 1, but actually roundoff errors is unavoidable in the process, and hence the solution of Problem I is generally obtained in more than st iteration steps.

Theorem 2.8. *Problem I is consistent if and only if there exists a nonnegative integer number k such that* $R_k = 0$ *or* $Q_{k,x} \neq 0$ *and* $Q_{k,y} \neq 0$.

Proof. Suppose that there exists a nonnegative integer number k such that $R_k = 0$. Then, Problem I is obviously consistent. If $Q_{k,x} \neq 0$ and $Q_{k,y} \neq 0$, then a mirror-symmetric solution of Problem I can be obtained within at most st iteration steps from Theorem 2.6, and so Problem I is also consistent.

Conversely, suppose that Problem I is consistent. Then, there exists a nonnegative integer number k, such that $R_k = 0$ or $Q_{k,x} \neq 0$ and $Q_{k,y} \neq 0$. Actually, if $R_k \neq 0$ and $Q_{k,x} = 0$, $Q_{k,y} = 0$, then Lemma 2.5 is contradicted. □

Remark 2.9. From Lemma 2.5, if there exists a nonnegative integer number k such that $Q_{k,x} = 0$ and $Q_{k,y} = 0$, but $R_k \neq 0$, then Problem

I is inconsistent. Hence, the solvability of Problem I can be judged automatically by Algorithm 1.

Lemma 2.10. *Problem I has mirror-symmetric solution if and only if the following linear matrix equations is consistent:*

$$\left\{\begin{array}{l} AXB + CYD = E, \\ AW_{(r,p)}XW_{(r,p)}B + CW_{(h,q)}YW_{(h,q)}D = E. \end{array}\right. \tag{2.3}$$

Proof. Suppose that Problem I has a mirror-symmetric solution $[X, Y]$. Then, $X = W_{(r,p)}XW_{(r,p)}$, $Y = W_{(h,q)}YW_{(h,q)}$, and

$$AXB + CYD = E,$$
$$AW_{(r,p)}XW_{(r,p)}B + CW_{(h,q)}YW_{(h,q)}D = AXB + CYD = E.$$

Hence, the mirror-symmetric solution $[X, Y]$ is a solution of the linear matrix equations (2.3); that is, the linear matrix equations (2.3) is consistent.

Conversely, suppose that the linear matrix equations (2.3) is consistent. Then, there exists a matrix pair $[\hat{X}, \hat{Y}]$ ($X \in R^{(2r+p)\times(2r+p)}$, $Y \in R^{(2h+q)\times(2h+q)}$) such that

$$\left\{\begin{array}{l} A\hat{X}B + C\hat{Y}D = E, \\ AW_{(r,p)}\hat{X}W_{(r,p)}B + CW_{(h,q)}\hat{Y}W_{(h,q)}D = E. \end{array}\right. \tag{2.4}$$

Let $X = \frac{\hat{X}+W_{(r,p)}\hat{X}W_{(r,p)}}{2}$ and $Y = \frac{\hat{Y}+W_{(h,q)}\hat{Y}W_{(h,q)}}{2}$. Then $X \in MS_{(r,p)}$, $Y \in MS_{(h,q)}$, and

$$\begin{aligned} AXB + CYD &= A\frac{\hat{X}+W_{(r,p)}\hat{X}W_{(r,p)}}{2}B + C\frac{\hat{Y}+W_{(h,q)}\hat{Y}W_{(h,q)}}{2}D \\ &= \frac{AXB+CYD+AW_{(r,p)}\hat{X}W_{(r,p)}B+CW_{(h,q)}\hat{Y}W_{(h,q)}D}{2} \\ &= \frac{E+E}{2} = E. \end{aligned}$$

Therefore, $[X, Y]$ is a mirror-symmetric solution group of Problem I. □

Remark 2.11. From Lemma 2.10, any mirror-symmetric solution of the linear matrix equations (2.3) must be a solution group of Problem I. Therefore, if we want to prove that $[X, Y]$ is the least Frobenius norm mirror-symmetric solution of Problem I, then it is enough to prove that $[X, Y]$ is the least Frobenius norm mirror-symmetric solution of the linear matrix equations (2.3).

The following lemma is derived from [16], and so we omit the proof.

Lemma 2.12. *Suppose that the consistent system of the linear equations* $Ax = b$ *has a solution* $x^* \in R(A^T)$*. Then,* x^* *is a unique least Frobenius norm solution of the system of linear equations.*

Theorem 2.13. *Suppose that Problem I is consistent. If we choose the initial mirror-symmetric matrices* $X_0 = A^T HB^T + W_{(r,p)}(A^T HB^T)W_{(r,p)}$, $Y_0 = C^T HD^T + W_{(h,q)}(C^T HD^T)W_{(h,q)}$, *H arbitrary, or more especially, let* $X_0 = 0$ *and* $Y_0 = 0$ *with suitable dimensions, then the mirror-symmetric solution* $[X^*, Y^*]$ *obtained by Algorithm 1 is the unique least Frobenius norm mirror-symmetric solution of Problem I.*

Proof. From Theorem 2.6, if we take

$$X_0 = A^T HB^T + W_{(r,p)}(A^T HB^T)W_{(r,p)},$$
$$Y_0 = C^T HD^T + W_{(h,q)}(C^T HD^T)W_{(h,q)},$$

where H is arbitrary, we can obtain the mirror-symmetric solution $[X^*, Y^*]$ of Problem I, and $[X^*, Y^*]$ can be expressed as

$$X^* = A^T TB^T + W_{(r,p)}(A^T TB^T)W_{(r,p)},$$
$$Y^* = C^T TD^T + W_{(h,q)}(C^T TD^T)W_{(h,q)},$$

where $T \in R^{s\times t}$. In the sequel, we will prove that $[X^*, Y^*]$ is the unique least Frobenius norm mirror-symmetric solution of Problem I. From Remark 3, we only prove that $[X^*, Y^*]$ is the least Frobenius norm mirror-symmetric solution of the linear matrix equations (2.3).

Linear matrix equations (2.3) is equivalent to the system of linear matrix equations

(2.5)
$$\begin{pmatrix} B^T \otimes A & D^T \otimes C \\ (B^T W_{(r,p)}) \otimes (AW_{(r,p)}) & (D^T W_{(h,q)}) \otimes (CW_{(h,q)}) \end{pmatrix} \begin{pmatrix} vec(X) \\ vec(Y) \end{pmatrix} = \begin{pmatrix} vec(E) \\ vec(E) \end{pmatrix},$$

Noting that

$$\begin{pmatrix} vec(X^*) \\ vec(Y^*) \end{pmatrix} = \begin{pmatrix} vec(A^TTB^T + W_{(r,p)}(A^TTB^T)W_{(r,p)}) \\ vec(C^TTD^T + W_{(h,q)}(C^TTD^T)W_{(h,q)}) \end{pmatrix}$$

$$= \begin{pmatrix} B \otimes A^T & (W_{(r,p)}B) \otimes (W_{(r,p)}A^T) \\ D \otimes C^T & (W_{(h,q)}D) \otimes (W_{(h,q)}C^T) \end{pmatrix} \begin{pmatrix} vec(T) \\ vec(T) \end{pmatrix}$$

$$= \begin{pmatrix} B^T \otimes A & D^T \otimes C \\ (B^TW_{(r,p)}) \otimes (AW_{(r,p)}) & (D^TW_{(h,q)}) \otimes (CW_{(h,q)}) \end{pmatrix}^T \begin{pmatrix} vec(T) \\ vec(T) \end{pmatrix}$$

$$\in R\left(\begin{pmatrix} B^T \otimes A & D^T \otimes C \\ (B^TW_{(r,p)}) \otimes (AW_{(r,p)}) & (D^TW_{(h,q)}) \otimes (CW_{(h,q)}) \end{pmatrix}^T \right).$$

Hence, from Lemma 2.12, $[vec(X^*), vec(Y^*)]$ is the unique least Frobenius norm mirror-symmetric solution of the matrix equations (2.5). Since vec operator is isomorphic, $[vec(X^*), vec(Y^*)]$ is the unique least Frobenius norm mirror-symmetric solution of the linear matrix equations (2.3), and thus it is also the unique least Frobenius norm mirror-symmetric solution group of Problem I. □

3. The solution of Problem II

Here, we show that the optimal approximation solution of Problem II can be derived by first finding the least norm mirror-symmetric solution of a new matrix equation.

Suppose that Problem I is consistent, and its mirror-symmetric solution set S_E is nonempty. Hence, for given mirror-symmetric matrix pair $[\bar{X}, \bar{Y}]$, we have

$$AXB + CYD = E \Leftrightarrow A(X - \bar{X})B + C(Y - \bar{Y})D = E - A\bar{X}B - C\bar{Y}D$$

Set $\ddot{X} = X - \bar{X}$, $\ddot{Y} = Y - \bar{Y}$, and $\ddot{E} = E - A\bar{X}B - C\bar{Y}D$. Then, Problem II is equivalent to find the least Frobenius norm mirror-symmetric solution of the linear matrix equation

$$A\ddot{X}B + C\ddot{Y}D = \ddot{E} \tag{3.1}$$

By using Algorithm 1, and letting the initial matrix be $X_0 = 0,\ Y_0 = 0$ in suitable dimensions, we can obtain the unique least Frobenius norm mirror-symmetric solution $[\tilde{X}^*,\ \tilde{Y}^*]$ of the linear matrix equations (3.1). Once the above $[\tilde{X}^*,\ \tilde{Y}^*]$ is obtained, the unique solution $[\tilde{X}, \tilde{Y}]$ of the matrix nearness Problem II can be computed. In this case, $\tilde{X}$ and $\tilde{Y}$ can be expressed as $\tilde{X} = \tilde{X^*} + \bar{X}$, $\tilde{Y} = \tilde{Y^*} + \bar{Y}$.

4. **Examples for the iterative method**

Here, we present numerical example to illustrate the efficiency of the proposed iteration method. All the tests are performed using Matlab 7.0 which has a machine unit round off precision of around 10^{-16}. Because of computing error, the process will not stop within a finite number of steps. Hence, we regard the approximation solution $[X_k, Y_k]$ to be a solution of Problem I if the corresponding residual satisfies $\|R_k\| \leq 10e - 010$.

Example 1. Let

$$A = \begin{pmatrix} 0 & -2 & 1.3 & 1.5 & 0 & -1 & -1 & 5 & 4 & 3.4 \\ -1 & -0.3 & 2 & 4 & -0.5 & 1 & -1 & 2 & 1 & 1.2 \\ 0 & -2 & 3 & 0.5 & 0 & -1 & -1 & 1 & 1.9 & -6 \\ -1 & -3 & 2 & 4 & -0.5 & 1 & -2 & 1 & 0.2 & 3 \\ 3 & 1.2 & 2 & 0 & 4.6 & 0 & 1 & 2 & 0 & 1.8 \\ -1 & -3 & 2 & 1 & -5 & 1 & -2 & 1 & 1 & 5 \\ 2 & -0.8 & 1 & 4.2 & 1.5 & 2.8 & 3.5 & 0.2 & 0.5 & 2.5 \\ -0.9 & 0.4 & 0.4 & -0.9 & 0.6 & 1 & 0.7 & 1.4 & -1.2 & -0.7 \end{pmatrix},$$

$$B = \begin{pmatrix} -0.9 & 0.4 & 0.4 & -0.9 & 0.6 & 0.8 & 5 \\ 1.8 & 1.2 & 0.3 & 0.8 & 0.8 & 5 & 1.8 \\ 1.5 & 0.7 & 0 & 0.5 & -1.2 & 2.1 & -4 \\ -5 & 1.5 & 0 & -0.3 & 2.0 & 0.8 & -3.6 \\ 0.7 & 5.6 & 1 & 0.7 & 1.4 & 4 & 3 \\ -1.2 & 0.6 & 0 & -1.2 & -0.7 & 1.7 & 5 \\ 2 & 4 & -0.5 & 1 & -2 & 1 & 2 \\ 0.4 & 0.4 & -0.9 & 0.6 & 1 & 0.7 & 1.4 \\ -1 & -0.3 & 2 & 4 & -0.5 & 1 & 4.1 \\ 1.5 & -0.7 & 0 & 0.5 & -1.6 & 2.1 & -4 \end{pmatrix},$$

$$E = \begin{pmatrix} 130 & 121 & 151 & -59 & 99.6 & 120 & 87 \\ 143 & -222 & 168 & 95 & 49 & 121 & 151 \\ 79 & 121 & 69 & 87 & -89 & -121 & 144 \\ 120 & 211 & 82 & -96 & 231 & -98 & 120 \\ -117 & 213 & 234 & 98 & -89 & 120 & 211 \\ 200 & -121 & 144 & 57 & 100 & 69 & 87 \\ 112 & 68 & -86 & 83 & 64 & 211 & 82 \\ 212 & 221 & 182 & -96 & 231 & -98 & -120 \end{pmatrix},$$

$$C = \begin{pmatrix} ones(3,4) & zeros(3,5) \\ zeros(5,4) & hankel(1:5) \end{pmatrix},$$

$$D = \begin{pmatrix} toeplitz(1:5) & ones(5,2) \\ zeros(4,5) & 3ones(4,2) \end{pmatrix},$$

$$W_{(r,p)} = \begin{pmatrix} & & J_3 \\ & I_4 & \\ J_3 & & \end{pmatrix}, \qquad W_{(h,q)} = \begin{pmatrix} & & J_3 \\ & I_3 & \\ J_3 & & \end{pmatrix},$$

where $toeplitz(1:n)$ and $hankel(1:n)$ denote the n-th order Toeplitz matrix and Hankel matrix whose first rows are $(1, 2, \cdots, n)$, respectively. $ones(n,m)$ and $zeros(n,m)$ respectively denote the $n \times m$ matrices whose elements are ones and zeros.

(I) Find the mirror-symmetric solution and the least-norm mirror-symmetric solution of the matrix equation $AXB + CYD = E$.

(II) Let S_E denote the set of all mirror-symmetric solution of matrix equation $AXB + CYD = E$. Denote $K = toeplitz(1:10)$, $L = magic(9)$, and let

$$\begin{aligned}\bar{X} &= K + W_{(r,p)} K W_{(r,p)} \\ &= \begin{pmatrix} 1 & 2 & 3 & 5.5 & 5.5 & 5.5 & 5.5 & 8 & 9 & 10 \\ 2 & 1 & 2 & 4.5 & 4.5 & 4.5 & 4.5 & 7 & 8 & 9 \\ 3 & 2 & 1 & 3.5 & 3.5 & 3.5 & 3.5 & 6 & 7 & 8 \\ 5.5 & 4.5 & 3.5 & 1 & 2 & 3 & 4 & 3.5 & 4.5 & 5.5 \\ 5.5 & 4.5 & 3.5 & 2 & 1 & 2 & 3 & 3.5 & 4.5 & 5.5 \\ 5.5 & 4.5 & 3.5 & 3 & 2 & 1 & 2 & 3.5 & 4.5 & 5.5 \\ 5.5 & 4.5 & 3.5 & 4 & 3 & 2 & 1 & 3.5 & 4.5 & 5.5 \\ 8 & 7 & 6 & 3.5 & 3.5 & 3.5 & 3.5 & 1 & 2 & 3 \\ 9 & 8 & 7 & 4.5 & 4.5 & 4.5 & 4.5 & 2 & 1 & 2 \\ 10 & 9 & 8 & 5.5 & 5.5 & 5.5 & 5.5 & 3 & 2 & 1 \end{pmatrix},\end{aligned}$$

$$\begin{aligned}\bar{Y} &= L + W_{(h,q)} L W_{(h,q)} \\ &= \begin{pmatrix} 8.2 & 8.2 & 8.2 & 15.0 & 8.2 & 1.4 & 8.2 & 8.2 & 8.2 \\ 8.2 & 8.2 & 8.2 & 6.9 & 8.2 & 9.5 & 8.2 & 8.2 & 8.2 \\ 8.2 & 8.2 & 8.2 & 6.0 & 8.2 & 10.4 & 8.2 & 8.2 & 8.2 \\ 14.3 & 6.2 & 7.1 & 4.0 & 6.2 & 8.4 & 7.1 & 6.2 & 14.3 \\ 8.2 & 8.2 & 8.2 & 6.0 & 8.2 & 10.4 & 8.2 & 8.2 & 8.2 \\ 2.1 & 10.2 & 9.3 & 8.0 & 10.2 & 12.4 & 9.3 & 10.2 & 2.1 \\ 8.2 & 8.2 & 8.2 & 6.0 & 8.2 & 10.4 & 8.2 & 8.2 & 8.2 \\ 8.2 & 8.2 & 8.2 & 6.9 & 8.2 & 9.5 & 8.2 & 8.2 & 8.2 \\ 8.2 & 8.2 & 8.2 & 15.0 & 8.2 & 1.4 & 8.2 & 8.2 & 8.2 \end{pmatrix}.\end{aligned}$$

From Lemma 2.1, we know that $\bar{X} \in MS_{(r,p)}$, $\bar{Y} \in MS_{(h,q)}$. Find $[\tilde{X}, \tilde{Y}] \in S_E$ such that

$$\|\tilde{X} - \bar{X}\|^2 + \|\tilde{Y} - \bar{Y}\|^2 = \min_{[X,Y] \in S_E} [\|X - \bar{X}\|^2 + [\|Y - \bar{Y}\|^2].$$

(I). Choose an arbitrary initial iterative matrix pair $[X_0, Y_0]$, where $X_0 \in MS_{(r,p)}$ and $Y_0 \in MS_{(h,q)}$, such as $X_0 = ones(10)$ and $Y_0 =$

$ones(9)$. By Algorithm 1, we have

$$X_{120} = \begin{pmatrix} -12.8344 & -1.4268 & -9.2204 & 4.5426 & -4.9021 & -9.6123 & 4.0051 & 3.2470 & 9.3157 & 6.6383 \\ 4.9134 & 11.0713 & 1.7931 & -6.4630 & -17.0482 & -8.9912 & -9.3492 & -2.0215 & 17.5962 & 11.4781 \\ -2.4879 & -9.0842 & -20.9229 & 4.5123 & -2.4561 & -6.7207 & 3.8151 & 2.4255 & 8.2197 & 11.5028 \\ -1.6214 & 4.2922 & 10.4121 & -4.9958 & 4.7209 & 4.4588 & 1.8047 & 10.4121 & 4.2922 & -1.6214 \\ -0.4631 & -4.1193 & -7.8290 & 5.0554 & 5.8670 & 15.0789 & -3.4014 & -7.8290 & -4.1193 & -0.4631 \\ 4.3806 & 6.2396 & 2.1064 & -5.0134 & -18.3562 & 5.7664 & -39.5967 & 2.1064 & 6.2396 & 4.3806 \\ -5.5744 & -7.6941 & 11.5926 & -0.5759 & 13.8524 & 0.1333 & 20.0304 & 11.5926 & -7.6941 & -5.5744 \\ 11.5028 & 8.2197 & 2.4255 & 4.5123 & -2.4561 & -6.7207 & 3.8151 & -20.9229 & -9.0842 & -2.4879 \\ 11.4781 & 17.5962 & -2.0215 & -6.4630 & -17.0482 & -8.9912 & -9.3492 & 1.7931 & 11.0713 & 4.9134 \\ 6.6383 & 9.3157 & 3.2470 & 4.5426 & -4.9021 & -9.6123 & 4.0051 & -9.2204 & -1.4268 & -12.8344 \end{pmatrix},$$

$$Y_{120} = \begin{pmatrix} -5.5669 & -6.7131 & -9.0979 & 9.1966 & -9.4912 & -6.2543 & 16.0266 & -3.0389 & -3.7257 \\ 2.8884 & 1.7423 & -0.6425 & -1.6379 & 5.8628 & 2.2010 & -2.7707 & -1.6631 & 1.5151 \\ 4.8196 & 3.6734 & 1.2886 & -6.9047 & 19.3670 & 4.1322 & -10.1262 & -1.2038 & 2.6174 \\ -0.7924 & -1.9386 & -4.3233 & 3.4571 & 14.5716 & -1.4798 & -4.3233 & -1.9386 & -0.7924 \\ 1.6962 & 0.3936 & -1.7968 & 19.3128 & -1.3390 & -7.8185 & -1.7968 & 0.3936 & 1.6962 \\ -5.9112 & -1.9349 & 11.5889 & -15.6114 & -1.1189 & 4.0328 & 11.5889 & -1.9349 & -5.9112 \\ 2.6174 & -1.2038 & -10.1262 & -6.9047 & 19.3670 & 4.1322 & 1.2886 & 3.6734 & 4.8196 \\ 1.5151 & -1.6631 & -2.7707 & -1.6379 & 5.8628 & 2.2010 & -0.6425 & 1.7423 & 2.8884 \\ -3.7257 & -3.0389 & 16.0266 & 9.1966 & -9.4912 & -6.2543 & -9.0979 & -6.7131 & -5.5669 \end{pmatrix}.$$

By concrete computations, we have

$$R_{120} = 2.9396e-011, \qquad \|X_{120}\| + \|Y_{120}\| = 156.8131,$$

and we can further get

$$\|X_{120} - W_{(r,p)}X_{120}W_{(r,p)}\| = 1.6220e-013,$$
$$\|Y_{120} - W_{(h,q)}Y_{120}W_{(h,q)}\| = 4.3010e-014.$$

So, $[X_{120}, Y_{120}]$ is regarded to be a mirror-symmetric solution of the matrix equation $AXB + CYD = E$.

If we let the initial matrix pair be

$$X_0 = A^T HB^T + W_{(r,p)}(A^T HB^T)W_{(r,p)},$$
$$Y_0 = C^T HD^T + W_{(h,q)}(C^T HD^T)W_{(h,q)},$$

where $H = eye(8,7)$, then by the Algorithm 1 we have

$$X_{121} = \left(\begin{matrix} -12.6654 & -1.5153 & -9.1239 & 4.7177 & -5.2634 & -9.2547 & 4.2326 \\ 4.0979 & 10.5115 & 1.4469 & -6.3556 & -17.2289 & -9.8456 & -9.1222 \\ -2.8617 & -8.6098 & -21.1405 & 3.9250 & -1.3432 & -7.4618 & 3.0554 \\ -1.6848 & 4.7034 & 10.5063 & -5.3446 & 5.3525 & 4.3139 & 1.2927 \\ -0.3882 & -3.9547 & -7.8273 & 4.6750 & 6.6536 & 15.1534 & -3.8814 \\ 3.7978 & 5.8318 & 1.7258 & -5.0566 & -18.2256 & 5.3286 & -39.5504 \\ -5.8683 & -7.8724 & 11.3957 & -0.3635 & 13.5397 & -0.5625 & 20.2510 \\ 11.0834 & 8.4042 & 2.3992 & 3.9250 & -1.3432 & -7.4618 & 3.0554 \\ 10.8405 & 17.2484 & -2.4373 & -6.3556 & -17.2289 & -9.8456 & -9.1222 \\ 6.9216 & 9.4288 & 3.3627 & 4.7177 & -5.2634 & -9.2547 & 4.2326 \end{matrix} \right.$$

$$\left. \begin{matrix} 3.3627 & 9.4288 & 6.9216 \\ -2.4373 & 17.2484 & 10.8405 \\ 2.3992 & 8.4042 & 11.0834 \\ 10.5063 & 4.7034 & -1.6848 \\ -7.8273 & -3.9547 & -0.3882 \\ 1.7258 & 5.8318 & 3.7978 \\ 11.3957 & -7.8724 & -5.8683 \\ -21.1405 & -8.6098 & -2.8617 \\ 1.4469 & 10.5115 & 4.0979 \\ -9.1239 & -1.5153 & -12.6654 \end{matrix} \right),$$

$$Y_{121} = \left(\begin{matrix} -6.1046 & -7.3241 & -9.7151 & 9.3482 & -9.8904 & -6.9936 & 15.9449 \\ 3.0297 & 1.8103 & -0.5808 & -1.5531 & 5.4715 & 2.1407 & -2.8471 \\ 4.5912 & 3.3717 & 0.9807 & -6.8005 & 19.2832 & 3.7022 & -10.1172 \\ -1.3837 & -2.6031 & -4.9942 & 2.7489 & 13.9215 & -2.2727 & -4.9942 \\ 1.5988 & 0.2951 & -1.7956 & 19.1879 & -1.3822 & -7.4342 & -1.7956 \\ -5.9820 & -1.9777 & 11.3988 & -15.7560 & -0.9684 & 4.3709 & 11.3988 \\ 2.7470 & -1.0788 & -10.1172 & -6.8005 & 19.2832 & 3.7022 & 0.9807 \\ 1.1654 & -1.6985 & -2.8471 & -1.5531 & 5.4715 & 2.1407 & -0.5808 \\ -3.9390 & -2.8659 & 15.9449 & 9.3482 & -9.8904 & -6.9936 & -9.7151 \end{matrix} \right.$$

$$\left. \begin{matrix} -2.8659 & -3.9390 \\ -1.6985 & 1.1654 \\ -1.0788 & 2.7470 \\ -2.6031 & -1.3837 \\ 0.2951 & 1.5988 \\ -1.9777 & -5.9820 \\ 3.3717 & 4.5912 \\ 1.8103 & 3.0297 \\ -7.3241 & -6.1046 \end{matrix} \right),$$

and

$$\begin{aligned} & R_{121} = 9.3519e-011, \\ & \|X_{121}\| + \|Y_{121}\| = 152.7857 < 156.8131, \\ & \|X_{121} - W_{(r,p)} X_{121} W_{(r,p)}\| = 1.7760e-013, \\ & \|Y_{121} - W_{(h,q)} Y_{121} W_{(h,q)}\| = 4.4697e-014. \end{aligned}$$

Hence, $[X_{121}, Y_{121}]$ is regarded to be the least-norm mirror-symmetric solution of matrix equation $AXB + CYD = E$.

If we let $X_0 = 0 \in R^{10\times 10}$ and $Y_0 = 0 \in R^{9\times 9}$, using Algorithm 1 we can obtain the least-norm mirror-symmetric solution of matrix equation

$AXB + CYD = E$ as follows:

$$X_{110} = \left(\begin{matrix} -12.6654 & -1.5153 & -9.1239 & 4.7177 & -5.2634 & -9.2547 & 4.2326 \\ 4.0979 & 10.5115 & 1.4469 & -6.3556 & -17.2289 & -9.8456 & -9.1222 \\ -2.8617 & -8.6098 & -21.1405 & 3.9250 & -1.3432 & -7.4618 & 3.0554 \\ -1.6848 & 4.7034 & 10.5063 & -5.3446 & 5.3525 & 4.3139 & 1.2927 \\ -0.3882 & -3.9547 & -7.8273 & 4.6750 & 6.6536 & 15.1534 & -3.8814 \\ 3.7978 & 5.8318 & 1.7258 & -5.0566 & -18.2256 & 5.3286 & -39.5504 \\ -5.8683 & -7.8724 & 11.3957 & -0.3635 & 13.5397 & -0.5625 & 20.2510 \\ 11.0834 & 8.4042 & 2.3992 & 3.9250 & -1.3432 & -7.4618 & 3.0554 \\ 10.8405 & 17.2484 & -2.4373 & -6.3556 & -17.2289 & -9.8456 & -9.1222 \\ 6.9216 & 9.4288 & 3.3627 & 4.7177 & -5.2634 & -9.2547 & 4.2326 \end{matrix}\right.$$

$$\left.\begin{matrix} 3.3627 & 9.4288 & 6.9216 \\ -2.4373 & 17.2484 & 10.8405 \\ 2.3992 & 8.4042 & 11.0834 \\ 10.5063 & 4.7034 & -1.6848 \\ -7.8273 & -3.9547 & -0.3882 \\ 1.7258 & 5.8318 & 3.7978 \\ 11.3957 & -7.8724 & -5.8683 \\ -21.1405 & -8.6098 & -2.8617 \\ 1.4469 & 10.5115 & 4.0979 \\ -9.1239 & -1.5153 & -12.6654 \end{matrix}\right),$$

$$Y_{110} = \left(\begin{matrix} -6.1046 & -7.3241 & -9.7151 & 9.3482 & -9.8904 & -6.9936 & 15.9449 \\ 3.0297 & 1.8103 & -0.5808 & -1.5531 & 5.4715 & 2.1407 & -2.8471 \\ 4.5912 & 3.3717 & 0.9807 & -6.8005 & 19.2832 & 3.7022 & -10.1172 \\ -1.3837 & -2.6031 & -4.9942 & 2.7489 & 13.9215 & -2.2727 & -4.9942 \\ 1.5988 & 0.2951 & -1.7956 & 19.1879 & -1.3822 & -7.4342 & -1.7956 \\ -5.9820 & -1.9777 & 11.3988 & -15.7560 & -0.9684 & 4.3709 & 11.3988 \\ 2.7470 & -1.0788 & -10.1172 & -6.8005 & 19.2832 & 3.7022 & 0.9807 \\ 1.1654 & -1.6985 & -2.8471 & -1.5531 & 5.4715 & 2.1407 & -0.5808 \\ -3.9390 & -2.8659 & 15.9449 & 9.3482 & -9.8904 & -6.9936 & -9.7151 \end{matrix}\right.$$

$$\left.\begin{matrix} -2.8659 & -3.9390 \\ -1.6985 & 1.1654 \\ -1.0788 & 2.7470 \\ -2.6031 & -1.3837 \\ 0.2951 & 1.5988 \\ -1.9777 & -5.9820 \\ 3.3717 & 4.5912 \\ 1.8103 & 3.0297 \\ -7.3241 & -6.1046 \end{matrix}\right),$$

and we have

$$\|R_{110}\| = 2.1067e - 011, \qquad \|X_{110}\| + \|Y_{110}\| = 152.7857.$$

(II) In order to find the optimal approximate solutions to given matrix pair $[\bar{X}, \bar{Y}]$, let $\ddot{X} = X - \bar{X}$, $\ddot{Y} = Y - \bar{Y}$, $\ddot{E} = E - A\bar{X}B - C\bar{Y}D$, and iterate 118 steps. Then, we can obtain the least-norm mirror-symmetric solution $[\widetilde{X^*}_{118},\ \widetilde{Y^*}_{118}]$ of the matrix equation $A\ddot{X}B + C\ddot{Y}D = \ddot{E}$ by choosing the initial iterative matrix $\ddot{X} = 0$, $\ddot{Y} = 0$ in suitable dimensions. To save space, we shall not report the date of $\widetilde{X^*}_{118}$ and $\widetilde{Y^*}_{118}$, but will make them available upon request. Then, the optimal approximate solution $[\tilde{X}, \tilde{Y}]$ can be expressed as:

$$
\begin{aligned}
\tilde{X} &= \widetilde{X^*}_{118} + \bar{X} \\
&= \begin{pmatrix}
-15.6252 & -0.8352 & -9.4674 & 3.7689 & -3.4859 & -10.8982 & 2.5761 & 4.2965 & 9.7053 & 5.3065 \\
8.6823 & 13.2198 & 2.5517 & -8.4811 & -14.4880 & -4.9077 & -11.3101 & 0.3863 & 18.3224 & 15.6661 \\
0.4917 & -12.2154 & -20.3244 & 7.9298 & -8.8334 & -4.3524 & 8.6673 & 1.7077 & 5.5555 & 13.1741 \\
0.4279 & 2.9758 & 9.4895 & -2.3833 & 0.3376 & 4.5902 & 6.0586 & 9.4895 & 2.9758 & 0.4279 \\
-0.5246 & -5.1232 & -9.1928 & 6.2355 & 3.1269 & 13.3938 & -0.9210 & -9.1928 & -5.1232 & -0.5246 \\
6.9876 & 6.8229 & 1.3293 & -5.2572 & -18.5363 & 4.9872 & -39.5965 & 1.3293 & 6.8229 & 6.9876 \\
-5.7068 & -6.2600 & 10.5866 & -1.4444 & 18.1366 & 0.4211 & 17.1217 & 10.5866 & -6.2600 & -5.7068 \\
13.1741 & 5.5555 & 1.7077 & 7.9298 & -8.8334 & -4.3524 & 8.6673 & -20.3244 & -12.2154 & 0.4917 \\
15.6661 & 18.3224 & 0.3863 & -8.4811 & -14.4880 & -4.9077 & -11.3101 & 2.5517 & 13.2198 & 8.6823 \\
5.3065 & 9.7053 & 4.2965 & 3.7689 & -3.4859 & -10.8982 & 2.5761 & -9.4674 & -0.8352 & -15.6252
\end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{Y} &= \widetilde{Y^*}_{118} + \bar{Y} \\
&= \begin{pmatrix}
-2.2171 & -2.8335 & -4.8400 & 12.0123 & -9.3036 & -7.1310 & 16.6278 & -2.6541 & -2.8855 \\
2.1991 & 1.5827 & -0.4238 & -3.1004 & 7.5822 & 5.3852 & -3.1926 & -2.3679 & 3.5065 \\
4.2704 & 3.6540 & 1.6475 & -7.0590 & 20.5729 & 8.3565 & -10.5838 & -2.4037 & 4.5109 \\
9.7553 & 1.0390 & -0.0675 & 4.4932 & 17.7133 & 5.7415 & -0.0675 & 1.0390 & 9.7553 \\
2.1843 & 0.7233 & -0.7712 & 16.8608 & 0.9408 & -6.9919 & -0.7712 & 0.7233 & 2.1843 \\
-8.3694 & -0.1331 & 12.7569 & -13.6853 & -2.6245 & 4.3545 & 12.7569 & -0.1331 & -8.3694 \\
4.5109 & -2.4037 & -10.5838 & -7.0590 & 20.5729 & 8.3565 & 1.6475 & 3.6540 & 4.2704 \\
3.5065 & -2.3679 & -3.1926 & -3.1004 & 7.5822 & 5.3852 & -0.4238 & 1.5827 & 2.1991 \\
-2.8855 & -2.6541 & 16.6278 & 12.0123 & -9.3036 & -7.1310 & -4.8400 & -2.8335 & -2.2171
\end{pmatrix},
\end{aligned}
$$

with the minimum,

$$\min_{[X,Y]\in S_E} [\|X - \bar{X}\|^2 + [\|Y - \bar{Y}\|^2] = \|\tilde{X} - \bar{X}\|^2 + \|\tilde{Y} - \bar{Y}\|^2 = 187.6832.$$

5. **Conclusions**

Mirror-symmetric matrices, which are the interaction matrices of mirror-symmetric structures, have already been thoroughly investigated by Li and Feng [2, 3, 4]. Here, we mainly studied the mirror-symmetric solution of the well-known matrix equation $AXB + CYD = E$.

With the proposed extended CG method (Algorithm 1), when the equation is consistent, it is shown that theoretically for any initial mirror-symmetric pair $[X_0, Y_0]$, a solution is obtained within st iterations, where s and t denote the size of the matrix E. Moreover, it was shown that the unique least Frobenius norm mirror-symmetric solution was obtained by special choices of initial mirror-symmetric iteration matrices, especially the zero matrices. Several numerical examples were presented to illustrate the theory in action.

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Jiao-fen Li
School of Mathematics and Computational Science, Guilin University of Electronic Technology, Guilin 541004, Peoples Republic of China.
Email: lixiaogui1290@163.com

Xi-yan Hu
College of Mathematics and Econometrics , Hunan University, Changsha 410082, Peoples Republic of China.

Xue-feng Duan
School of Mathematics and Computational Science, Guilin University of Electronic Technology, Guilin 541004, Peoples Republic of China.
Email: duanxuefenghd@yahoo.com.cn

Lei Zhang
College of Mathematics and Econometrics , Hunan University, Changsha 410082, Peoples Republic of China.