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THE STRUCTURE AND AMENABILITY OF ℓ^P-MUNN ALGEBRAS

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Abstract. We introduce the notion of $\mathcal{LM}_I^p(\mathcal{A})$, where $\mathcal{A}$ is a Banach space, I is an index set and $1 \leq p < \infty$. We find necessary and sufficient conditions for which $\mathcal{LM}_I^p(\mathcal{A})$ is a Banach algebra and investigate amenability of this Banach algebra. Applications to $\ell^p(S)$ $(1 \leq p < \infty)$, where S is a Brandt semigroup, are also given.

1. Introduction

Some properties of ℓ^1-Munn algebras were investigated by Esslamzadeh [3] , where the author introduced the notion and used them as a tool for studying certain semigroup algebras. For more information, see [2-4]. Our aim here is to introduce and investigate the properties of ℓ^p-Munn algebras. It enables us to study some properties of l^p-spaces on Brandt semigroups. This paper is organized as follows. Our notations are introduced in the present section. In section 2, we introduce and investigate the structure of $\mathcal{L}M_I^p(\mathcal{A})$, for the Banach space $\mathcal{A}$, the index set I, and $1 \leq p < \infty$. The Banach space $\mathcal{LM}_I^p(\mathcal{A})$ is the vector space of all $I \times I$-matrices A over $\mathcal{A}$ such that $\|A\|_p = \left(\sum_{i,j \in I} \|A_{ij}\|^p \right)^{\frac{1}{p}} < \infty$. We find

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necessary and sufficient conditions for which $\mathcal{L}M_I^p(\mathcal{A})$ is a Banach algebra. We prove that if $\mathcal{A}$ is a unital Banach algebra, then $\mathcal{L}M_I^p(\mathcal{A})$ is a Banach algebra if and only if $1 \leq p \leq 2$. Moreover, it is proved that if G is a group and S is a Brandt semigroup over G with index set I, then the Banach space $\ell^p(S)/\mathbb{C}\delta_0$ is isometrically isomorphic with $\mathcal{LM}_I^p(\ell^p(G))$. Moreover, if G is a finite group, and I is finite, then $(\ell^p(S),*)/\mathbb{C}\delta_0$ is isometrically isomorphic with $\mathcal{LM}_I^p(\ell^p(G),*)$. Finally, in Section 3 we study the amenability of the Banach algebra $\mathcal{L}M_I^p(\mathcal{A})$ $(1 \leq p \leq 2)$ over a Banach algebra $\mathcal{A}$ with a unit. We prove that $\mathcal{L}M_I^p(\mathcal{A})$ $(1 \leq p \leq 2)$ is amenable, if and only if $\mathcal{A}$ is amenable, and I is finite.

The following are some of the notations which we use here.

Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{A}$ admits a unit $e_{\mathcal{A}}$ $(ae_{\mathcal{A}} = e_{\mathcal{A}}a = a$, for all $a \in \mathcal{A})$ and $\|e_{\mathcal{A}}\| = 1$, we say that $\mathcal{A}$ is a *unital normed algebra.* For a Banach algebra $\mathcal{A}$, an $\mathcal{A}$-bimodule will always refer to a *Banach $\mathcal{A}$-bimodule* X; that is, a Banach space which is algebraically an $\mathcal{A}$-bimodule, and for which there is a constant $C_X \geq 0$ such that for $a \in \mathcal{A},\ x \in X,\ \|a.x\| \leq C_X\|a\|\|x\|,\ \|x.a\| \leq C_X\|x\|\|a\|$. A *derivation* $D : \mathcal{A} \longrightarrow X$ is a linear map, always taken to be continuous, satisfying $D(ab) = D(a).b + a.D(b)$, for $a, b \in \mathcal{A}$. For every $x \in X$, we define ad_x by $ad_x(a) = a.x - x.a$, for $a \in \mathcal{A}$. Note that ad_x is a derivation which is called an *inner derivation.* A Banach algebra $\mathcal{A}$ is called *amenable* if and only if, for any $\mathcal{A}$ -bimodule X, every derivation $D : \mathcal{A} \longrightarrow X^*$ is inner.

2. The structure of the Banach space $\mathcal{LM}_I^p(\mathcal{A})$ $(1 \leq p < \infty)$ over a Banach algebra $\mathcal{A}$

Definition 2.1. Let $\mathcal{A}$ be a Banach space, $1 \leq p < \infty$, and I be an arbitrary index set, and let $\mathcal{LM}_I^p(\mathcal{A})$ be the vector space of all $I \times I$-matrices A over $\mathcal{A}$ such that

$$\|A\|_p = \Big(\sum_{i,j \in I} \|A_{ij}\|^p \Big)^{\frac{1}{p}} < \infty.$$

Then, it is easy to check that $\mathcal{LM}_I^p(\mathcal{A})$ with scaler multiplication, matrix addition, and the norm $\|.\|_p$ is a Banach space. This Banach space is called *ℓ^p-Munn Banach space* over $\mathcal{A}$. If $\mathcal{LM}_I^p(\mathcal{A})$ is a Banach algebra, then $\mathcal{LM}_I^p(\mathcal{A})$ is called the *ℓ^p-Munn Banach algebra* over $\mathcal{A}$ with index set I.

The space $\mathcal{LM}_I^1(\mathcal{A})$ over a unital Banach algebra $\mathcal{A}$ is called the ℓ^1-*Munn Banach algebra* over $\mathcal{A}$ with index set I (see [3]). If I is finite, then $\|.\|_{\mathcal{L}M_I^2(\mathbb{C})}$ is called *Frobenius norm.*

A Brandt semigroup S over a group G with index set I consists of all canonical $I\times I$ matrix units over $G\bigcup\{0\}$ and a zero matrix 0. Note that an $I\times I$ matrix whose entries are zero except one, is called a canonical matrix unit.

Let G be a group and S be a Brandt semigroup over G. For $f\in\ell^p(S)$, and $i,j\in I$, define $f_{ij}:G\longrightarrow\mathbb{C}$ by

$$f_{ij}(g)=f((g)_{ij}),$$

where $(g)_{ij}$ is the matrix with (k,l)-entry equal to g if $(k,l)=(i,j)$ and 0 if $(k,l)\neq(i,j)$. Since for every $i,j\in I$,

$$\sum_{g\in G}|f((g)_{ij})|^p\leq\sum_{s\in S}|f(s)|^p<\infty,$$

then we have $f_{ij}\in\ell^p(G)$. It is clear that if $A=[f_{ij}]$, then $A\in\mathcal{LM}_I^p(\ell^p(G))$. Now, as in Proposition 5.6 of [3], let

$$\Phi:\ell^p(S)\longrightarrow\mathcal{LM}_I^p(\ell^p(G)):\ f\mapsto[f_{ij}].$$

It is clear that Φ is a well-defined linear map with $\|\Phi\|\leq1$. Suppose $A\in\mathcal{LM}_I^p(\ell^p(G))$ and $A=[f_{ij}]$. Define $f:S\longrightarrow\mathbb{C}$ by $f(0)=0$ and $f((g)_{ij})=f_{ij}(g)$, for $g\in G$ and $i,j\in I$. Since

$$\sum_{s\in S}|f(s)|^p=\sum_{i,j\in I}\sum_{g\in G}|f((g)_{ij})|^p=\sum_{i,j\in I}\|f_{ij}\|_p^p<\infty,$$

Then $f\in\ell^p(S)$. Clearly $\Phi(f)=A$. Hence, Φ is onto. Therefore, there is an isometrical isomorphism from $\ell^p(S)/\mathbb{C}\delta_0$ onto $\mathcal{LM}_I^p(\ell^p(G))$. Thus, we have the following result.

Proposition 2.2. *Let G be a group and S be a Brandt semigroup over G with the index set I. Then, the Banach space $\ell^p(S)/\mathbb{C}\delta_0$ is isometrically isomorphic with $\mathcal{LM}_I^p(\ell^p(G))$.*

For the rest of the paper, we assume that $\mathcal{A}$ is a Banach algebra.

Theorem 2.3. *Let $1\leq p\leq2$. The Banach space $\mathcal{LM}_I^p(\mathcal{A})$ with matrix multiplication and norm $\|.\|_p$ is a Banach algebra.*

Proof. Let $A,B\in\mathcal{LM}_I^p(\mathcal{A})$, and $i,j\in I$. Since $1\leq p\leq2$, then for q with $\frac{1}{p}+\frac{1}{q}=1$, we have $q\geq2\geq p$. Hence, $\ell^p(I)\subseteq\ell^q(I)$ and

$\|f\|_q^p \leq \|f\|_p^p$ $(f \in \ell^p(I))$. We denote the function $f : I \to \mathbb{C}$, by $(f(i))_i$. Now, we have

$$\begin{aligned}
\left(\sum_{k\in I}\|A_{ik}\|\|B_{kj}\|\right)^p &= \|(\|A_{ik}\|)_k(\|B_{kj}\|)_k\|_1^p \\
&\leq \|(\|A_{ik}\|)_k\|_p^p\|(\|B_{kj}\|)_k\|_q^p \\
&\leq \|(\|A_{ik}\|)_k\|_p^p\|(\|B_{kj}\|)_k\|_p^p \\
&= \left(\sum_{k\in I}\|A_{ik}\|^p\right)\left(\sum_{l\in I}\|B_{lj}\|^p\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|AB\|_p^p &= \sum_{i,j\in I}\left\|\sum_{k\in I}A_{ik}B_{kj}\right\|^p \\
&\leq \sum_{i,j\in I}\left(\sum_{k\in I}\|A_{ik}\|\|B_{kj}\|\right)^p \\
&\leq \sum_{i,j\in I}\left(\sum_{k\in I}\|A_{ik}\|^p\right)\left(\sum_{l\in I}\|B_{lj}\|^p\right) \\
&= \left(\sum_{i,k\in I}\|A_{ik}\|^p\right)\left(\sum_{j,l\in I}\|B_{lj}\|^p\right) \\
&= \|A\|_p^p\|B\|_p^p.
\end{aligned}$$

Hence, $\|AB\|_p \leq \|A\|_p\|B\|_p$. This shows that $\|.\|_p$ is an algebra norm. Hence, $\mathcal{LM}_I^p(\mathcal{A})$ is a Banach algebra. □

Example 2.4. Let $\mathcal{A}$ be a non-zero Banach space. Define

$$a.b = 0 \quad (a, b \in \mathcal{A}).$$

With this multiplication $\mathcal{A}$ is a Banach algebra. Now, let I be an arbitrary set and $1 \leq p < \infty$. Then for each $A, B \in \mathcal{LM}_I^p(\mathcal{A})$, $AB = 0$. Hence, $\mathcal{LM}_I^p(\mathcal{A})$ is a Banach algebra.

Proposition 2.5. *Let I be an infinite set and $\mathcal{A}$ be a Banach algebra such that $\mathcal{A}^2 \neq 0$. Then, for each $2 < p < \infty$, $\mathcal{LM}_I^p(\mathcal{A})$ is not an algebra.*

Proof. Since $\mathcal{A}^2 \neq 0$, then there exist $a, b \in \mathcal{A}$ such that $ab \neq 0$. Let $\{i_n\}_{n\in\mathbb{N}}$ be an infinite subset of distinct elements of I. Define the $I \times I$-matrix A over $\mathcal{A}$ by $A_{i_1 i_n} = \frac{1}{\sqrt{n}} a$ $(n \in \mathbb{N})$ and $A_{ij} = 0$, for other $i, j \in I$. Also, define the $I \times I$-matrix B over $\mathcal{A}$ by $B_{i_n i_1} = \frac{1}{\sqrt{n}} b$ $(n \in \mathbb{N})$ and $B_{ij} = 0$, for other $i, j \in I$. It is easy to see that $A, B \in \mathcal{LM}_I^p(\mathcal{A})$. But AB is not even well defined, since

$$(AB)_{i_1 i_1} = \sum_{n\in\mathbb{N}} A_{i_1 i_n} B_{i_n i_1} = \left(\sum_{n\in\mathbb{N}} \frac{1}{n} \right) ab.$$

$\square$

Proposition 2.6. *Let I be a set with at least two elements, and $\mathcal{A}$ be a unital Banach algebra. Then, $\mathcal{L}M_I^p(\mathcal{A})$ is a Banach algebra if and only if $1 \leq p \leq 2$.*

Proof. By Theorem 2.3, if $1 \leq p \leq 2$, then $\mathcal{L}M_I^p(\mathcal{A})$ is a Banach algebra. By Proposition 2.5, if I is infinite, and $2 < p < \infty$, then $\mathcal{L}M_I^p(\mathcal{A})$ is not a Banach algebra. Now, suppose I is finite. Let $i_1, i_2 \in I$ and $i_1 \neq i_2$. Define the $I \times I$-matrix A over $\mathcal{A}$ by $A_{i_1 i_1} = A_{i_1 i_2} = e_{\mathcal{A}}$ and $A_{ij} = 0$, for other $i, j \in I$. Also, define the $I \times I$-matrix B over $\mathcal{A}$ by $B_{i_1 i_1} = B_{i_2 i_1} = e_{\mathcal{A}}$ and $B_{ij} = 0$, for other $i, j \in I$. Then,

$$\|AB\|_p = 2 > 2^{\frac{2}{p}} = 2^{\frac{1}{p}} 2^{\frac{1}{p}} = \|A\|_p \|B\|_p,$$

and so $\|.\|_p$ is not an algebra norm. Hence, $\mathcal{L}M_I^p(\mathcal{A})$ is not a Banach algebra. $\square$

Remark 2.7. (a) Let I be finite and $\mathcal{A}$ be a Banach algebra with the unit $e_{\mathcal{A}}$. Suppose $Card(I) = m$. If $(A_1, \ldots, A_m) \in \mathcal{A}^m$, $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|(A_i)_i\|_1 = \|(A_i e_{\mathcal{A}})_i\|_1 \leq \|(A_i)_{i\in I}\|_p \|(e_{\mathcal{A}})_{i\in I}\|_q = m^{\frac{1}{q}} \|e_{\mathcal{A}}\| \|(A_i)_{i\in I}\|_p.$$

Thus, for arbitrary $I \times I$ matrices A, B on $\mathcal{A}$,

$$\begin{aligned}\|AB\|_p^p &= \sum_{i,j\in I}\left\|\sum_{k\in I} A_{ik}B_{kj}\right\|^p \leq \sum_{i,j\in I}\left(\sum_{k\in I}\|A_{ik}\|\|B_{kj}\|\right)^p \\ &= \sum_{i,j\in I}\left\|(\|A_{ik}\|)_k(\|B_{kj}\|)_k\right\|_1^p \leq \sum_{i,j\in I}\left\|(\|A_{ik}\|)_k\right\|_1^p\left\|(\|B_{lj}\|)_l\right\|_1^p \\ &\leq m^{\frac{2p}{q}}\|e_{\mathcal{A}}\|^{2p}\sum_{i,j\in I}\left\|(\|A_{ik}\|)_k\right\|_p^p\left\|(\|B_{lj}\|)_l\right\|_p^p = m^{\frac{2p}{q}}\|e_{\mathcal{A}}\|^{2p}\|A\|_p^p \\ &\quad \|B\|_p^p.\end{aligned}$$

Hence, $\|AB\|_p \leq m^{\frac{2}{q}}\|e_{\mathcal{A}}\|^2\|A\|_p\|B\|_p$.

(b) Let I be finite and $\mathcal{A}$ be a Banach algebra with the unit $e_{\mathcal{A}}$. Suppose $Card(I) = m$. By (a), it is easy to see that $(\mathcal{L}M_I^p(\mathcal{A}), m^{\frac{2}{q}}\|e_{\mathcal{A}}\|^2\|.\|_p)$ is a Banach algebra.

(c) Let I be finite and $\mathcal{A}$ be a Banach algebra with the unit $e_{\mathcal{A}}$. Suppose $Card(I) = m$. Define the norm $|||.|||$ on $\mathcal{A}$ by $|||a||| = C\|a\|$ ($a \in \mathcal{A}$), where $C \geq m^{\frac{2}{q}}\|e_{\mathcal{A}}\|^2$. Let $\widetilde{\mathcal{A}}$ denote the algebra $\mathcal{A}$ with the norm $|||.|||$ and A be an $I \times I$-matrix over $\widetilde{\mathcal{A}}$. Then,

$$\|A\|_{\mathcal{L}M_I^p(\widetilde{\mathcal{A}})} = C\|A\|_{\mathcal{L}M_I^p(\mathcal{A})}.$$

From this equality and (a), for each $A, B \in \mathcal{L}M_I^p(\widetilde{\mathcal{A}})$ we obtain,

$$\begin{aligned}\|AB\|_{\mathcal{L}M_I^p(\widetilde{\mathcal{A}})} &= C\|AB\|_{\mathcal{L}M_I^p(\mathcal{A})} \leq Cm^{\frac{2}{q}}\|e_{\mathcal{A}}\|^2\|A\|_{\mathcal{L}M_I^p(\mathcal{A})}\|B\|_{\mathcal{L}M_I^p(\mathcal{A})} \\ &= \frac{m^{\frac{2}{q}}\|e_{\mathcal{A}}\|^2}{C}\|A\|_{\mathcal{L}M_I^p(\widetilde{\mathcal{A}})}\|B\|_{\mathcal{L}M_I^p(\widetilde{\mathcal{A}})} \leq \|A\|_{\mathcal{L}M_I^p(\widetilde{\mathcal{A}})} \\ &\quad \|B\|_{\mathcal{L}M_I^p(\widetilde{\mathcal{A}})}.\end{aligned}$$

Therefore, $\mathcal{L}M_I^p(\widetilde{\mathcal{A}})$ is a Banach algebra.

Example 2.8. The algebra $\mathcal{A} = \mathbb{C}$ with the norm $\|A\| = 3|A|$ ($A \in \mathcal{A}$) is a Banach algebra with a the unit that is not unital (since $\|1\| = 3 \neq 1$). Then, by notations of Remark 2.7, $\mathcal{A} = \mathbb{C}$ with $C = 3$. Let $I = \{1, 2\}$. Since $C \geq 2^{2\frac{2}{3}}|1|$, then by remark 2.7, $\mathcal{L}M_I^3(\mathcal{A})$ is a Banach algebra. This example shows that we can not replace the condition "$\mathcal{A}$ is unital" by "$\mathcal{A}$ has a unit" in the Proposition 2.6.

Proposition 2.9. *Let G be a finite group with $Card(G) = m$, $1 < p < \infty$, and S be a Brandt semigroup over G with the index set I. Then, $\ell^p(S)$ is closed under convolution if and only if I is finite. Moreover, if I is finite, then there exists a constant C such that $\ell^p(S)$ with the product*

$$\delta_s * \delta_t = \delta_{st} \quad (s, t \in S),$$

and the norm $C\|.\|_p$ defines a Banach algebra. Also, $\ell^p(G)$ with the norm $C\|.\|_p$ is a Banach algebra under convolution, and $\ell^p(S)/\mathbb{C}\delta_0$ is an isometric Banach algebra-isomorphic with $\mathcal{LM}_I^p(\ell^p(G))$.

Proof. Suppose I is infinite. Let $\{i_n\}_{n\in\mathbb{N}}$ be an infinite subset of distinct elements of I. Let $f = \sum_{n=1}^{\infty} \frac{1}{n}\delta_{(e)_{i_1 i_n}}$, and $g = \sum_{n=1}^{\infty} \frac{1}{n}\delta_{(e)_{i_n i_1}}$. Clearly, $f, g \in \ell^p(S)$. But

$$f * g(0) = \sum_{m,n\in\mathbb{N}, m\neq n} \frac{1}{mn} = \infty.$$

Hence, $\ell^p(S)$ is not closed under convolution.

Suppose I is finite with $Card(I) = l$. It is easy to see that the Banach space $\ell^p(G)$ with the norm $\|.\|_{\ell^p(G)} = m^{1-\frac{1}{p}}\|.\|_p$ and the product

$$\delta_x * \delta_y = \delta_{xy} \quad (x, y \in G),$$

defines a convolution Banach algebra. Note that δ_e (e is the unit of G) is the unit of $\ell^p(G)$ with $\|\delta_e\|_{\ell^p(G)} = m^{1-\frac{1}{p}}$.

By Theorem 2.3, for $p \leq 2$, $\mathcal{LM}_I^p(\ell^p(G))$ defines a Banach algebra. In this case, let $C = m^{1-\frac{1}{p}}$ For $p > 2$, by Remark 2.7(c),

$|||.||| = l^{\frac{2}{q}}\|\delta_e\|^2_{\ell^p(G)}\|.\|_{\ell^p(G)} = l^{\frac{2}{q}}\|\delta_e\|^2_{\ell^p(G)}m^{1-\frac{1}{p}}\|.\|_p = l^{\frac{2}{q}}m^{3(1-\frac{1}{p})}\|.\|_p$.

Thus, $\mathcal{LM}_I^p(\ell^p(G))$ defines a Banach algebra. In this case, let $C = l^{\frac{2}{q}}m^{3(1-\frac{1}{p})}$. It is easy to see that for the mapping

$$\Phi : (\ell^p(S), C\|.\|_p) \longrightarrow \mathcal{LM}_I^p((\ell^p(G), C\|.\|_p)) : \ f \mapsto [f_{ij}],$$

$\Phi(\delta_s * \delta_t) = \Phi(\delta_s)\Phi(\delta_t)$. Hence, Φ is an algebra homomorphism. Therefore, by Proposition 2.2, the Banach algebra $(\ell^p(S), C\|.\|_p)/\mathbb{C}\delta_0$ is isometrically algebra isomorphic with $\mathcal{LM}_I^p((\ell^p(G), C\|.\|_p))$. □

Remark 2.10. By Proposition 5.6 of [3], for a Brandt semigroup S over a group G with an index set I, $\ell^1(S)/\mathbb{C}\delta_0$ is isometrically algebra isomorphic with $\mathcal{LM}_I(\ell^1(G))$.

3. Amenability of the Banach algebra $\mathcal{LM}_I^p(\mathcal{A})$ $(1 \leq p \leq 2)$ over a Banach algebra $\mathcal{A}$ with unit

Throughout this section, we suppose $\mathcal{A}$ has a unit which we denote by $e_{\mathcal{A}}$.

Lemma 3.1. *Let $\mathcal{A}$ be a Banach algebra with unit $e_{\mathcal{A}}$, and $1 \leq p \leq 2$. The following conditions are equivalent:*
(1) *$\mathcal{LM}_I^p(\mathcal{A})$ has a bounded approximate identity.*
(2) *I is finite.*

Proof. (1)⇒(2) Suppose on the contrary that I is infinite and $(E_\alpha)_\alpha$ is an approximate identity for $\mathcal{LM}_I^p(\mathcal{A})$. For every finite subset F of I, define E_F by $(E_F)_{ii} = e_{\mathcal{A}}$ if $i \in F$, $(E_F)_{ii} = 0$ if $i \in I - F$ and $(E_F)_{ij} = 0$ if $i \neq j$. Then,

$$\begin{aligned} (CardF)^{\frac{1}{p}} &= (\sum_{i \in F} \|e_{\mathcal{A}}\|^p)^{\frac{1}{p}} = \|E_F\|_p \\ &= \lim_\alpha \|E_F E_\alpha\|_p = \lim_\alpha (\sum_{i \in F, j \in I} \|(E_\alpha)_{ij}\|^p)^{\frac{1}{p}} \\ &\leq \liminf \|E_\alpha\|_p. \end{aligned}$$

Therefore, $\lim_\alpha \|E_\alpha\|_p = \infty$. Thus, $\mathcal{LM}_I^p(\mathcal{A})$ does not have a bounded approximate identity.

(2)⇒(1) Suppose I is finite. Then, it is clear that E_I is a unit for $\mathcal{LM}_I^p(\mathcal{A})$. □

Theorem 3.2. *Let $\mathcal{A}$ be a Banach algebra with a unit and $1 \leq p \leq 2$. The following conditions are equivalent:*
(i) *$\mathcal{LM}_I^p(\mathcal{A})$ is amenable.*
(ii) *$\mathcal{A}$ is amenable and I is finite.*

Proof. (i)⇒(ii): Since $\mathcal{LM}_I^p(\mathcal{A})$ is amenable, then by Proposition(2.2.1) of [5], $\mathcal{LM}_I^p(\mathcal{A})$ has a bounded approximate identity and by Lemma 3.1, I is a finite set. By Corollary 4 of Section 4 of [1], there exists an equivalent norm $|||.|||$ on $\mathcal{A}$ such that $\widetilde{\mathcal{A}} = (\mathcal{A}, |||.|||)$ is unital. Since I is finite, then the identity map $A \mapsto A$; $\mathcal{LM}_I^p(\mathcal{A}) \longrightarrow \mathcal{LM}_I(\widetilde{\mathcal{A}})$ is continuous. Indeed, if $c\|a\| \leq |||a||| \leq C\|a\|$ $(a \in \mathcal{A})$, then by Remark

2.7(a), $\|A\|_{\mathcal{LM}_I(\widetilde{\mathcal{A}})} \leq \frac{C}{c^2}(Card(I))^{\frac{2}{q}}\|e_{\mathcal{A}}\|^2\|A\|_{\mathcal{LM}_I^p(\mathcal{A})}$. So, $\mathcal{LM}_I^p(\mathcal{A})$ is Banach algebra isomorphic with $\mathcal{LM}_I(\widetilde{\mathcal{A}})$. Hence, $\mathcal{LM}_I(\widetilde{\mathcal{A}})$ is amenable, and so by Theorem 4.1 of [3], $\widetilde{\mathcal{A}}$ is amenable. Therefore, $\mathcal{A}$ is amenable.
(ii)⇒(i): We apply the notations of the above paragraph. By Theorem 4.1 of [3], $\mathcal{LM}_I(\widetilde{\mathcal{A}})$ is amenable. Hence, $\mathcal{LM}_I^p(\mathcal{A})$ is amenable. □

Remark 3.3. The above theorem remains valid, if we replace the condition "$1 \leq p \leq 2$" by "$\mathcal{LM}_I^p(\mathcal{A})$ is a Banach algebra".

Example 3.4. Let S be a Brandt semigroup over a finite group G with a finite index I. Then, by Proposition 2.9, Theorem 3.2, and Remark 3.3, the convolution Banach algebra $\ell^p(S)$ is amenable.

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References

[1] F. Bonsall, J. Duncan, Complete Normed Algebras, Springer-Verlag, New York, 1973.
[2] H. G. Dales, A. T. M. Lau D. Strauss, *Banach algebras on semigroups and their compactifications*, To appear in the Memoirs of AMS.
[3] G. H. Esslamzadeh, *Banach algebra structure and amenability of a class of matrix algebras with applications*, J. Funct. Anal. **161** (1999) 364-383.
[4] G. H. Esslamzadeh, *Ideals and representations of certain semigroup algebras*, Semigroup Forum **69** (2004) 51-56.
[5] V. Runde, Lectures on Amenability, Lecture Notes in Mathematics, Vol. **1774**, Springer, Berlin, 2002.

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