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SEQUENTIALLY COHEN-MACAULAY GRAPHS OF FORM $\theta_{n_1,\dots,n_k}$

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Abstract. Let k be an integer greater than 2 and $n_1, \dots, n_k$ be a sequence of positive integers with at most one of them being equal to 1. Let $\theta_{n_1,\dots,n_k}$ be a graph consisting of k paths, having only their endpoints in common. We characterize all sequentially Cohen-Macaulay graphs of this type. We also show for these types of graphs the notions of vertex decomposable, shellable and sequentially Cohen-Macaulay are equivalent.

1. Introduction

Let G be a finite simple graph. To G with vertex set $[n] = \{1, \dots, n\}$ and edge set $\mathrm{E}(G)$, one can associate an ideal $\mathcal{I}(G) \subset R = K[x_1, \dots, x_n]$, called the edge ideal of G, which is generated by all monomials $x_i x_j$ such that $\{i, j\} \in \mathrm{E}(G)$. Here, K is an arbitrary field. The independence complex Δ_G of a graph G is defined by

$$\Delta_G = \{A \subseteq V \mid A \text{ is an independent set in } G\},$$

where, A is an independent set in G if none of its elements are adjacent. Note that Δ_G is precisely the simplicial complex associated with $\mathcal{I}(G)$.

It is a well-known consequence of Menger's Theorem [5, Theorem 3.3.5] that each 3-connected graph has an induced subgraph of the form

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$\theta_{p,q,r}$, for some natural numbers p, q and r. This was our motivation to study sequentially Cohen-Macaulay graphs of the form $\theta_{n_1,\ldots,n_k}$.

A graded R-module M is called *sequentially Cohen-Macaulay* (over K) if there exists a finite filtration of graded R-modules,

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M,$$

such that each M_i/M_{i-1} is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing; that is,

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}).$$

A graph G is said to be sequentially Cohen-Macaulay, if $R/\mathcal{I}(G)$ is a sequentially Cohen-Macaulay R-module.

On the other hand, a simplicial complex Δ is called *shellable*, in the sense of Björner and Wachs [1], if the facets (maximal faces) of Δ can be ordered as $F_1, \ldots, F_s$ such that for all $1 \leqslant i < j \leqslant s$, there exists some $v \in F_j \setminus F_i$ and some $l \in \{1, \ldots, j-1\}$ with $F_j \setminus F_l = \{v\}$. A graph G is called shellable, if Δ_G is a shellable simplicial complex. In [12], Stanley showed that every shellable simplicial complex was sequentially Cohen-Macaulay, but the converse was not true.

Studying shellable or sequentially Cohen-Macaulay graphs has attracted significant attentions of researchers working in the borderline of combinatorial commutative algebra and algebraic combinatorics; see [1, 6, 7, 8, 10, 14, 16]. In [8], Francisco and Van Tuyl characterized all sequentially Cohen-Macaulay cycles. They showed that the n-cycle C_n was sequentially Cohen-Macaulay if and only if $n \in \{3, 5\}$ (see [8, Proposition 4.1]). In [6], Faridi showed that simplicial trees were sequentially Cohen-Macaulay. Moreover, in [10], sequentially Cohen-Macaulay cacti graphs (a cactus is a connected graph in which each edge belongs to at most one cycle) were characterized. In addition, in [14], Van Tuyl and Villarreal showed that a bipartite graph G was shellable if and only if it was sequentially Cohen-Macaulay (see [14, Theorem 3.8]).

Here, we determine all sequentially Cohen-Macaulay graphs of the form $\theta_{n_1,\ldots,n_k}$, where $\{n_1, \ldots, n_k\} \neq \{2, 5\}$. For $\{n_1, \ldots, n_k\} \neq \{2, 5\}$, we show in Theorem 2.6 that $\theta_{n_1,\ldots,n_k}$ is sequentially Cohen-Macaulay if and only if $\{1, 2\} \subseteq \{n_1, \ldots, n_k\}$ or $\{2, 3\} \subseteq \{n_1, \ldots, n_k\}$ or $\{n_1, \ldots, n_k\} = \{1, 4\}$. Moreover, as a result of this theorem, in Theorem 2.7 we show those graphs of the form $\theta_{n_1,\ldots,n_k}$, which satisfy each one of the latter relations, are sequentially Cohen-Macaulay if and only if they are shellable or vertex decomposable.

Finally, in Proposition 2.8, we show that for $\{n_1, \ldots, n_k\} = \{2, 5\}$, the graph $\theta_{n_1,\ldots,n_k}$ is not vertex decomposable. Therefore, we characterize all vertex decomposable graphs of the form $\theta_{n_1,\ldots,n_k}$ in Theorem 2.9. In Proposition 2.10, by direct computation, we show that for $k = 3$ and $\{n_1, \ldots, n_k\} = \{2, 5\}$, the graph $\theta_{n_1,\ldots,n_k}$ is not even sequentially Cohen-Macaulay. This result and computational evidences from some other examples lead us to conjecture that all graphs of the form $\theta_{n_1,\ldots,n_k}$, for which $\{n_1, \ldots, n_k\} = \{2, 5\}$, are not sequentially Cohen-Macaulay.

Characterizing vertex decomposable, shellable and sequentially Cohen-Macaulay graphs of the form $\theta_{n_1,\ldots,n_k}$ with [13, Lemma 2.4] and [14, Theorem 2.9] enable us to get more examples of vertex decomposable, shellable and sequentially Cohen-Macaulay graphs.

2. Sequentially Cohen-Macaulay graphs of the form $\theta_{n_1,\ldots,n_k}$

Let k be an integer greater than 1 and $n_1, \ldots, n_k$ be a sequence of positive integers. Let $\theta_{n_1,\ldots,n_k}$ be the graph constructed by k paths of length $n_1, \ldots, n_k$, with only their endpoints being in common. By length of a path, we mean the number of edges in the path. Since the graphs are assumed simple, at most one of the n_is in $\theta_{n_1,\ldots,n_k}$ can be equal to one. If $k = 2$, then $\theta_{n_1,\ldots,n_k}$ would be a cycle of length $n_1 + n_2$. The vertex decomposable and sequentially Cohen-Macaulay graphs of these types are completely studied in [8, 16]. Here, we assume $k > 2$ and characterize all vertex decomposable, shellable and sequentially Cohen-Macaulay graphs of the form $\theta_{n_1,\ldots,n_k}$.

Given a simplicial complex Δ on $[n]$, the *Alexander dual* complex $\Delta^\vee$ is defined by $\Delta^\vee = \{[n] \setminus F | F \notin \Delta\}$. Unless otherwise stated, when we discuss the Alexander dual $\Delta^\vee$ of a simplicial complex Δ, we assume that $[n] \setminus i \notin \Delta$, for all $i \in [n]$. Thus, $\Delta^\vee$ is again a simplicial complex on $[n]$.

Let $I = (x_{1,1} \cdots x_{1,s_1}, \ldots, x_{t,1} \cdots x_{t,s_t})$ be a square-free monomial ideal. The ideal

$$I^\vee = (x_{1,1}, \ldots, x_{1,s_1}) \cap \ldots \cap (x_{t,1}, \ldots, x_{t,s_t})$$

is called the Alexander dual of I. These two ideals are related in the following way. If I is the Stanley-Reisner ideal of a simplicial complex Δ, then the Stanley-Reisner ideal of its Alexander dual $\Delta^\vee$ is $I^\vee$.

Another related notion is componentwise linear ideals, introduced by Herzog and Hibi, to characterize sequentially Cohen-Macaulay ideals.

Let I be a graded ideal of R and let $I_{<d>}$ be the ideal generated by all homogeneous polynomials of degree d of I. A graded ideal I of R is called *componentwise linear* if $I_{<d>}$ has a linear resolution, for every d. Let I be a square-free monomial ideal in a polynomial ring. The ideal generated by the square-free monomials of degree d of I is denoted by $I_{[d]}$. Herzog and Hibi in [9, Proposition 1.5] showed that the square-free ideal I was componentwise linear if and only if $I_{[d]}$ had a linear resolution for every d.

Let G be a graph with vertex set $\mathrm{V}(G)$ and edge set $\mathrm{E}(G)$. A subset $C \subseteq \mathrm{V}(G)$ is a *minimal vertex cover* of G if: (1) every edge of G is incident with one vertex in C, and (2) there is no proper subset of C with the first property. In [8], Francisco and Van Tuyl showed that if $\mathcal{I}(G)$ was the ideal of a graph G, then

$$\mathcal{I}(G)^{\vee}_{[d]} = (\{x_{i_1} \cdots x_{i_d} | \{x_{i_1}, \ldots, x_{i_d}\} \text{ is a vertex cover of G of size } d\}).$$

In [9], Herzog and Hibi showed the following theorem to be used in the proof of Proposition 2.4.

Theorem A. Let I be a square-free monomial ideal in a polynomial ring. Then $I^{\vee}$ is componentwise linear if and only if R/I is sequentially Cohen-Macaulay.

Let $N(v)$ be the set of all adjacent vertices of v and let $N[v] = N(v) \cup \{v\}$. Vertex decomposability was introduced by Provan and Billera [11] in the pure case, and extended to the non-pure case by Björner and Wachs [2]. We will use the following definition of vertex decomposable graphs which is an interpretation of the definition of vertex decomposable for the independence complex of a graph, as stated in [13, 16].

Definition 2.1. The independence complex of G is vertex decomposable if G is a totally disconnected graph (with no edges), or if

- $G \setminus v$ and $G \setminus N[v]$ are both vertex decomposable, and
- No independent set in $G \setminus N[v]$ is a maximal independent set in $G \setminus v$.

A vertex v which satisfies in these conditions is called a shedding vertex.

The graph G is called vertex decomposable if its independence complex is vertex decomposable. It is known that the any vertex decomposable graph is shellable and so is sequentially Cohen-Macaulay (see [16]).

For characterizing vertex decomposable, shellable and sequentially Cohen-Macaulay graphs of the form $\theta_{n_1,\ldots,n_k}$, we have to distinguish among some cases, depending on $n_1, \ldots, n_k$, as follows.

Proposition 2.2. *If $\{1, 2\} \subseteq \{n_1, \ldots, n_k\}$, then $\theta_{n_1,\ldots,n_k}$ is vertex decomposable and so is shellable and sequentially Cohen-Macaulay.*

Proof. Two paths of length one and two form a triangle. Let v, u and w be its vertices such that $\deg(v) = 2$. The graphs $\theta_{n_1,\ldots,n_k} \setminus \{u\}$ and $\theta_{n_1,\ldots,n_k} \setminus N[u]$ are chordal and so they are vertex decomposable, by [16, Theorem 1]. For any independent set F in $\theta_{n_1,\ldots,n_k} \setminus N[u]$, $F \cup \{v\}$ is an independent set in $\theta_{n_1,\ldots,n_k} \setminus \{u\}$. Therefore, $\theta_{n_1,\ldots,n_k}$ fulfills the conditions of Definition 2.1, which completes the proof. □

Remark 2.3. If in the above proposition, one assumes $\{n_1, \ldots, n_k\} = \{1, 2\}$, then the associated graph, $\theta_{n_1,\ldots,n_k}$, is chordal. These types of graphs are known to be vertex decomposable, by [16, Theorem 1].

A chordless path in a graph G is a path $v_1, v_2, \ldots, v_k$ in G with no edge v_iv_j with $j \neq i + 1$. A simplicial k-path in G is a chordless path $v_1, v_2, \ldots, v_k$ which cannot be extended on both endpoints to a chordless path $v_0, v_1, \ldots, v_k, v_{k+1}$ in G.

Proposition 2.4. *Let $\{2, 3\} \subseteq \{n_1, \ldots, n_k\}$. Then, $\theta_{n_1,\ldots,n_k}$ is vertex decomposable and consequently shellable and sequentially Cohen-Macaulay.*

Proof. Let $P_1 : u, x, v$ and $P_2 : u, y, z, v$ be two paths of length two and three in $\theta_{n_1,\ldots,n_k}$. Since the path $P : x, u, y$ is a simplicial 3-path, which is not a subgraph of any chordless C_4, by [16, Lemma 4.3] we deduce that G is vertex decomposable. □

Proposition 2.5. *Let $\{n_1, \ldots, n_k\} = \{1, 4\}$. Then, $\theta_{n_1,\ldots,n_k}$ is vertex decomposable and consequently shellable and sequentially Cohen-Macaulay.*

Proof. Each cycle other than C_5 in $\theta_{n_1,\dots,n_k}$ has a chord and so, by [16, Theorem 1], it is vertex decomposable. □

The following theorem is one of the main results of this paper which characterizes all sequentially Cohen-Macaulay graphs of the form $\theta_{n_1,\dots,n_k}$, where $\{n_1,\dots,n_k\} \neq \{2,5\}$.

Theorem 2.6. *Let $n_1,\dots,n_k \neq \{2,5\}$. Then, $\theta_{n_1,\dots,n_k}$ is sequentially Cohen-Macaulay if and only if one of the following holds:*

(1) $\{1,2\} \subseteq \{n_1,\dots,n_k\}$.
(2) $\{2,3\} \subseteq \{n_1,\dots,n_k\}$.
(3) $\{1,4\} = \{n_1,\dots,n_k\}$.

Proof. "If". Suppose that one of (1) to (3) holds. Then, by Proposition 2.2, Proposition 2.4 and Proposition 2.5, the result holds.

"Only if". Let $G = \theta_{n_1,\dots,n_k}$ be a sequentially Cohen-Macaulay graph. The proof is by induction on k. If $k = 2$, then the graph is a cycle and so the result holds by [8, Proposition 4.1]. Let $k > 2$, $n_1 \leq \cdots \leq n_k$ and $P_i : x, x_{i,1}, \dots, x_{i,n_i-1}, y$, for $1 \leq i \leq k$, be the paths which construct G. If $n_t \geqslant 6$, for some $t \geq 3$, then

$$H = G \setminus \bigcup_{i=t}^{k} (N[x_{i,2}] \cup N[x_{i,n_i-2}])$$

has a component of the form $\theta_{n_1,\dots,n_{t-1}}$. So, by the induction hypothesis, (1) or (2) or (3) holds, for $\theta_{n_1,\dots,n_{t-1}}$. If (1) or (2) holds for $\theta_{n_1,\dots,n_{t-1}}$, then this holds, for $\theta_{n_1,\dots,n_k}$. Let (3) holds for $\theta_{n_1,\dots,n_{t-1}}$, but $\{n_1,\dots,n_k\} \neq \{1,4\}$. Let $S = \{j; n_j = 4\}$ and $H' = G \setminus \bigcup_{j\in S} N[x_{j,2}]$. Since $n_2 = 4$, then H' has no path of length two, three and four. By the induction hypothesis, H' is not sequentially Cohen-Macaulay, which is a contradiction by [14, Theorem 3.3].

So, we can assume that $n_k < 6$. Since G has no vertex of degree one, it is not a bipartite graph by [14, Lemma 2.8]. Therefore, for $n_k = 2$, we have $n_1 = 1$ and so (1) holds. Similarly, If $n_k = 3$, then $n_i = 2$, for some i, and so (2) holds. If $n_k = 4$, then $G \setminus N[x_{k,2}]$ is $\theta_{n_1,\dots,n_{k-1}}$. If (1), (2) or (3) holds, for $\theta_{n_1,\dots,n_{k-1}}$, then the similar statement holds for G. So, assume that $n_k = 5$. Since G is not bipartite, for some i we have $n_i = 2$ or 4. If $n_i = 4$ for some i, then $H = G \setminus N[x_{i,2}]$ is sequentially Cohen-Macaulay and so (1) or (2) holds, which completes the result.

Otherwise, the assumption $\{n_1,\dots,n_k\} \neq \{2,5\}$ shows that $n_j = 1$ or 3, for some j, and so (1) or (2) holds. □

Recently, Van Tuyl showed that in bipartite graphs, the three concepts vertex decomposability, shellability and sequentially Cohen-Macaulayness are equivalent; see [13, Theorem 2.10]. Using the proof of the above theorem, we have the same property for $\theta_{n_1,\dots,n_k}$, where $\{n_1,\dots,n_k\} \neq \{2,5\}$.

Theorem 2.7. *Let $n_1,\dots,n_k \neq \{2,5\}$. Then, the followings are equivalent:*

(i) $\theta_{n_1,\dots,n_k}$ *is sequentially Cohen-Macaulay.*
(ii) $\theta_{n_1,\dots,n_k}$ *is shellable.*
(iii) $\theta_{n_1,\dots,n_k}$ *is vertex decomposable.*

Proof. Note that (iii) ⇒ (ii) ⇒ (i) always holds for any graph. It is enough to show that for these type of graphs, (i) ⇒ (iii). Let $\theta_{n_1,\dots,n_k}$ be a sequentially Cohen-Macaulay graph. Then, Theorem 2.6 shows that $\theta_{n_1,\dots,n_k}$ satisfies one of the relations of Theorem 2.6. Therefore, by Proposition 2.2, Proposition 2.4 and Proposition 2.5, we deduce that $\theta_{n_1,\dots,n_k}$ is vertex decomposable. □

In the following, we consider the case $\{n_1,\dots,n_k\} = \{2,5\}$.

Proposition 2.8. *Let $\{n_1,\dots,n_k\} = \{2,5\}$. Then, $\theta_{n_1,\dots,n_k}$ is not vertex decomposable.*

Proof. Let $P_1,\dots,P_s$ be the paths of length two in $G = \theta_{n_1,\dots,n_k}$ and $P_{s+1},\dots,P_k$ be the paths of length five in G. Consider the labeling for G such that $P_j:\ u,\alpha_j,v$, for $1 \le j \le s$, and $P_j:\ u,x_{j,1},x_{j,2},x_{j,3},x_{j,4},v$, for $s+1 \le j \le k$. We claim that no vertex of G is a shedding vertex to deduce that G is not vertex decomposable. For any $s+1 \le j \le k$, the independent set $\{u,x_{s+1,4},\dots,x_{k,4}\}$ is maximal in both graphs $G \setminus x_{j,2}$ and $G \setminus N[x_{j,2}]$. For the other vertices of G, the similar arguments hold. Therefore, G is not vertex decomposable. □

Proposition 2.8 and Theorem 2.6 imply the following characterization of the vertex decomposable graphs of the form $\theta_{n_1,\dots,n_k}$.

Theorem 2.9. *Let $n_1,\dots,n_k$ be a sequence of positive integers. Then, $\theta_{n_1,\dots,n_k}$ is vertex decomposable if and only if one of the followings holds:*

(1) $\{1, 2\} \subseteq \{n_1, \ldots, n_k\}$.
(2) $\{2, 3\} \subseteq \{n_1, \ldots, n_k\}$.
(3) $\{1, 4\} = \{n_1, \ldots, n_k\}$.

The next result extends Proposition 2.8 to show that for $k = 3$, those graphs are not even sequentially Cohen-Macaulay.

Proposition 2.10. *The graphs $\theta_{2,2,5}$ and $\theta_{2,5,5}$ are not sequentially Cohen-Macaulay.*

Proof. Consider the labeling for $\theta_{2,2,5}$ and $\theta_{2,5,5}$ as given in Figure 1 and Figure 2. By [8, Lemma 2.3], the minimal generators of $\mathcal{I}(\theta_{2,2,5})^\vee$, correspond to the minimal vertex covers of $\theta_{2,2,5}$ and these minimal vertex covers correspond precisely to minimal prime ideals of $\mathcal{I}(\theta_{2,2,5})$. Therefore, by finding the minimal prime ideals of $\mathcal{I}(\theta_{2,2,5})$, the monomials $x_1x_2x_4x_6, x_1x_3x_4x_6, x_2x_4x_6x_7x_8, x_1x_3x_5x_6, x_2x_4x_5x_7x_8, x_2x_3x_5x_7x_8,$ $x_1x_3x_5x_7x_8$, generate the ideal $\mathcal{I}(\theta_{2,2,5})^\vee$. With computation by CoCoA, we see that $\mathcal{I}(\theta_{2,2,5})^\vee_{[5]}$ has the minimal graded free resolution as

$$0 \to R^3(-8) \to R^{12}(-7)(+)R(-8) \to R^{23}(-6) \to R^{14}(-5) \to R.$$

Thus, it does not have a linear resolution. Therefore, $\theta_{2,2,5}$ is not sequentially Cohen-Macaulay, by Theorem A.

Similarly, the minimal prime ideals of $\mathcal{I}(\theta_{2,5,5})$ generate the ideal $\mathcal{I}(\theta_{2,5,5})^\vee$. By computation, we deduce that $\mathcal{I}(\theta_{2,5,5})^\vee_{[7]}$ has the minimal graded free resolution as:

$$\cdots \to R^{55}(-10)(+)R(-11) \to R^{121}(-9) \to R^{124}(-8) \to R^{49}(-7) \to R.$$

Thus, $\mathcal{I}(\theta_{2,5,5})^\vee_{[7]}$ does not have a linear resolution and so $\mathcal{I}(\theta_{2,5,5})^\vee$ is not componentwise linear. Therefore, $\theta_{2,5,5}$ is not sequentially Cohen-Macaulay by Theorem A.

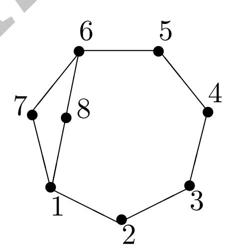

Figure 1

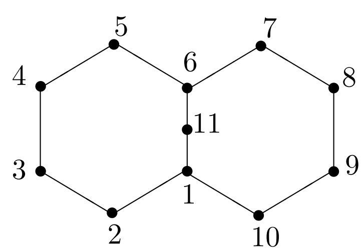

Figure 2

□

In view of Proposition 2.8 and Proposition 2.10, we conjecture that the answer to the following questions is positive.

Question 2.11. Let $K > 2$ and $\{n_1, \ldots, n_k\} = \{2, 5\}$. Is $\theta_{n_1,\ldots,n_k}$ not shellable? Is $\theta_{n_1,\ldots,n_k}$ not sequentially Cohen-Macaulay?

Theorem 2.6 with [14, Theorem 2.9] enable us to get more examples of shellable and sequentially Cohen-Macaulay graphs.

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