

ON THE RESTRICTION OF CHARACTERS OF SPECIAL LINEAR GROUPS OF DIMENSION THREE

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ABSTRACT. Let G be the special linear group $SL(3, q)$, where q is power of a prime p . Here, we show that χ_P has a linear constituent with multiplicity one for each irreducible character χ and Sylow p -subgroup P of G . Furthermore, if $\text{cf}(G)$ is the vector space of class functions of G , we show that the restriction of a subset of irreducible characters of G on P is a basis for the vector space of class functions defined on P spanned by $\{\phi_P \mid \phi \in \text{cf}(G)\}$.

1. Introduction

Steinberg asserts, in particular, that for any finite Chevalley group G , each nonprincipal linear character of a maximal unipotent subgroup H (a Sylow p -subgroup where p is the characteristic of G) of G is a constituent of χ_H with multiplicity at most 1 for every irreducible character χ of G [8, Theorem 49]. Moreover, in an earlier work, Gel'fand and Graev [2] showed the same results for groups $SL(n, q)$ for arbitrary n with a particular attention to the case $n = 3$. If q is a power of a prime p , by constructing the primitive central idempotents of the complex group algebra $\mathbb{C}G = \mathbb{C}SL(3, q)$, Guzel [3] shows that the restriction of χ to a Sylow p -subgroup of G has a linear constituents with multiplicity 1, for

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each irreducible character χ of G . This result has been referred by the author of this manuscript in [1] without any details.

Here, we provide another explicit proof of this result by using values of the irreducible characters of G on p -elements and writing them down as *integral* linear combinations of some specific characters.

Theorem 1.1. *Let $G = \text{SL}(3, q)$, where $q > 2$ is a power of a prime p . Let P be a Sylow p -subgroup of G . Then, for all irreducible characters χ of G , there exists a linear character φ of P such that $\langle \chi_P, \varphi \rangle = 1$.*

In the following section, we describe the structure of conjugacy classes and irreducible characters of G and their restrictions to the Sylow p -subgroup P . Section 3 contains the proof of Theorem 1.1. Finally, in section 4 we conclude that a subset of restricted irreducible characters of G on P is a basis for the vector space of class functions defined on P and spanned by $\{\phi_P \mid \phi \in \text{cf}(G)\}$, where $\text{cf}(G)$ is the vector space of class functions of G .

2. Structure of characters

The special linear group $G = \text{SL}(3, q)$, where q is a power of a prime p , of dimension 3 over the finite field $\mathbb{F}_q = \text{GF}(q)$, is the set of all nonsingular 3×3 matrices with determinant 1.

Let $\text{LT}(a, b, c)$ denote a 3×3 lower triangular matrix with diagonal entries being 1 and the entries at the positions $(2, 1)$, $(3, 1)$ and $(3, 2)$ being a, b and c , respectively. The set P of all matrices $\text{LT}(a, b, c)$ with $a, b, c \in \mathbb{F}_q$ is a Sylow p -subgroup of G of order q^3 . We use the character values of G restricted to P to show that for each irreducible character χ of G there exists a linear character φ of P such that $\langle \chi_P, \varphi \rangle = 1$.

The conjugacy classes and the character table of G are given in [7]. We use notations defined in [7]. We shall use that table to get the values of characters on the different conjugacy classes of G which contain the elements of P .

Table 1 is a part of Table 1a of [7] that shows the structure of conjugacy classes of G which contain some elements of the Sylow p -subgroup P . Let $d = \gcd(3, q - 1)$, ω be a cube root of unity and $\epsilon^3 \neq 1$, for $\epsilon \in \text{GF}(q)$.

Based on the structure of the elements of P and the fact that ω is a cube root of unity, the elements of P are contained only in the conjugacy classes $\mathcal{C}_1^{(0)}$, $\mathcal{C}_2^{(0)}$ and $\mathcal{C}_3^{(0,l)}$ of G . The centre $Z(P) = \{\text{LT}(0, z, 0) \mid z \in$

$\mathbb{F}_q\}$ is an elementary abelian p -group of order q . Using the canonical representative elements of conjugacy classes $\mathcal{C}_1^{(0)}$, $\mathcal{C}_2^{(0)}$ and $\mathcal{C}_3^{(0,l)}$, we see that the minimal polynomials of elements of these conjugacy classes have degrees 1, 2 and 3, respectively. The minimal polynomials of nontrivial elements of $Z(P)$ have degree 2, and so nontrivial elements of $Z(P)$ are contained in the conjugacy class $\mathcal{C}_2^{(0)}$.

TABLE 1: Conjugacy classes of $\text{SL}(3, q)$
which contain elements of the Sylow p -subgroup P for $d = 1, 3$.

Conjugacy class	Canonical representative	Parameters
$\mathcal{C}_1^{(k)}$	$\begin{pmatrix} \omega^k & 0 & 0 \\ 0 & \omega^k & 0 \\ 0 & 0 & \omega^k \end{pmatrix}$	$0 \leq k \leq (d-1)$
$\mathcal{C}_2^{(k)}$	$\begin{pmatrix} \omega^k & 0 & 0 \\ 1 & \omega^k & 0 \\ 0 & 0 & \omega^k \end{pmatrix}$	$0 \leq k \leq (d-1)$
$\mathcal{C}_3^{(k,l)}$	$\begin{pmatrix} \omega^k & 0 & 0 \\ \epsilon^l & \omega^k & 0 \\ 0 & \epsilon^l & \omega^k \end{pmatrix}$	$0 \leq k, l \leq (d-1)$

The following lemma gives some properties of P .

Lemma 2.1. *Suppose $G = \text{SL}(3, q)$, where q is a power of a prime p . If P is a Sylow p -subgroup of G , then we have:*

- (1) P has $q^2 + q - 1$ conjugacy classes.
- (2) P has q^2 linear characters and $q - 1$ non-linear characters of degree q such that their values on nontrivial elements of $Z(P)$ are 1 and $q\omega^i$, for some $1 \leq i \leq p$, respectively, where ω is a primitive p^{th} root of unity.
- (3) If τ is an irreducible character of degree q of P , then $\tau(x) = 0$, for $x \notin Z(P)$, and $\sum_{1 \neq z \in Z(P)} \tau(z) = -q$.

Proof. First of all we show $P/Z(P)$ is abelian. Let $x, y \in P$. It is enough to show $x^{-1}y^{-1}xy \in Z(P)$. Let $x = \text{LT}(a, b, c)$ and $y =$

$\text{LT}(d, e, f)$. Then, $x^{-1}y^{-1}xy = \text{LT}(0, af - dc, 0)$. Hence, $P/Z(P)$ is abelian and $P' \subseteq Z(P)$, where P' is the derived subgroup of P . Conversely, if $z = \text{LT}(0, t, 0) \in Z(P)$, then $z = x^{-1}y^{-1}xy \in P'$, where $x = \text{LT}(t, b, c)$ and $y = \text{LT}(0, 1, e)$, for $b, c, e \in \mathbb{F}_q$. Therefore, $P' = Z(P)$.

Now, suppose $h = \text{LT}(h_1, h_2, h_3) \in P \setminus Z(P)$ so that at least one of h_1, h_3 is not 0. Then $x^{-1}hx = h^x = \text{LT}(h_1, h_1c - ah_3 - h_2, h_3)$.

As x runs over P , $h_1c - ah_3 - h_2$ runs over \mathbb{F}_q . Thus, the conjugacy class $\{h^x \mid x \in P\}$ has order q . Therefore, each conjugacy class of P has order 1 or q and P has q single element conjugacy classes, since $|Z(P)| = q$. If n is the number of conjugacy classes of order q , then $|P| = (q \times 1) + (n \times q)$ and so $n = q^2 - 1$. Thus, P has $q^2 + q - 1$ conjugacy classes.

Since $|P : P'| = q^2$, then P has q^2 linear characters and since the number of conjugacy classes of P is $q^2 + q - 1$, then P has $q - 1$ non-linear characters. Let τ be a non-linear irreducible character of P . Since $Z(P) \subseteq Z(\tau)$ and by [5, Corollary 2.30],

$$(2.1) \quad \tau^2(1) \leq |P : Z(\tau)| \leq |P : Z(P)| = q^2,$$

then $\tau(1) \leq q$. On the other hand, the number of conjugacy classes of P is $q^2 + q - 1$ and the order of P is q^3 , and thus

$$q^3 = |P| = \sum_{i=1}^{q^2} \varphi_i(1)^2 + \sum_{j=1}^{q-1} \tau_j(1)^2,$$

where φ_i and τ_j are linear and non-linear irreducible characters of P , respectively. Since $\tau_j(1) \leq q$, then $\tau_j(1) = q$ and (2.1) implies $Z(P) = Z(\tau)$. Since $P' = Z(P)$, then the value of all linear characters of P on $Z(P)$ is 1. Also, for an irreducible character τ of degree q , if ρ is the representation which affords τ , then $\rho(z)$ is a scalar for all $1 \neq z \in Z(P)$ and thus $\tau(z) = q\omega^j$, for some $1 \leq j \leq p$, where ω is a primitive p^{th} -root of unity.

Since $\tau^2(1) = q^2 = |P : Z(P)|$, [5, Corollary 2.30] shows that $\tau(x) = 0$, for all $x \notin Z(P)$. Using the first orthogonality relation, we get

$$\frac{1}{|P|} \sum_{x \in P} \tau(x) 1(x^{-1}) = \frac{1}{|P|} \sum_{x \in P} \tau(x) = \frac{1}{|P|} \sum_{z \in Z(P)} \tau(z) = 0.$$

Therefore, $\tau(1) = q$ implies

$$(2.2) \quad \sum_{1 \neq z \in Z(P)} \tau(z) = -q,$$

and this completes the proof. \square

The following lemmas are simple consequences of Clifford's Theorem [5, Theorem 6.2] and Frattini's argument [6, Lemma 1.13].

Lemma 2.2. *Let H be a subgroup of any group G , $x \in N_G(H)$ and ϑ and ψ be characters of H . Then, $\langle \vartheta^x, \psi^x \rangle = \langle \vartheta, \psi \rangle$. In particular, taking $\psi = \vartheta$, ϑ^x is irreducible if and only if ϑ is irreducible.*

Lemma 2.3. *Let G be a normal subgroup of a group L and H be a Sylow p -subgroup of G . Let χ and ϑ be irreducible characters of G and H , respectively. Let $l \in L$. Then,*

$$\langle \chi_H, \vartheta \rangle = \langle \chi_H^l, \vartheta^x \rangle \text{ for some } x \in N_L(H).$$

In particular, $\langle \chi_H, \mathbf{1} \rangle = \langle \chi_H^l, \mathbf{1} \rangle$.

Tables 2 and 3 show the values of the restriction of the irreducible characters of the groups $\text{SL}(3, q)$ on the elements of Sylow p -subgroup P when $d = 1$ and $d = 3$, respectively (see [7] Table 1b).

Lemma 2.4. *Let $G = \text{SL}(3, q)$, where $q > 2$ is a power of a prime p and let P be the Sylow p -subgroup of G and ψ be the irreducible character of degree $q^2 + q$ of G . Then,*

- (1) $\langle \psi_P, \mathbf{1} \rangle = 2$.
- (2) $\langle \psi_P, \tau \rangle = 1$, for each irreducible character τ of degree q of P .
- (3) *There exist some non-principal linear characters φ and ϕ of P such that $\langle \psi_P, \varphi \rangle = 0$ and $\langle \psi_P, \phi \rangle = 1$.*

Proof. Suppose $x = \text{LT}(a, b, c) \in P$ is contained in the conjugacy class $\mathcal{C}_2^{(0)}$ of G . Since each element in $\mathcal{C}_2^{(0)}$ has a minimal polynomial of degree 2, $(x - 1)^2 = \text{LT}(0, ac, 0) = 0$. This, together with $x \notin Z(P)$, implies $a = 0$ or $c = 0$ but not both. Therefore the number of possibilities for the elements x with the above properties is $2q(q - 1)$. The elements of $Z(P)$ are also contained in $\mathcal{C}_2^{(0)}$ and the values of ψ on $\mathcal{C}_1^{(0)}$, $\mathcal{C}_2^{(0)}$ and $\mathcal{C}_3^{(0,0)}$ are $q^2 + q$, q and 0, respectively. Thus, we have

$$\begin{aligned} \langle \psi_P, \mathbf{1} \rangle &= \frac{1}{|P|} \sum_{x \in P} \psi_P(x) \mathbf{1}(x) \\ &= \frac{1}{q^3} (\psi_P(1) + \sum_{1 \neq z \in Z(P)} \psi_P(z) + \sum_{z \notin Z(P)} \psi_P(z)) \end{aligned}$$

$$= \frac{1}{q^3}((q^2 + q) + (q - 1)q + 2q(q - 1)q) = 2.$$

This proves the first assertion.

Now, suppose τ is an irreducible character of degree q of P . By using Table 2 for the value of ψ on the conjugacy class $\mathcal{C}_2^{(0)}$ of G which contains the elements of $Z(P)$ and Lemma 2.1, we have

$$\begin{aligned} \langle \psi_P, \tau \rangle &= \frac{1}{|P|} \sum_{x \in P} \psi_P(x) \overline{\tau(x)} \\ &= \frac{1}{q^3} (\psi_P(1)\tau(1) + \sum_{1 \neq z \in Z(P)} \psi_P(z) \overline{\tau(z)} + \sum_{z \notin Z(P)} \psi_P(z) \overline{\tau(z)}) \\ &= \frac{1}{q^3} ((q^2 + q)q - q^2 + 0) = 1, \end{aligned}$$

where $\overline{\tau(x)}$ is the complex conjugate of the value $\tau(x)$. Therefore, for each irreducible character τ of degree q of P ,

$$\langle \psi_P, \tau \rangle = 1,$$

as claimed.

Now, since $\langle \psi_P, \tau \rangle = 1$ for each irreducible character τ of degree q of P , then $\psi_P = \sum_{i=1}^{q-1} \tau_i + \sum_{j=1}^t m_j \phi_j$, where the ϕ_j are linear characters of P with the multiplicities m_j . Since $\psi(1) = q^2 + q$ and $\sum_{i=1}^{q-1} \tau_i(1) = q^2 - q$, we have $\sum_{j=1}^t m_j \phi_j(1) = 2q$. Since P possesses $q^2 - 1$ non-principal linear characters, there exists at least one non-principal linear character φ such that $\langle \psi_P, \varphi \rangle = 0$.

By the first assertion, $\langle \psi_P, \mathbf{1} \rangle = 2$. Hence, $\sum_{j=1}^t m_j \phi_j(1) = 2q - 2 > 1$, where \sum' runs over $\phi_j \neq \mathbf{1}$. This means there exists some non-principal linear character ϕ of P such that $\langle \psi_P, \phi \rangle \neq 0$. Note that $(\nu_t)_P = \rho_P - \psi_P + \mathbf{1}$ is a character of P and that ρ_P is the regular character of P . It follows that any nonprincipal linear constituent ϕ of ψ_P has multiplicity 1. This completes the proof. \square

By the values of characters ω_m and γ_n on the conjugacy classes $\mathcal{C}_1^{(0)}$, $\mathcal{C}_2^{(0)}$ and $\mathcal{C}_3^{(0,l)}$ in the Table 1b of [7], we have

$$(2.3) \quad \{(\omega_1)_P, (\omega_2)_P, (\omega_3)_P\} = \{(\gamma_1)_P, (\gamma_2)_P, (\gamma_3)_P\}.$$

TABLE 2: VALUES OF CHARACTERS OF $SL(3, q)$ ON ELEMENTS OF P WHEN $d = 1$, WHERE $1 \leq i, j \leq q - 2$, $1 \leq r \leq (q^2 - 5q + 6)/6$, $1 \leq s \leq (q^2 - q)/2$ AND $1 \leq t \leq (q^2 + q)/3$.

	$C_1^{(0)}$	$C_2^{(0)}$	$C_3^{(0,0)}$
1	1	1	1
ψ	$q^2 + q$	q	0
ρ	q^3	0	0
ζ_i	$q^2 + q + 1$	$q + 1$	1
η_j	$q^3 + q^2 + q$	q	0
ε_r	$q^3 + 2q^2 + 2q + 1$	$2q + 1$	1
μ_s	$q^3 - 1$	-1	-1
ν_t	$q^3 - q^2 - q + 1$	$1 - q$	1

TABLE 1: VALUES OF CHARACTERS OF $SL(3, q)$ ON ELEMENTS OF P WHEN $d = 3$,WHERE $1 \leq i, j \leq q - 2$, $1 \leq r \leq (q^2 - 5q + 4)/6$, $1 \leq s \leq (q^2 - q)/2$, $1 \leq t \leq (q^2 + q - 2)/3$ AND $1 \leq k, m, n \leq 3$.

	$C_1^{(0)}$	$C_2^{(0)}$	$C_3^{(0,l)}$
1	1	1	1
ψ	$q^2 + q$	q	0
ρ	q^3	0	0
ζ_i	$q^2 + q + 1$	$q + 1$	1
η_j	$q^3 + q^2 + q$	q	0
θ_k	$(q^3 + 2q^2 + 2q + 1)/3$	$(2q + 1)/3$ OR $(1 - q)/3$	$(2q + 1)/3$ OR $(1 - q)/3$
ε_r	$q^3 + 2q^2 + 2q + 1$	$2q + 1$	1
μ_s	$q^3 - 1$	-1	-1
ν_t	$q^3 - q^2 - q + 1$	$1 - q$	1
ω_m	$(q^3 - q^2 - q + 1)/3$	$(1 - q)/3$ OR $(2q + 1)/3$	$(1 - q)/3$ OR $(2q + 1)/3$
γ_n	$(q^3 - q^2 - q + 1)/3$	$(1 - q)/3$ OR $(2q + 1)/3$	$(1 - q)/3$ OR $(2q + 1)/3$

3. Proof

Proof of Theorem 1. By Table 2, the characters ρ and ψ have degrees q^3 and $q^2 + q$, respectively. Now, if we restrict them to P we see that

for all nontrivial $x \in P$, we have $\rho_P(x) = 0$ and $\psi_P(x) = q$ or 0 . Thus, from the values of the other characters of G on P , we get

$$(3.1) \quad (\zeta_i)_P = \psi_P + \mathbf{1}$$

$$(3.2) \quad (\eta_j)_P = \rho_P + \psi_P$$

$$(3.3) \quad (\varepsilon_r)_P = \rho_P + 2\psi_P + \mathbf{1}$$

$$(3.4) \quad (\mu_s)_P = \rho_P - \mathbf{1}$$

and

$$(3.5) \quad (\nu_t)_P = \rho_P - \psi_P + \mathbf{0}$$

Since $\rho(1) = q^3$ is the order of P and $\rho_P(x) = 0$, for all $x \neq 1$ in P , thus ρ_P is the regular character of P and $\rho_P = \sum_{v \in \text{Irr}(P)} v(1)v$. On the other hand, by Lemma 2.4 there exists a non-principal linear character φ of P such that $\langle \psi_P, \varphi \rangle = 0$. Then, since $\langle \rho_P, \varphi \rangle = 1$, we have

$$\begin{aligned} \langle (\eta_j)_P, \varphi \rangle &= \langle \rho_P + \psi_P, \varphi \rangle = 1, \\ \langle (\varepsilon_r)_P, \varphi \rangle &= \langle \rho_P + 2\psi_P + \mathbf{1}, \varphi \rangle = 1, \\ \langle (\mu_s)_P, \varphi \rangle &= \langle \rho_P - \mathbf{1}, \varphi \rangle = 1 \end{aligned}$$

and

$$\langle (\nu_t)_P, \varphi \rangle = \langle \rho_P - \psi_P + \mathbf{1}, \varphi \rangle = 1.$$

Also, by Lemma 2.4 there exists a non-principal linear character ϕ of P such that $\langle \psi_P, \phi \rangle = 1$. Thus,

$$\langle (\zeta_i)_P, \phi \rangle = \langle \psi_P + \mathbf{1}, \phi \rangle = 1.$$

For the case that $d = 3$, the only remaining characters to consider are θ_k, ω_m and γ_n , for $1 \leq k, m, n \leq 3$.

Using the Frobenius reciprocity, we have

$$\langle \eta_j, \varphi^G \rangle = \langle \varepsilon_r, \varphi^G \rangle = \langle \mu_s, \varphi^G \rangle = \langle \nu_t, \varphi^G \rangle = 1,$$

and $\langle \zeta_i, \varphi^G \rangle = 0$. Also, if

$$\langle (\theta_k)_P, \varphi \rangle = K_k, \langle (\omega_m)_P, \varphi \rangle = M_m \text{ and } \langle (\gamma_n)_P, \varphi \rangle = N_n,$$

then

$$\langle \theta_k, \varphi^G \rangle = K_k, \langle \omega_m, \varphi^G \rangle = M_m \text{ and } \langle \gamma_n, \varphi^G \rangle = N_n,$$

for $1 \leq k, m, n \leq 3$. Therefore, if we induce φ to G , we get

$$\varphi^G = \rho + (q-2)\eta_j + ((q^2-5q+4)/6)\varepsilon_r + ((q^2-q)/2)\mu_s$$

$$+((q^2 + q - 2)/3)\nu_t + \sum_{k=1}^3 K_k \theta_k + \sum_{m=1}^3 M_m \omega_m + \sum_{n=1}^3 N_n \gamma_n.$$

Using the fact that $\varphi^G(1) = |G : P|\varphi(1)$, we calculate the value at 1 and simplifying the above equation, we have

$$|G : P| = -q^2 - 2q^3 + q^5 + \sum_{k=1}^3 K_k \theta_k(1) + \sum_{m=1}^3 M_m \omega_m(1) + \sum_{n=1}^3 N_n \gamma_n(1).$$

Since $|G : P| = q^5 - q^3 - q^2 + 1$, we get

$$\sum_{k=1}^3 K_k \theta_k(1) + \sum_{m=1}^3 M_m \omega_m(1) + \sum_{n=1}^3 N_n \gamma_n(1) = q^3 + 1.$$

Since

$$\theta_k(1) = (q^3 + 2q^2 + 2q + 1)/3$$

and

$$\omega_m(1) = \gamma_n(1) = (q^3 - q^2 - q + 1)/3,$$

we have

$$\left(\sum_{k=1}^3 K_k\right)((q^3 + 2q^2 + 2q + 1)/3) + \left(\sum_{m=1}^3 M_m + \sum_{n=1}^3 N_n\right)((q^3 - q^2 - q + 1)/3) = q^3 + 1.$$

Hence, by considering $K = \sum_{k=1}^3 K_k$, $M = \sum_{m=1}^3 M_m$ and $N = \sum_{n=1}^3 N_n$, we get

$$K((q^3 + 2q^2 + 2q + 1)/3) + (M + N)((q^3 - q^2 - q + 1)/3) = q^3 + 1,$$

and so

$$(K + M + N)q^3 + (2K - (M + N))q^2 + ((2K - (M + N))q + (K + M + N)) = 3(q^3 + 1).$$

Thus,

$$(3.6) \quad (A - 3)(q^3 + 1) = -B(q^2 + q),$$

where $A = K + M + N$ and $B = 2K - (M + N)$. Since K , M and N are non negative integers and are not all equal to 0, then A is a positive integer. Since $q \mid -B(q^2 + q)$, then $q \mid A - 3$ and this means that $A - 3 = tq$, for some integer t . Hence, simplifying equation (3.6) implies $-B = t(q^2 - q + 1)$. Thus,

$$0 \leq 3K = A + B = 3 - t(q - 1)^2.$$

Since $d = \gcd(3, q - 1) = 3$, then we can consider $q > 3$, which in this case $A = 3 + tq > 0$ implies $t \geq 0$ and $A + B = 3 - t(q - 1)^2 \geq 0$

shows $t \leq 0$. Thus, $t = 0$, $A = 3$ and $B = 0$, which yield $K = 1$ and $M + N = 2$. Therefore, $\sum_{k=1}^3 K_k = 1$ and $\sum_{m=1}^3 M_m + \sum_{n=1}^3 N_n = 2$. So, for some k , $K_k = 1$ and $\langle(\theta_k)_P, \varphi\rangle = 1$.

Let $\langle(\theta_1)_P, \varphi\rangle = 1$. Then, the characters θ_1, θ_2 and θ_3 are conjugate in $L = \text{GL}(3, q)$ (see [7, Sec. 4]). Hence, by Lemma 2.3, we have

$$\langle(\theta_1)_P, \varphi\rangle = \langle(\theta_2)_P, \varphi^x\rangle = \langle(\theta_3)_P, \varphi^y\rangle = 1,$$

for some $x, y \in N_L(P)$. On the other hand, by Lemma 2.2, φ^x and φ^y are linear characters of P and so the restriction of characters θ_1, θ_2 and θ_3 to P have at least a constituent of degree one with multiplicity one.

Also, by (2.3),

$$\{(\omega_1)_P, (\omega_2)_P, (\omega_3)_P\} = \{(\gamma_1)_P, (\gamma_2)_P, (\gamma_3)_P\},$$

and so $\sum_{m=1}^3 M_m = 1$ and $\sum_{n=1}^3 N_n = 1$. Therefore, for some m and n we have $N_n = 1$ and $M_m = 1$, which means $\langle(\omega_m)_P, \varphi\rangle = \langle(\gamma_n)_P, \varphi\rangle = 1$. Without any ambiguity, we can suppose $\langle(\omega_1)_P, \varphi\rangle = \langle(\gamma_1)_P, \varphi\rangle = 1$. Since the elements of each set of characters $\{\omega_1, \omega_2, \omega_3\}$ and $\{\gamma_1, \gamma_2, \gamma_3\}$ are conjugate in $L = \text{GL}(3, q)$ (see [7, Sec. 4]), then by Lemma 2.3 and Lemma 2.2, there exist $r, s, t, u \in N_L(G)$ such that $\varphi^r, \varphi^s, \varphi^t$ and φ^u are linear characters of P and

$$\langle(\omega_2)_P, \varphi^r\rangle = \langle(\omega_3)_P, \varphi^s\rangle = \langle(\gamma_2)_P, \varphi^t\rangle = \langle(\gamma_3)_P, \varphi^u\rangle = 1.$$

Hence, for $1 \leq m, n \leq 3$, the characters $(\omega_m)_P$ and $(\gamma_n)_P$ have a linear constituent with multiplicity 1. This completes the proof. \square

4. Basis for class functions

If G is a finite group with n conjugacy classes such that t of these classes are p -elements, then the $n \times n$ matrix X constructed from the character table of G has an $n \times t$ submatrix M whose columns correspond to the p -elements. Since X is invertible, the columns of M are linearly independent and so M has rank t . Thus, there exist t irreducible characters of G such that for every irreducible character χ of G , the restriction of χ on p -elements is a linear combination of these t characters. Now, suppose $G = \text{SL}(3, q)$. If $d = 1$, then equations (3.1) to (3.5) show that the class function ϕ_P of P is an integral linear combination of characters ρ_P, ψ_P and $\mathbf{1}$, for each generalized character ϕ of G . If $d = 3$, then using the values of the characters θ_k, ω_m and γ_n on p -elements, we have

$$\rho_P = (\theta_k)_P + (\omega_m)_P + (\gamma_n)_P - \mathbf{1}.$$

Now, considering equations (3.1) to (3.5), we have

$$\begin{aligned}(\zeta_i)_P &= \psi_P + \mathbf{1}, \\(\eta_j)_P &= (\theta_k)_P + (\omega_m)_P + (\gamma_n)_P + \psi_P - \mathbf{1}, \\(\varepsilon_r)_P &= (\theta_k)_P + (\omega_m)_P + (\gamma_n)_P + 2\psi_P, \\(\mu_s)_P &= (\theta_k)_P + (\omega_m)_P + (\gamma_n)_P - 2 \cdot \mathbf{1},\end{aligned}$$

and

$$(\nu_t)_P = (\theta_k)_P + (\omega_m)_P + (\gamma_n)_P - \psi_P.$$

Thus, $\{(\theta_k)_P, (\omega_m)_P, (\gamma_n)_P, \psi_P, \mathbf{1}_P\}$ is a basis for the vector space of class functions defined on P with integer coefficients. This is analogous to the theory of π -partial characters of solvable groups, developed by Isaacs, for the case $\pi = \{p\}$ (see [4]).

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