Bulletin of the Iranian Mathematical Society Vol. 36 *No.* 2 (2010), *pp 145-155.*

COFINITENESS OF LOCAL COHOMOLOGY BASED ON A NON–CLOSED SUPPORT DEFINED BY A PAIR OF IDEALS

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Communicated by Siamak Yassemi

Abstract. Let I, J be ideals of a commutative Noetherian ring R and let t be a non–negative integer. Let M be an R–module such that $\operatorname{Ext}^t_R(R/I, M)$ is a finite R–module. If t is the first integer such that the local cohomology module with respect to (I, J) is non–(I, J)–cofinite, then we show that $\operatorname{Hom}_R(R/I, \mathrm{H}^t_{I,J}(M))$ is finite. Also, we study the finiteness of $\operatorname{Ext}^i_R(R/I, \mathrm{H}^t_{I,J}(M))$, for $i = 1, 2$. In addition, for a finite R–module M, we show that the associated primes of $\mathrm{H}^t_{I,J}(M)$ have an equal grade, when $t = \inf\{i|\mathrm{H}^i_{I,J}(M) \neq 0\}$.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring and I, J are ideals of R. The generalized local cohomology module with respect to a pair of ideals I, J of R is introduced by Takahashi–Yoshino [12].

We are concerned with the subsets

$$W(I, J) = \{\ p \in \operatorname{Spec}(R)|I^n \subseteq p + J, \text{for an integer } n \gg 1\}$$

MSC(2010): Primary: 13D45, 13D22; Secondary: 13E99, 13D07.
Keywords: Local cohomology, cofinite module, associated prime.
Received: 29 June 2009, Accepted: 29 August 2009.
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of $Spec(R)$ and $\tilde{W}(I,J) = \{a \trianglelefteq R | I^n \subseteq a + J$, for an integer $n \gg 1\}$. In general, $W(I,J)$ is closed under specialization, but not necessarily a closed subset of Spec(R). For an R–module M, we consider the (I,J)–torsion submodule $\Gamma_{I,J}(M)$ of M which consists of all elements x of M with $\mathrm{Supp}\,(Rx) \subseteq W(I,J)$. Furthermore, for an integer i, we define the local cohomology functor $\mathrm{H}^i_{I,J}(-)$ with respect to (I,J) to be the i–th right derived functor of $\Gamma_{I,J}(-)$. Note that if $J = 0$, then $\mathrm{H}^i_{I,J}(-)$ coincides with the ordinary local cohomology functor $\mathrm{H}^i_I(-)$, with the support in the closed subset $V(I)$. On the other hand, if J contains I, then $\Gamma_{I,J}$ is the identity functor and $\mathrm{H}^i_{I,J}(-) = 0$, for $i > 0$.

There are many questions about classical local cohomology modules. In particular, Grothendieck proposed the following conjecture.

CONJECTURE 1. Let M be a finite module over a ring R, and let I be an ideal of R. Then, the module $\mathrm{Hom}_R(R/I, \mathrm{H}^j_I(M))$ is finite, for all $j \geq 0$.

Hartshorne later refined this conjecture, and proposed the following one.

CONJECTURE 2. Let M be a finite R–module, and let I be an ideal of R. Then, $\mathrm{Ext}^i_R(R/I, \mathrm{H}^j_I(M))$ is finite, for every $i \geq 0$ and $j \geq 0$.

Using the derived category, Hartshorne showed that if M is a finitely generated R–module, where R is a complete regular local ring, then $\mathrm{H}^j_I(M)$ is I–cofinite in two cases:

(i) I is non–zero principal ideal.
(ii) I is a prime ideal with dimension 1.

Kawasaki [9] proved (i) for any Noetherian ring and Marley–Delfino [1] proved (ii) for any Noetherian ring.

In Section 2, we study the finiteness condition of $\mathrm{Ext}^i_R(R/I, \mathrm{H}^j_{I,J}(M))$, for $i = 0, 1, 2$. More precisely, we show the following.

Theorem 2.3. Let t be a non–negative integer. Let M be an R–module such that $\mathrm{Ext}^t_R(R/I, M)$ is a finite R–module and $\mathrm{H}^i_{I,J}(M)$ is (I,J)–cofinite, for every $i < t$. If $N \subseteq \mathrm{H}^t_{I,J}(M)$ is such that $\mathrm{Ext}^1_R(R/I, N)$ is finite, then $\mathrm{Hom}_R(R/I, \mathrm{H}^t_{I,J}(M)/N)$ is a finite R–module.

Theorem 2.5. Let t be a non–negative integer. Let M be an R–module such that $\mathrm{H}^i_{I,J}(M)$ is (I,J)–cofinite, for every $i<t$. Then, the following statements hold.

(a) If $\mathrm{Ext}^{t+1}_R(R/I,M)$ is a finite R–module, then $\mathrm{Ext}^1_R(R/I,\mathrm{H}^t_{I,J}(M))$ is finite.
(b) If $\mathrm{Ext}^i_R(R/I,M)$ is finite, for all $i\geq 0$, then $\mathrm{Hom}_R(R/I,\mathrm{H}^{t+1}_{I,J}(M))$ is finite if and only if $\mathrm{Ext}^2_R(R/I,\mathrm{H}^t_{I,J}(M))$ is finite.

We recall that an important problem in commutative algebra is determining the set of associated primes of local cohomology modules. Huneke [8] raised the following conjecture: If M is a finitely generated R–module, then the set of associated primes of $\mathrm{H}^i_I(M)$ is finite, for every ideals I of R and every $i\geq 0$. Singh [11] gives a counter–example to this conjecture. On the other hand, Brodmann and Lashgari [2] have shown that the first non–finite local cohomology module $\mathrm{H}^i_I(M)$ of a finite module M has only finitely many associated primes. Also, Dibaei and Yassemi [5], by using cofinitness, found a condition for finiteness of associated primes of local cohomology.

In Section 3, we study the above results for local cohomology with respect to a pair of ideals I, J of R and as a consequence of Theorem 2.3, We show that the set of associated primes of local cohomology are finite. Also, we prove that all associated prime ideals of the first non–zero local cohomology module have an equal grade.

2. **Cofiniteness**

Definition 2.1. *An R–module M is called (I,J)–cofinite if $Supp\,(M)\subseteq W(I,J)$ and $Ext^i_R(R/I,M)$ is a finite R–module, for every $i\geq 0$.*

Remark 2.2. Let M be an R–module and let E be the injective hull of the R–module $M/\Gamma_{I,J}(M)$. Let $L=E/(M/\Gamma_{I,J}(M))$. Since $Ass\mathrm{Hom}_R(R/I,E)=V(I)\cap Ass(E)\subseteq W(I,J)\cap Ass(M/\Gamma_{I,J}(M))=\phi$, the modules $Hom_R(R/I,E)$ and $\Gamma_{I,J}(E)$ are zero. Also, from the exact sequence

$$0\longrightarrow M/\Gamma_{I,J}(M)\longrightarrow E\longrightarrow L\longrightarrow 0,$$

by applying $Hom_R(R/I,-)$, we have $\mathrm{Ext}^i_R(R/I,L)\cong\mathrm{Ext}^{i+1}_R(R/I,M/\Gamma_{I,J}(M))$ and $\mathrm{H}^i_{I,J}(L)\cong\mathrm{H}^{i+1}_{I,J}(M)$, for every $i\geq 0$.

Theorem 2.3. *Let t be a non–negative integer. Let M be an R–module such that $Ext^t_R(R/I, M)$ is a finite R–module and $H^i_{I,J}(M)$ is (I, J)–cofinite, for every $i < t$. If $N \subseteq H^t_{I,J}(M)$ is such that $Ext^1_R(R/I, N)$ is finite, then $Hom_R(R/I, H^t_{I,J}(M)/N)$ is a finite R–module.*

Proof. First assume that $N = 0$. We use induction on t. Let $t = 0$. Then, $\mathrm{Hom}_R(R/I, \Gamma_{I,J}(M))$ is equal to the finite R–module $\mathrm{Hom}_R(R/I, M)$.

Suppose that $t > 0$ and the case $t - 1$ is settled. Since $\Gamma_{I,J}(M)$ is (I, J)–cofinite, $\mathrm{Ext}^i_R(R/I, \Gamma_{I,J}(M))$ is finite, for every i. By using the exact sequence

$$0 \longrightarrow \Gamma_{I,J}(M) \longrightarrow M \longrightarrow M/\Gamma_{I,J}(M) \longrightarrow 0,$$

we get that $\mathrm{Ext}^t_R(R/I, M/\Gamma_{I,J}(M))$ is finite. Now, by Remark 2.2, the R–module $\mathrm{Ext}^t_R(R/I, L)$ is finite and $\mathrm{H}^i_{I,J}(L)$ is (I, J)–cofinite for every $i < t-1$. Thus, by induction hypothesis, $\mathrm{Hom}_R(R/I, \mathrm{H}^{t-1}_{I,J}(L))$ is finite, which implies that $\mathrm{Hom}_R(R/I, \mathrm{H}^t_{I,J}(M))$ is finite.

Now, assume that $N \neq 0$. By considering the exact sequence

$$0 \longrightarrow N \longrightarrow \mathrm{H}^t_{I,J}(M) \longrightarrow \mathrm{H}^t_{I,J}(M)/N \longrightarrow 0,$$

and applying $\mathrm{Hom}_R(R/I, -)$ to that, we obtain the exact sequence

$$\mathrm{Hom}_R(R/I, \mathrm{H}^t_{I,J}(M)) \longrightarrow \mathrm{Hom}_R(R/I, \mathrm{H}^t_{I,J}(M)/N) \longrightarrow \mathrm{Ext}^1_R(R/I, N).$$

Since the left hand (by case $N = 0$) and the right hand sides are finite, we have that $\mathrm{Hom}_R(R/I, \mathrm{H}^t_{I,J}(M)/N)$ is finite.

The next result was shown by Dibaei and Yassemi in [5], and it generalized [2, Theorem 2.2]

Corollary 2.4. *Let I be an ideal of a Noetherian ring R. Let t be a non–negative integer. Let M be an R–module such that $Ext^t_R(R/I, M)$ is a finite R–module. If $H^i_I(M)$ is I–cofinite, for all $i < t$, then $Hom_R(R/I, H^t_I(M))$ is finite.*

Theorem 2.5. *Let t be a non–negative integer. Let M be an R–module such that $H^i_{I,J}(M)$ is (I, J)–cofinite, for all $i < t$. Then, the following statements hold:*

(a) *If $Ext^{t+1}_R(R/I, M)$ is a finite R–module, then $Ext^1_R(R/I, H^t_{I,J}(M))$ is finite.*

(b) *If* $Ext^i_R(R/I, M)$ *is finite, for all* $i \geq 0$*, then* $Hom_R(R/I, H^{t+1}_{I,J}(M$ $))$ *is finite if and only if* $Ext^2_R(R/I, H^t_{I,J}(M))$ *is finite.*

Proof. (a) We use induction on t. Let $t = 0$. Then, the short exact sequence

$$(*) \qquad 0 \longrightarrow \Gamma_{I,J}(M) \longrightarrow M \longrightarrow M/\Gamma_{I,J}(M) \longrightarrow 0$$

implies that $\operatorname{Ext}^1_R(R/I, Gamma_{I,J}(M))$ is finite.

Suppose that $t > 0$ and the case $t - 1$ is settled. Since $\Gamma_{I,J}(M)$ is (I, J)–cofinite, the R–module $\operatorname{Ext}^i_R(R/I, \Gamma_{I,J}(M))$ is finite, for every i, and so by (*), $\operatorname{Ext}^{t+1}_R(R/I, M/\Gamma_{I,J}(M))$ is finite. Now, by Remark 2.2, the R–module $\operatorname{Ext}^t_R(R/I, L)$ is finite and $\mathrm{H}^i_{I,J}(L)$ is (I, J)–cofinite, for every $i < t-1$. Thus, by the induction hypothesis, $\operatorname{Ext}^1_R(R/I, \mathrm{H}^{t-1}_{I,J}(L))$ is finite, and so $\operatorname{Ext}^1_R(R/I, \mathrm{H}^t_{I,J}(M))$ is finite.

(b) ($\Rightarrow$) We use induction on t. Let $t = 0$. Then, the short exact sequence (*) induces the following exact sequence

$$\operatorname{Ext}^1_R(R/I, M/\Gamma_{I,J}(M)) \longrightarrow \operatorname{Ext}^2_R(R/I, \Gamma_{I,J}(M)) \longrightarrow \operatorname{Ext}^2_R(R/I, M).$$

To show that $\operatorname{Ext}^2_R(R/I, \Gamma_{I,J}(M))$ is finite, it is enough to show that $\operatorname{Ext}^1_R(R/I, M/\Gamma_{I,J}(M))$ is finite. By Remark 2.2, we have

$$\begin{aligned}\operatorname{Ext}^1_R(R/I, M/\Gamma_{I,J}(M)) &\cong \operatorname{Hom}_R(R/I, L)\\ &\cong \operatorname{Hom}_R(R/I, \Gamma_{I,J}(L))\\ &\cong \operatorname{Hom}_R(R/I, \mathrm{H}^1_{I,J}(M)).\end{aligned}$$

Now, the assertion holds.

Suppose $t > 0$ and the case $t - 1$ is settled. Since $\Gamma_{I,J}(M)$ is (I, J)–cofinite, the R–module $\operatorname{Ext}^i_R(R/I, \Gamma_{I,J}(M))$ is finite, for every i. Using the exact sequence (*), we get that $\operatorname{Ext}^i_R(R/I, M/\Gamma_{I,J}(M))$ is finite, for every i. By Remark 2.2, $\operatorname{Ext}^i_R(R/I, L)$ is finite, for every i and also $\operatorname{Hom}_R(R/I, \mathrm{H}^t_{I,J}(L)) \cong \operatorname{Hom}_R(R/I, \mathrm{H}^{t+1}_{I,J}(M))$ is finite. By the induction hypothesis, the R–module $Ext^2_R(R/I, \mathrm{H}^{t-1}_{I,J}(L))$ is finite and hence, we have $\operatorname{Ext}^2_R(R/I, \mathrm{H}^t_{I,J}(M))$ is finite.

($\Leftarrow$) We use induction on t. Let $t = 0$. The short exact sequence (*) induces the following exact sequence,

$$\operatorname{Ext}^1_R(R/I, M) \longrightarrow \operatorname{Ext}^1_R(R/I, M/\Gamma_{I,J}(M)) \longrightarrow \operatorname{Ext}^2_R(R/I, \Gamma_{I,J}(M)).$$

Thus, $\mathrm{Ext}^1_R(R/I, M/\Gamma_{I,J}(M))$ is finite. By Remark 2.2, $\mathrm{Hom}_R(R/I, L)$ is finite and hence the R–module $\mathrm{Hom}_R(R/I, \Gamma_{I,J}(L))$ is finite. Thus $\mathrm{Hom}_R(R/I, \mathrm{H}^1_{I,J}(M))$ is finite.

Now, let $t > 0$ and the case $t-1$ be settled. Remark 2.2 implies that the modules $\mathrm{Ext}^2_R(R/I, \mathrm{H}^{t-1}_{I,J}(L))$ and $\mathrm{Ext}^i_R(R/I, L)$ are finite, for all i. By the induction hypothesis, the R–module $\mathrm{Hom}_R(R/I, \mathrm{H}^t_{I,J}(L))$ is finite and hence $\mathrm{Hom}_R(R/I, \mathrm{H}^{t+1}_{I,J}(M))$ is finite.

The following corollary generalizes Dibaei and Yassemi's result [6].

Corollary 2.6. *Let M be a finite R–module and $t = \inf\{i | H^i_{I,J}(M) \neq 0\}$. Then, the following hold:*

(a) *$Ext^1_R(R/I, H^t_{I,J}(M))$ is finite.*

(b) *$Ext^2_R(R/I, H^t_{I,J}(M))$ is finite if and only if $Hom_R(R/I, H^{t+1}_{I,J}(M))$ is finite.*

3. **Associated primes**

Let M be a finite R–module. Let $t = \inf\{i | \mathrm{H}^i_{I,J}(M) \neq 0\}$ and $N \subseteq \mathrm{H}^t_{I,J}(M)$ be such that $Ext^1_R(R/I, N)$ is a finite R–module. If $\mathrm{H}^t_{I,J}(M)/N$ is an I–torsion, then by Theorem 2.3, $\mathrm{H}^t_{I,J}(M)/N$ has finitely many associated primes. In particular, $Ass\mathrm{H}^t_{I,J}(M)$ is a finite set if and only if $AssN$ is a finite set.

Remark 3.1. Let M be a finite R–module and i be an integer. Suppose that $p \in \mathrm{Ass}\,\mathrm{H}^i_{I,J}(M)$. Then, $pR_p \in \mathrm{Ass}\,(H^i_{I,J}(M))_p$ implies that $\mathrm{Hom}_{R_p}(R_p/pR_p, (\mathrm{H}^i_{I,J}(M))_p) \neq 0$. By [12, Theorem 3.2], we have

$$\mathrm{Hom}_{R_p}(R_p/pR_p, (\varinjlim_{a \in \tilde{W}(I,J)} (\mathrm{H}^i_a(M)) \otimes_{R_p} R_p) \cong$$

$$\varinjlim_{a \in \tilde{W}(I,J)} (\mathrm{Hom}_{R_p}(R_p/pR_p, \mathrm{H}^i_{aR_p}(M_p)).$$

So, there exists $a \in \tilde{W}(I,J)$ such that $\mathrm{Hom}_{R_p}(R_p/pR_p, H^i_{aR_p}(M_p)) \neq 0$. Hence, $pR_p \in \mathrm{Ass}\,(H^i_{aR_p}(M_p))$, which implies that $p \in \mathrm{Ass}\,H^i_a(M)$, for an $a \in \tilde{W}(I,J)$. Therefore,

$$\mathrm{Ass}\,\mathrm{H}^i_{I,J}(M) \subseteq \bigcup_{a \in \tilde{W}(I,J)} \mathrm{Ass}\,\mathrm{H}^i_a(M).$$

Proposition 3.2. *Let M be a finite R–module. If $t = \inf\{i | H^i_{I,J}(M) \neq 0\}$, then*

$$Ass\, H^t_{I,J}(M) \subseteq \bigcup_{\substack{\mathfrak{a}\in \tilde{W}(I,J) \\ grade_M \mathfrak{a}=t}} Ass\, H^t_{\mathfrak{a}}(M). \tag{1}$$

Proof. By Remark 3.1, it is enough to show that $\operatorname{grade}_M \mathfrak{a} = t$. Since $V(\mathfrak{a}) \subseteq W(I,J)$, by [12, Theorem 4.1], we have $\operatorname{grade}_M \mathfrak{a} = \inf\{\operatorname{depth} M_{\mathfrak{p}} | \mathfrak{p} \in \operatorname{Supp}(M/\mathfrak{a}M)\} \geq \inf\{\operatorname{depth} M_{\mathfrak{p}} | \mathfrak{p} \in W(I,J)\} = t$. On the other hand, $\mathrm{H}^t_{\mathfrak{a}}(M) \neq 0$ implies that $\operatorname{grade}_M \mathfrak{a} = t$.

Now, we show that we can replace the set $\tilde{W}(I,J)$ by $W(I,J)$ in (1).

Lemma 3.3. *Let M be a finite R–module and $\mathfrak{a}, \mathfrak{b}$ be ideals of R such that $\mathfrak{a} \subseteq \mathfrak{b}$ and $grade_M \mathfrak{a} = grade_M \mathfrak{b} = t$. Then,*

$$Ass\, H^t_{\mathfrak{b}}(M) \subseteq Ass\, H^t_{\mathfrak{a}}(M).$$

Proof. By choosing $x \in \mathfrak{b} \setminus \mathfrak{a}$ and considering the following $Mayer-Vietoris$ sequence, $0 \longrightarrow \mathrm{H}^t_{\mathfrak{a}+xR}(M) \longrightarrow \mathrm{H}^t_{\mathfrak{a}}(M) \longrightarrow \mathrm{H}^t_{\mathfrak{a}}(M_x)$, we obtain that $\operatorname{Ass} \mathrm{H}^t_{\mathfrak{a}+xR}(M) \subseteq \operatorname{Ass} \mathrm{H}^t_{\mathfrak{a}}(M)$. Now, the assertion follows by induction.

Proposition 3.4. *Let M be a finite R–module and $t = \inf\{i | H^i_{I,J}(M) \neq 0\}$. Then,*

$$Ass\, H^t_{I,J}(M) \subseteq \bigcup_{\substack{\mathfrak{q}\in W(I,J) \\ grade_M \mathfrak{q}=t}} Ass\, H^t_{\mathfrak{q}}(M).$$

Proof. By Proposition 3.2, for all $\mathfrak{p} \in \operatorname{Ass} \mathrm{H}^t_{I,J}(M)$, there exists $\mathfrak{a} \in \tilde{W}(I,J)$ such that $\operatorname{grade}_M \mathfrak{a} = t$ and $\mathfrak{p} \in \operatorname{Ass} \mathrm{H}^t_{\mathfrak{a}}(M)$. Now, consider the non–empty set

$$\Sigma_{\mathfrak{p}} = \{\mathfrak{a} \in \tilde{W}(I,J) | \operatorname{grade}_M \mathfrak{a} = t, \mathfrak{p} \in \operatorname{Ass} \mathrm{H}^t_{\mathfrak{a}}(M)\},$$

for a prime ideal $\mathfrak{p} \in \operatorname{Ass} \mathrm{H}^t_{I,J}(M)$. Since R is a Noetherian ring, $\Sigma_{\mathfrak{p}}$ has a maximal element $\mathfrak{q}$. We claim that $\mathfrak{q}$ is a prime ideal. Let $x, y \in R$ be such that $xy \in \mathfrak{q}$, but $x, y \notin \mathfrak{q}$. Therefore $xR + \mathfrak{q}, yR + \mathfrak{q} \notin \Sigma_{\mathfrak{p}}$. On the other hand, $\mathfrak{q}^2 \subseteq (xR + \mathfrak{q})(yR + \mathfrak{q}) \subseteq \mathfrak{q}$ implies that

$\operatorname{grade}_M \mathfrak{q} = \operatorname{grade}_M (xR+\mathfrak{q})(yR+\mathfrak{q})$. So, we have $\operatorname{grade}_M(xR+\mathfrak{q}) \geq t$ and $\operatorname{grade}_M(yR+\mathfrak{q}) \geq t$. Now, from the exact sequence $0 \longrightarrow \mathrm{H}^t_{xR+\mathfrak{q}}(M) \longrightarrow \mathrm{H}^t_{\mathfrak{q}}(M) \longrightarrow \mathrm{H}^t_{\mathfrak{q}}(M_x)$, we obtain that $\mathfrak{p} \in \operatorname{Ass} \mathrm{H}^t_{\mathfrak{q}}(M) \subseteq \operatorname{Ass} \mathrm{H}^t_{xR+\mathfrak{q}}(M) \cup \operatorname{Ass} \mathrm{H}^t_{\mathfrak{q}}(M_x)$. If $\operatorname{grade}_M(xR+\mathfrak{q}) > t$, then $\operatorname{Ass} \mathrm{H}^t_{\mathfrak{q}}(M) \subseteq \operatorname{Ass} \mathrm{H}^t_{\mathfrak{q}}(M_x)$. So, assume that $\operatorname{grade}_M(xR+\mathfrak{q}) = t$. Then, by maximality of $\mathfrak{q}$, we have $\mathfrak{p} \in \operatorname{Ass} \mathrm{H}^t_{\mathfrak{q}}(M_x)$. Similarly, $\mathfrak{p} \in \operatorname{Ass} \mathrm{H}^t_{\mathfrak{q}}(M_y)$. Thus, $\mathfrak{p} \in \operatorname{Supp}(M_x) \cap \operatorname{Supp}(M_y)$. On the other hand, $xy \in \mathfrak{p}$ implies that $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$, which is contradictory to $\operatorname{Supp}_R(M_x) = \{\mathfrak{p} \in \operatorname{Supp}(M) | x$ is not in $\mathfrak{p}\}$. Hence, $\mathfrak{q}$ is a prime ideal.

Corollary 3.5. *Let M be a finite R–module and $t = \inf\{i | H^i_{I,J}(M) = 0\}$. If $\mathfrak{q}$ is the maximal element of $\Sigma_{\mathfrak{p}}$, then $\mathfrak{p} = \mathfrak{q}$.*

Proof. Suppose that $\mathfrak{q} \subset \mathfrak{p}$ and consider $x \in \mathfrak{p} \setminus \mathfrak{q}$. By the exact sequence $0 \longrightarrow \mathrm{H}^t_{xR+\mathfrak{q}}(M) \longrightarrow \mathrm{H}^t_{\mathfrak{q}}(M) \longrightarrow \mathrm{H}^t_{\mathfrak{q}}(M_x)$, we obtain that $\mathfrak{p} \in \operatorname{Ass} \mathrm{H}^t_{\mathfrak{q}}(M) \subseteq \operatorname{Ass} \mathrm{H}^t_{xR+\mathfrak{q}}(M) \cup \operatorname{Ass} \mathrm{H}^t_{\mathfrak{q}}(M_x)$. Since $x \in \mathfrak{p}$, we have $\mathfrak{p} \in \operatorname{Ass} \mathrm{H}^t_{xR+\mathfrak{q}}(M)$, which is a contradiction, by maximality of $\mathfrak{q}$. So, $\mathfrak{p} = \mathfrak{q}$.

Now, we can state our main theorem here.

Theorem 3.6. *Let M be a finite R–module and $t = \inf\{i | H^i_{I,J}(M) \neq 0\}$. Then, for all $\mathfrak{p} \in Ass\, H^t_{I,J}(M)$, $grade_M \mathfrak{p} = t$.*

Proof. This follows from Proposition 3.4 and Corollary 3.5.

Corollary 3.7. *Let M be a finite R–module and $t = \inf\{i | H^i_{I,J}(M) \neq 0\}$. If $Hom_R(R/I, H^t_{I,J}(M)) \neq 0$, then $grade_M I = t$.*

Proof. Let $\mathfrak{p} \in \operatorname{Ass} \mathrm{H}^t_{I,J}(M) \cap V(I)$. So, by Theorem 3.6, $\operatorname{grade}_M I \leq t$. On the other hand, $V(I) \subseteq W(I,J)$ implies that $\operatorname{grade}_M I \geq t$. Hence, $\operatorname{grade}_M I = t$.

Theorem 3.8. *Let M be a finite R–module. Let I, J be ideals of R such that $grade_M I = n$ and $\Gamma_{I,J}(M/(x_1,...,x_{n-1})M) = 0$, for a maximal M–sequence $x_1,...,x_n$ in I. Then,*

$$Ass\,\Gamma_{I,J}(M/(x_1,...,x_n)M) = \{\mathfrak{p} \in Ass\,(M/(x_1,...,x_n)M) \cap W(I,J) | grade_M \mathfrak{p} = n\}.$$

Proof. We prove using induction on n. Let $n = 1$. Then, for a non–zero divisor $x \in I$, the following short exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0,$$

induces the long exact sequence, $0 \longrightarrow \Gamma_{I,J}(M/xM) \longrightarrow \mathrm{H}^1_{I,J}(M) \xrightarrow{x} \mathrm{H}^1_{I,J}(M)$. Since $\Gamma_I(M/xM) \neq 0, \Gamma_{I,J}(M/xM) \neq 0$ and $\mathrm{Ass}\,\Gamma_{I,J}(M/xM) \subseteq \mathrm{Ass}\,\mathrm{H}^1_{I,J}(M)$, therefore by Theorem 3.6,

$$\mathrm{Ass}\,\Gamma_{I,J}(M/xM) = \{\mathfrak{p} \in \mathrm{Ass}\,(M/xM) \cap W(I,J) | \mathrm{grade}_{\,M}\mathfrak{p} = 1\}.$$

Suppose that $n > 1$ and the case $n-1$ is settled. Then, $\mathrm{grade}_{\,M/x_1M} I = n-1$ implies:

$$\Gamma_{I,J}(M/x_1M/(x_2,...,x_{n-1})M/x_1M) \cong \Gamma_{I,J}(M/(x_1,x_2,...,x_{n-1})M) = 0,$$

$$\mathrm{H}^1_{I,J}(M/x_1M/(x_2,...,x_{n-1})M/x_1M) \cong \mathrm{H}^1_{I,J}(M/(x_1,x_2,...,x_{n-1})M) \neq 0.$$

So, by the induction hypothesis, we have,

$$\mathrm{Ass}\,\Gamma_{I,J}(M/x_1M/(x_2,...,x_n)M/x_1M) = \{\mathfrak{p} \in \mathrm{Ass}\,(M/x_1M/(x_2,...,x_n) \\ M/x_1M) \cap W(I,J) | \mathrm{grade}_{\,M/x_1M}\mathfrak{p} = n-1\}.$$

Therefore,

$$\mathrm{Ass}\,\Gamma_{I,J}(M/(x_1,...,x_n)M) = \\ \{\mathfrak{p} \in \mathrm{Ass}\,(M/(x_1,...,x_n)M) \cap W(I,J) | \mathrm{grade}_{\,M}\mathfrak{p} = n\}.$$

Now, we can conclude the same result for local cohomology module with respect to an ideal.

Corollary 3.9. *Let M be a finite R–module and let I be an ideal of R such that $\mathrm{grade}_{\,M} I = n$ is a non–zero integer. Then, for a maximal M–sequence $x = x_1,...,x_n$ in I,*

$$Ass\,\Gamma_I(M/xM) = \{\mathfrak{p} \in Ass\,(M/xM) \cap V(I) | grade_{\,M}\mathfrak{p} = n\}.$$

Theorem 3.10. *Let M be a finite R–module and let I be an ideal of R such that $\mathrm{grade}_{\,M} I = n$ is a non–zero integer. Then, for a maximal M–sequence $x = x_1,...,x_n$ in I,*

$$Ass\,H^n_I(M) = \{\mathfrak{p} \in Ass\,(M/xM) \cap V(I) | grade_{\,M}\mathfrak{p} = n\}.$$

Proof. It is enough to prove the case $n = 1$. Consider $x \in I \setminus Z(M)$. From the short exact sequence $0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$, we obtain the exact sequence $0 \longrightarrow \Gamma_I(M/xM) \longrightarrow \mathrm{H}^1_I(M) \xrightarrow{x} \mathrm{H}^1_I(M)$. Since

$$\operatorname{Ass}\Gamma_I(M/xM) = \operatorname{Ass}(0 :_{\mathrm{H}^1_I(M)} x) = \operatorname{Ass}\mathrm{H}^1_I(M),$$

by Corollary 3.9, we get

$$\operatorname{Ass}\mathrm{H}^1_I(M) = \{\mathfrak{p} \in \operatorname{Ass}(M/xM) \cap V(I) | \operatorname{grade}_M \mathfrak{p} = 1\}.$$

Now, the result follows by induction on $\operatorname{grade}_M I$.

Corollary 3.11. *Let $(R, \mathfrak{m})$ be a local ring, and let M be a finite Cohen–Macaulay R–module. Let I be an ideal of R such that $dim M/IM > 0$ and $grade_M I = n$. Then, $m \notin Ass H^n_I(M)$.*

Acknowledgments

The authors are deeply grateful to Professor S. Yassemi for his constant help, advice and, in particular, for suggesting many of the topics considered in this paper.

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