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THE BRUNN-MINKOWSKI-TYPE INEQUALITY

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Abstract. We establish the Brunn-Minkowski-type inequality for L_p-dual mixed volumes of star duality of mixed intersection bodies.

1. **Introduction**

The intersection operator and the class of intersection bodies were defined by Lutwak [14]. The closure of the class of intersection bodies was studied by Goody et al. [6]. The intersection operator and the class of intersection bodies played a critical role in Gardner [3] and Zhang [23] solutions of the famous Busemann-Petty problem in three dimensions and four dimensions, respectively. (See also Gardner et al. [5].)

Just as the period from the mid 60's to the mid 80's was a time of great advances in the understanding of the projection operator and the class of projection bodies, during the past 30 years significant advances have been made in our understanding of the intersection operator and the class of intersection bodies by Koldobsky, Campi, Goodey, Gardner, Lutwak, Grinberg, Fallert, Weil, Zhang, Ludwig and others (see, e.g., [1-2], [4], [6-10],[12], [14], [23]).

As Lutwak [14] shows (and as is further elaborated in Gardner's book [4]), there is a duality between projection and intersection bodies (which

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at present is not yet understood). Consider the following illustrative example: It is well known that the projections (onto lower dimensional subspaces, the dimension lower than $n-1$) of projection bodies are the projection bodies themselves. Lutwak conjectured the "duality": When intersection bodies are intersected with lower dimensional subspaces, the results are intersection bodies (within the lower dimensional subspaces). This was proven by Fallert et al. [2]. In [17] (see also [15] and [16]), Lutwak introduced mixed projection bodies and established the following Brunn-Minkowski inequality for mixed projection bodies. If K and L are convex bodies, then

$$V(\mathbf{\Pi}(K+L))^{1/n(n-1)} \geq V(\mathbf{\Pi}K)^{1/n(n-1)} + V(\mathbf{\Pi}L)^{1/n(n-1)},$$

with equality if and only if K and L are homothetic.

Following Lutwak, the following Brunn-Minkowski inequality for polar of mixed projection bodies was established as follows [24]. If K and L are convex bodies, then

$$V(\mathbf{\Pi}^*(K+L))^{-1/n(n-1)} \geq V(\mathbf{\Pi}^*K)^{-1/n(n-1)} + V(\mathbf{\Pi}^*L)^{-1/n(n-1)}, \tag{1.1}$$

with equality if and only if K and L are homothetic.

Moreover, the Brunn-Minkowski inequality for mixed intersection bodies was established as follows [25].

If K and L are star bodies, then

$$V(\mathbf{I}(K\tilde{+}L))^{1/n(n-1)} \geq V(\mathbf{I}K)^{1/n(n-1)} + V(\mathbf{I}L)^{1/n(n-1)},$$

with equality if and only if K and L are dilates, where $\tilde{+}$ is the radial Minkowski sum.

Here, we intend to establish Brunn-Minkowski inequality for star duality of intersection bodies as follows.

Theorem 1.1. *If $K, L \in \varphi^n$, and $-\infty < p < -1$, then*

$$\tilde{V}_p(\mathbf{I}^\circ(K\tilde{+}L))^{-1/p(n-1)} \leq \tilde{V}_p(\mathbf{I}^\circ K)^{-1/p(n-1)} + \tilde{V}_p(\mathbf{I}^\circ L)^{-1/p(n-1)}, \tag{1.2}$$

with equality if and only if K and L are dilates, where $\mathbf{I}^\circ K$ denotes the star dual of K.

Taking $p=-n$ in (1.2), the following result is at hand.

Corollary 1.2. *If $K, L \in \varphi^n$, then*

$$V(\mathbf{I}^\circ(K\tilde{+}L))^{1/-n(n-1)} \leq V(\mathbf{I}^\circ K)^{1/-n(n-1)} + V(\mathbf{I}^\circ L)^{1/-n(n-1)},$$

with equality if and only if K and L are dilates.

This is just a dual form of inequality (1.1).

2. **Background**

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n(n > 2)$. Let $\mathbb{C}^n$ denote the set of non-empty convex figures (compact, convex subsets) and $\mathcal{K}^n$ denote the subset of $\mathbb{C}^n$ consisting of all convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B is reserved for the unit ball centered at the origin. The surface of B is S^{n-1}. For $u \in S^{n-1}$, let E_u denote the hyperplane, through the origin, that is orthogonal to u. We will use K^u to denote the image of K under an orthogonal projection onto the hyperplane E_u. We use $V(K)$ for the n-dimensional volume of convex body K. The support function of $K \in \mathcal{K}^n$, $h(K,\cdot)$, defined on $\mathbb{R}^n$ by $h(K,\cdot) = Max\{x \cdot y : y \in K\}$. Let δ denote the Hausdorff metric on $\mathcal{K}^n$; i.e., for $K, L \in \mathcal{K}^n$, $\delta(K,L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions, $C(S^{n-1})$.

Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, its radial function $\rho(K,\cdot) : S^{n-1} \to \mathbb{R}$, is defined for $u \in S^{n-1}$, by $\rho(K,u) = Max\{\lambda \geq 0 : \lambda u \in K\}$. If $\rho(K,\cdot)$ is positive and continuous, then K will be called a star body. Let φ^n denote the set of star bodies with 0 in $\mathbb{R}^n$.

2.1 *Star duality and polar*

In [20], Moszyńska introduced the notion of star duality of star body (see also Moszyńska [21]) as follows.

For the star bodies with 0 in the kernel and positive continuous radial function, a duality ($\circ$) was introduced, named the *star duality.* Let $i : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$ be inversion with respect to S^{n-1}:

$$i(x) := \frac{x}{\|x\|^2}.$$

Definition 2.1. [20] For every $K \in \varphi^n$,

$$K^\circ := \mathrm{cl}(\mathbb{R}^n \setminus i(K)).$$

Definition 2.2. [20] For every $K \in \varphi^n$,

$$\rho(K^\circ, u) = \frac{1}{\rho(K, u)}. \tag{2.1}$$

If K is a convex body that contains the origin in its interior, the polar body of K, K^*, is defined by

$$K^* := \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in K\}.$$

2.2 *Dual mixed volumes*

If $K_1, \ldots, K_r \in \varphi^n$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$, then the radial Minkowski linear combination, $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$, is defined by $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r = \{\lambda_1 x_1 \tilde{+} \cdots \tilde{+} \lambda_r x_r : x_i \in K_i\}$.

The following property will be used later. If $K, L \in \varphi^n$ and $\lambda, \mu \geq 0$, then

$$\rho(\lambda K \tilde{+} \mu L, \cdot) = \lambda \rho(K, \cdot) + \mu \rho(L, \cdot). \tag{2.2}$$

For $K_1, \ldots, K_r \in \varphi^n$ and $\lambda_1, \ldots, \lambda_r \geq 0$, the volume of the radial Minkowski liner combination $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$ is a homogeneous nth-degree polynomial in the λ_i,

$$V(\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r) = \sum \tilde{V}_{i_1, \ldots, i_n} \lambda_{i_1} \cdots \lambda_{i_n}, \tag{2.3}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ whose entries are positive integers not exceeding r. If we require the coefficients of the polynomial in (2.3) to be symmetric in their arguments, then they are uniquely determined. The coefficient $\tilde{V}_{i_1, \ldots, i_n}$ is nonnegative and depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$. It is written as $\tilde{V}(K_{i_1}, \ldots, K_{i_n})$ and is called the *dual mixed volume* of $K_{i_1}, \ldots, K_{i_n}$. If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = L$, the dual mixed volume is written as $\tilde{V}_i(K, L)$. The dual mixed volume $\tilde{V}_i(K, B)$ is written as $\tilde{W}_i(K)$. If $K_i \in \varphi^n (i = 1, 2, \ldots, n-1)$, then the dual mixed volume of $K_i \cap E_u$ $(i = 1, 2, \ldots, n-1)$ will be denoted by $\tilde{v}(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u)$. If $K_1 = \ldots = K_{n-1-i} = K$ and $K_{n-i} = \ldots = K_{n-1} = L$, then $\tilde{v}(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u)$ is written as $\tilde{v}_i(K \cap E_u, L \cap E_u)$. If $L = B$, then $\tilde{v}_i(K \cap E_u, B \cap E_u)$ is written as $\tilde{w}_i(K \cap E_u)$.

Let $K, L \in \varphi^n$ and $p \neq 0$, and define a star body $K \tilde{+}_p L$ by

$$\rho(K \tilde{+}_p L, u)^p = \rho(K, u)^p + \rho(L, u)^p. \tag{2.4}$$

The operation $\tilde{+}_p$ is called L_p-radial addition. The radial addition $\tilde{+}$ is the special case of the L_p-radial addition.

The L_p-dual volume is defined to be:

$$(2.5) \quad \tilde{V}_p(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^p dS(u),\ -\infty < p < +\infty.$$

2.3 *Intersection bodies and its star duality*

For $K \in \varphi^n$, there is a unique star body $\mathbf{I}K$ whose radial function satisfies, for $u \in S^{n-1}$,

$$(2.6) \quad \rho(\mathbf{I}K, u) = v(K \cap E_u),$$

This is called the *intersection bodies* of K. From a result due to Busemann, it follows that $\mathbf{I}K$ is convex if K is convex and centrally symmetric with respect to the origin. Clearly, any intersection body is centred. Volume of the intersection bodies is given by $V(\mathbf{I}K) = \frac{1}{n}\int_{S^{n-1}} v(K \cap E_u)^n dS(u)$.

The mixed intersection bodies of $K_1, \ldots, K_{n-1} \in \varphi^n$, is $\mathbf{I}(K_1, \ldots, K_{n-1})$, whose radial function is defined by

$$(2.7) \quad \rho(\mathbf{I}(K_1, \ldots, K_{n-1}), u) = \tilde{v}(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u),$$

where $\tilde{v}$ is the $(n-1)$-dimensional dual mixed volume.

If $K \in \varphi^n$ with $\rho(K,u) \in C(S^{n-1})$, and $i \in \mathbb{R}$ is positive, then the *intersection body of order* i of K is the centered star body $\mathbf{I}_i K$ such that $\rho(\mathbf{I}_i K) = \frac{1}{n-1}\int_{S^{n-1}} \rho(K,u)^{n-i-1} dS(u)$, for $u \in S^{n-1}$, where $\mathbf{I}_i K = \mathbf{I}(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{B, \ldots, B}_{i})$. If $K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = L$, then $\mathbf{I}(K_1, \ldots, K_{n-1})$ is written as $\mathbf{I}_i(K, L)$. If $L = B$, then $\mathbf{I}_i(K, L)$ written as $\mathbf{I}_i K$ is called the ith intersection body of K. For $\mathbf{I}_0 K$, simply write $\mathbf{I}K$.

The star duality of the mixed intersection bodies of $K_1, \ldots, K_{n-1} \in \varphi^n$ will be written as $\mathbf{I}^\circ(K_1, \ldots, K_{n-1})$. If $K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = L$, then $\mathbf{I}^\circ(K_1, \ldots, K_{n-1})$ is written as $\mathbf{I}_i^\circ(K, L)$. If $L = B$, then $\mathbf{I}_i^\circ(K, L)$ written as $\mathbf{I}_i^\circ K$ is called the star duality ith intersection body of K. For $\mathbf{I}_0^\circ K$, simply write $\mathbf{I}^\circ K$.

The following property will be used later: If $K, L, M, K_1, \ldots, K_{n-1} \in \varphi^n$, and $\lambda, \mu, \lambda_1, \ldots, \lambda_{n-1} > 0$, then

$$(2.8) \quad \mathbf{I}(\lambda K \tilde{+} \mu L, M) = \lambda \mathbf{I}(K, M) \tilde{+} \mu \mathbf{I}(L, M),\ M = (K_1, \ldots, K_{n-2}).$$

2.4 *Dual width-integrals of index i*

In [11], Li et al. introduced the chord-integrals of star bodies. In fact, there are some similar notions as follows next.

For $u \in S^{n-1}$, $b(K,u) =: \frac{1}{2}(h(K,u) + h(K,-u))$ is defined to be half the width of K in the direction u. Two convex bodies K and L are said to have similar width if there exists a constant $\lambda > 0$ such that $b(K,u) = \lambda b(L,u)$, for all $u \in S^{n-1}$. Width-integrals were first considered by Blaschke. The width-integral of index i is defined by Lutwak [13], For $K \in \mathcal{K}^n, i \in \mathbb{R}$, $B_i(K) = \frac{1}{n}\int_{S^{n-1}} b(K,u)^{n-i} dS(u)$.

Here, we introduce *dual width-integrals* of star bodies.

For $K \in \varphi^n, u \in S^{n-1}$,

$$\tilde{b}(K,u) =: \frac{1}{2}(\rho(K,u) + \rho(K,-u)) \tag{2.9}$$

is defined to be half the dual width of star body K in the direction u. Two star bodies K and L are said to have similar dual width if there exists a constant $\lambda > 0$ such that $\rho(K,u) = \lambda\rho(L,u)$, for all $u \in S^{n-1}$.

We also introduce *dual width-integral of index* i as follows.

For $K \in \varphi^n, i \in \mathbb{R}$,

$$\tilde{B}_i(K) = \frac{1}{n}\int_{S^{n-1}} \tilde{b}(K,u)^{n-i} dS(u). \tag{2.10}$$

The dual width-integral of index i is a map $\tilde{B}_i : \varphi^n \to \mathbb{R}$. It is positive, continuous, homogeneous of degree $n-i$ and invariant under motion.

3. Proof of Theorem 1.1

The following lemmas are needed to prove the theorem.

Lemma 3.1. *If $K, L \in \varphi^n$, $-\infty < p < \infty$, then*

$$\tilde{V}_p(\mathbf{I}^\circ K) = \frac{1}{n}\int_{S^{n-1}} v(K \cap u)^{-p} dS(u), \tag{3.1}$$

$$\tilde{V}_p(\mathbf{I}_j^\circ K) = \frac{1}{n}\int_{S^{n-1}} \tilde{w}_j(K \cap u)^{-p} dS(u), \tag{3.2}$$

$$\tilde{V}_p(\mathbf{I}_j^\circ(K,L)) = \frac{1}{n}\int_{S^{n-1}} \tilde{v}_j(K \cap u, L \cap u)^{-p} dS(u). \tag{3.3}$$

Proof. From (2.1), (2.5) and (2.7), we obtain:

$$\begin{aligned}\tilde{V}_p(\mathbf{I}_j^\circ(K,L)) &= \frac{1}{n}\int_{S^{n-1}} \rho(\mathbf{I}_j^\circ(K,L),u)^p dS(u)\\ &= \frac{1}{n}\int_{S^{n-1}} \rho(\mathbf{I}_j(K,L),u)^{-p} dS(u)\end{aligned}$$

$$= \frac{1}{n}\int_{S^{n-1}} \tilde{v}_j(K\cap u, L\cap u)^{-p} dS(u).$$

The proof of the identity (3.3) is complete.

Taking $K = L$ in (3.3), (3.3) changes to (3.1). Taking $L = B$ in (3.3), (3.3) changes to (3.2). □

Lemma 3.2. [26] *If If $K, L \in \varphi^n$ and $i < n-1$, then*

$$\tilde{W}_i(K,L)^{n-i} \leq \tilde{W}_i(K)^{n-i-1}\tilde{W}_i(L), \tag{3.4}$$

with equality if and only if K is a dilation of L.

The inequality is reversed, for $i > n$ or $(n-1) < i < n$.

Lemma 3.3. *Let $K, L \in \varphi^n$, $n \in N, n > 1$ and $-\infty < p < 0$.*

(i) *If $0 < j < n-1$, then*

$$\tilde{V}_p(\mathbf{I}_j^\circ(K,L))^{n-1} \leq \tilde{V}_p(\mathbf{I}^\circ K)^{n-j-1}\tilde{V}_p(\mathbf{I}^\circ L)^j,$$

with equality if and only if K and L are dilates.

(ii) *If $j > n-1$, then*

$$\tilde{V}_p(\mathbf{I}_j^\circ(K,L))^{n-1} \geq \tilde{V}_p(\mathbf{I}^\circ K)^{n-j-1}\tilde{V}_p(\mathbf{I}^\circ L)^j,$$

with equality if and only if K and L are dilates.

Proof. We first give the proof of the case $j > n-1$ as follows.

In view of the reverse of inequality (3.4), we obtain:

$$\tilde{v}_j(K\cap E_u, L\cap E_u)^{-p} \geq v(K\cap E_u)^{\frac{-p(n-j-1)}{n-1}} v(L\cap E_u)^{\frac{-jp}{n-1}}, \tag{3.5}$$

with equality if and only if $K\cap E_u$ and $L\cap E_u$ are dilates, which follows if and only if K and L are dilates.

From Lemma 3.1, (3.5) and in view of the reverse of Hölder's inequality for integral, we have

$$\begin{aligned} \tilde{V}_p(\mathbf{I}_j^\circ(K,L)) &= \tfrac{1}{n}\textstyle\int_{S^{n-1}} \tilde{v}_j\left(K\cap E_u, L\cap E_u\right)^{-p} dS(u) \\ &\geq \frac{1}{n}\int_{S^{n-1}} v(K\cap E_u)^{\frac{-p(n-j-1)}{n-1}} v(L\cap E_u)^{\frac{-jp}{n-1}} dS(u) \end{aligned} \tag{3.6}$$

$$\geq \left(\frac{1}{n}\int_{S^{n-1}} v(K\cap E_u)^{-p} dS(u)\right)^{\frac{(n-j-1)}{n-1}} \left(\frac{1}{n}\int_{S^{n-1}} v(L\cap E_u)^{-p} dS(u)\right)^{\frac{j}{n-1}}$$

$$= \tilde{V}_p(\mathbf{I}^\circ K)^{\frac{(n-j-1)}{n-1}} \tilde{V}_p(\mathbf{I}^\circ L)^{\frac{j}{n-1}}.$$

In view of the equality case of (3.5) and Hölder's inequality for integral, it follows that the equality holds if and only if K and L are dilates.

Similarly, from Lemma 3.1, Lemma 3.2 and in view of Hölder's inequality for integral, the proof of the case $0 < j < (n-1)$ can be completed by the same steps as in the proof of the case $j > (n-1)$ with suitable changes. Here, we omit the details.

The proof of Lemma 3.3 is thus complete. □

Lemma 3.4. Assume $K \in \varphi^n$.

(i) If $-\infty < p < -1$ *or* $0 < p < \infty$, *then*

$$\tilde{B}_{n+p}(K) \leq \tilde{V}_p(K^\circ),$$

with equality if and only if K *is centered.*

(ii) If $-1 < p < 0$, *then*

$$\tilde{B}_{n+p}(K) \geq \tilde{V}_p(K^\circ),$$

with equality if and only if K *is centered.*

Proof. For $-\infty < p < -1$, from (2.9) and (2.10), we have

$$\tilde{B}_{n+p}^{-1/p}(K) = \left(\frac{1}{n}\int_{S^{n-1}} (\frac{1}{2}\rho(K,u) + \frac{1}{2}\rho(K,-u))^{-p} dS(u)\right)^{-1/p}.$$

Apply Minkowski inequality for integral and in view of (2.1) and (2.5), we obtain:

$$\tilde{B}_{n+p}(K) \leq \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{-p} dS(u) = \frac{1}{n}\int_{S^{n-1}} \rho(K^\circ,u)^{p} dS(u) = \tilde{V}_p(K^\circ),$$

with equality if and only if $\rho(K,u) = \rho(K,-u)$, for $u \in S^{n-1}$. This is the desired result.

On the other hand, in view of the reverse Minkowski inequality for integral, the cases $0 < p < \infty$ and $-1 < p < 0$ easily follow.

For the special case $p = n$:

If $K \in \varphi^n$, then

$$\tilde{B}_{2n}(K) \leq V(K^\circ),$$

with equality if and only if K is centered.

Lemma 3.5. *If* $K, L \in \varphi^n$, $\lambda > 0, \mu > 0$, *then*

$$\tilde{b}(\lambda K \tilde{+} \mu L, u) = \lambda \tilde{b}(K,u) + \mu \tilde{b}(L,u).$$

Form (2.2) and (2.9), this easily follows.

Proof of Theorem. Assume that $C = (L_1, \ldots, L_{n-2})$, where $L_1, \ldots, L_{n-2} \in \varphi^n$. From (2.8)-(2.10) and in view of Minkowski inequality for integral, we obtain that for $j < n-1 (j \in \mathbb{R})$,

$$(3.7) \quad \tilde{B}_j(\mathbf{I}(K \tilde{+} L, C))^{1/(n-j)} = n^{-1/(n-j)} \|\tilde{b}(\mathbf{I}(K \tilde{+} L, C), u)\|_{n-j}$$
$$= n^{-1/(n-j)} \|\tilde{b}(\mathbf{I}(K, C), u) + \tilde{b}(\mathbf{I}(L, C), u)\|_{n-j}$$
$$\leq n^{-1/(n-j)} \left(\|\tilde{b}(\mathbf{I}(K, C), u)\|_{n-j} + \|\tilde{b}(\mathbf{I}(L, C), u) \right) \|_{n-j}$$
$$= \tilde{B}_j(\mathbf{I}(K, C))^{1/(n-j)} + \tilde{B}_j(\mathbf{I}(L, C))^{1/(n-j)},$$

with equality if and only if $\mathbf{I}(K, C)$ and $\mathbf{I}(L, C)$ have similar dual width. In view of intersection bodies being centered, it follows that $\mathbf{I}(K, C)$ and $\mathbf{I}(L, C)$ are dilates.

Taking $j = n + p$ in (3.7), we have

$$\tilde{B}_{n+p}(\mathbf{I}(K \tilde{+} L, C))^{-1/p} \leq \tilde{B}_{n+p}(\mathbf{I}(K, C))^{-1/p} + \tilde{B}_{n+p}(\mathbf{I}(L, C))^{-1/p}.$$

Using Lemma 3.4, we obtain:

$$(3.8) \quad \tilde{V}_p(\mathbf{I}^\circ(K \tilde{+} L, C))^{-1/p} \leq \tilde{V}_p(\mathbf{I}^\circ(K, C))^{-1/p} + \tilde{V}_p(\mathbf{I}^\circ(L, C))^{-1/p},$$

with equality if and only if $\mathbf{I}(K, C)$ and $\mathbf{I}(L, C)$ are dilates.

Taking $L_1 = \cdots = L_{n-2} = K \tilde{+} L$ in (3.8), we obtain:

$$\tilde{V}_p(\mathbf{I}^\circ(K \tilde{+} L))^{-1/p} \leq \tilde{V}_p(\mathbf{I}^\circ_{n-2}(K, K \tilde{+} L))^{-1/p} + \tilde{V}_p(\mathbf{I}^\circ_{n-2}(L, K \tilde{+} L))^{-1/p}.$$

Now, apply Lemma 3.3 twice in right sides of the above inequality, and get

$$\tilde{V}_p(\mathbf{I}^\circ(K \tilde{+} L))^{-1/p} \leq \tilde{V}_p(\mathbf{I}^\circ K)^{-1/p(n-1)} \tilde{V}_p(\mathbf{I}^\circ(K \tilde{+} L))^{-(n-2)/p(n-1)}$$

$$(3.9) \qquad + \tilde{V}_p(\mathbf{I}^\circ L)^{-1/p(n-1)} \tilde{V}_p(\mathbf{I}^\circ(K \tilde{+} L))^{-(n-2)/p(n-1)},$$

and with equality if and only if K, L and $K \tilde{+} L$ are dilates, along with $\mathbf{I}(K, C)$ and $\mathbf{I}(L, C)$ being dilates, the equality holds if and only if K and L are dilates.

Dividing both sides of (3.9) by $\tilde{V}_p(\mathbf{I}^\circ(K \tilde{+} L))^{-(n-2)/p(n-1)}$, we get

$$\tilde{V}_p(\mathbf{I}^\circ(K \tilde{+} L))^{-1/p(n-1)} \leq \tilde{V}_p(\mathbf{I}^\circ K)^{-1/p(n-1)} + \tilde{V}_p(\mathbf{I}^\circ L)^{-1/p(n-1)},$$

with equality if and only if K and L are dilates.

The proof is now complete. $\square$

Taking $p = -(n - i)$ in (1.2), we have the following result.

Corollary 3.1. *If $K, L \in \varphi^n$, and $0 \leq i < n$, then*

$$\tilde{W}_i(\mathbf{I}(K\tilde{+}L))^{1/(n-i)(n-1)} \leq \tilde{W}_i(\mathbf{I}K)^{1/(n-i)(n-1)} + \tilde{W}_i(\mathbf{I}L)^{1/(n-i)(n-1)},$$

with equality if and only if K and L are dilates.

This is just the Brunn-Minkowski inequality for intersection bodies which was given in [27]. This is also a dual form of the following inequality which was given by Lutwak [15].

The Brunn-Minkowski inequality for mixed projection bodies: If $K, L \in \mathcal{K}^n$, *and $0 \leq i < n$, then*

$$W_i(\mathbf{\Pi}(K+L))^{-1/(n-i)(n-1)} \geq W_i(\mathbf{\Pi}K)^{-1/(n-i)(n-1)} + W_i(\mathbf{\Pi}L)^{-1/(n-i)(n-1)},$$

with equality if and only if K and L are homothetic.

See [11], [19] and [22] for some recent interrelated results.

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