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EXTENSIONS OF NILPOTENT P.P. RINGS

L. OUYANG

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Abstract. We introduce the notion of nilpotent *p.p.* rings, and prove that the nilpotent *p.p.* condition is preserved over polynomial rings and skew polynomial rings.

1. Introduction

Throughout this paper, R denotes an associative ring with unity, $\alpha : R \longrightarrow R$ is an endomorphism, and δ an α-derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b+\alpha(a)\delta(b)$, for $a, b \in R$. We denote $S = R[x;\alpha,\delta]$ the Ore extension whose elements are the polynomials over R, the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x+\delta(a)$, for any $a \in R$. Recall that a ring R is called:

$$\begin{array}{rl} \textit{reduced} & \text{if } a^2 = 0 \Rightarrow a = 0, \text{for all } a \in R, \\ \textit{reversible} & \text{if } ab = 0 \Rightarrow ba = 0, \text{ for all } a, b \in R, \\ \textit{semicommutative} & \text{if } ab = 0 \Rightarrow aRb = 0, \text{ for all } a, b, \in R. \end{array}$$

The following implications hold:

$$\textit{reduced} \Rightarrow \textit{reversible} \Rightarrow \textit{semicommutative}.$$

In general, each of these implications is irreversible (see [14]).

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Let α be an endomorphism and δ an $\alpha-$derivation of a ring R. Following Hashemi and Moussavi [6], a ring R is said to be $\alpha-$compatible if for each $a, b \in R, ab = 0 \Leftrightarrow a\alpha(b) = 0$. Moreover, R is called to be $\delta-$compatible if for each $a, b \in R, ab = 0 \Rightarrow a\delta(b) = 0$. If R is both $\alpha-$compatible and $\delta-$compatible, then R is said to be $(\alpha, \delta)-$compatible.

For a nonempty subset X of a ring R, we write $r_R(X) = \{r \in R \mid Xr = 0\}$ and $l_R(X) = \{r \in R \mid rX = 0\}$, which are called the right annihilator of X in R and the left annihilator of X in R, respectively. The concept of annihilators has been the focus of a number of research papers (see [1, 2, 3, 5, 8, 15, 16]). As a generalization of annihilators, here we introduce the notion of nilpotent annihilators. Let R be a ring and $nil(R)$ be the set of all nilpotent elements of R. For a nonempty subset X of a ring R, we define $N_R(X) = \{a \in R \mid xa \in nil(R)$, for all $x \in X\}$, which is called a nilpotent annihilator of X in R. Obviously, for any nonempty subset X of a ring R, we have $r_R(X) \subseteq N_R(X)$ and $l_R(X) \subseteq N_R(X)$. So, a nilpotent annihilator is a natural generalization of an annihilator. If R is reduced, then $r_R(X) = N_R(X) = l_R(X)$.

In [10], Kaplansky introduced the Baer rings as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. Closely related to Baer rings are *p.p.* rings. A ring R is called a right *p.p.* ring if the right annihilator of each element of R is generated by an idempotent. A ring R is called a *p.p.* ring if it is both a right and a left *p.p.* ring [9, 13]. These concepts have their roots in functional analysis, having close links to $C^* - algebras$ and von Neumann algebras [4, 10]. Large classes of rings satisfy the Baer property-examples include right self-injective von Neumann regular rings, von Neumann algebras, and the endomorphisms rings of semisimple modules. Examples of p.p. rings also include large classes, such as all Baer rings. Motivated by their work, in this note we initiate the study of nilpotent *p.p.* rings. A ring R is said to be a nilpotent *p.p.* ring if the nilpotent annihilator of each element of R does not equal R, then it is generated as a right ideal by a nilpotent. Recently, the surge of interest in quantum groups and quantized algebras has brought renewed interest in general skew polynomials rings, due to the fact that many of these quantized algebras and their representations can be expressed in terms of iterated skew polynomial rings. So, in this note we mainly investigate the nilpotent p.p. condition over polynomial extensions and skew polynomial extensions.

For a polynomial $f(x) = a_0 + a_1x + \cdots + a_tx^t \in R[x]$. If $f(x)$ is a nilpotent element of $R[x]$, then we say that $f(x) \in nil(R[x])$.

2. Polynomial extensions over nilpotent *p.p.* rings

Definition 2.1. Let R be a ring. For a subset X of a ring R, we define $N_R(X) = \{a \in R \mid xa \in nil(R), \text{ for all } x \in X\}$, which is called the nilpotent annihilator of X in R. If X is a singleton, say $X = \{r\}$, we use $N_R(r)$ in place of $N_R(\{r\})$. Clearly, for any nonempty subset X of R, we have $N_R(X) = \{a \in R \mid xa \in nil(R), \text{ for all } x \in X\} = \{b \in R \mid bx \in nil(R), \text{ for all } x \in X\}$.

Example 2.2. Let Z be the ring of integers and $T_2(Z)$ the 2×2 upper triangular matrix ring over Z. We consider the subset $X = \left\{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right\}$.

Clearly, $r_{T_2(Z)}(X) = 0$, and $N_{T_2(Z)}(X) = \left\{\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}, | m \in Z\right\}$. Thus, $r_{T_2(Z)}(X) \neq N_{T_2(Z)}(X)$. Hence, a nilpotent annihilator is a nontrivial generalization of an annihilator.

Proposition 2.3. *Let X, Y be subsets of R. Then, we have the followings:*

(1) $X \subseteq Y$ *implies* $N_R(X) \supseteq N_R(Y)$.
(2) $X \subseteq N_R(N_R(X))$.
(3) $N_R(X) = N_R(N_R(N_R(X)))$.

Proof. proofs of (1) and (2) are really easy.

(3) Applying (2) to $N_R(X)$, we obtain $N_R(X) \subseteq N_R(N_R(N_R(X)))$. Since $X \subseteq N_R(N_R(X))$, we have $N_R(X) \supseteq N_R(N_R(N_R(X)))$, by (1). Therefore, $N_R(X) = N_R(N_R(N_R(X)))$. □

Lemma 2.4. *Let R be a subring of S. Then, for any subset X of R, we have $N_R(X) = N_S(X) \cap R$.*

Proof. Let $r \in N_R(X)$. Then, $r \in R$ and $xr \in nil(R)$, for each $x \in X$, and so $xr \in nil(S)$, for each $x \in X$. Hence, $r \in N_S(X) \cap R$ and so $N_R(X) \subseteq N_S(X) \cap R$. Assume that $a \in N_S(X) \cap R$. Then, $a \in R$ and $xa \in nil(S)$, for each $x \in X$. Note that $X \subseteq R$. We have $xa \in nil(R)$, for each $x \in X$. Thus $a \in N_R(X)$ and so $N_R(X) \supseteq N_S(X) \cap R$. Therefore, $N_R(X) = N_S(X) \cap R$. □

Definition 2.5. A ring R is said to be a nilpotent *p.p.* ring if for any element $p \in R$ with $N_R(p) \neq R$, $N_R(p)$ is generated as a right ideal by a nilpotent element.

Let R be a ring and let

$$T_n(R) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} \mid a_i \in R \right\}$$

with $n \geq 2$. Then, $T_n(R)$ is a ring with the usual matrix addition and multiplication.

Proposition 2.6. *If R is a domain, then $T_n(R)$ is a nilpotent p.p. ring.*

Proof. Let $p = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} \in T_n(R)$, with $N_{T_n(R)}(p)$ $\neq T_n(R)$. If $a_1 = 0$, then $N_{T_n(R)}(p) = T_n(R)$. This is contrary to the fact that $N_{T_n(R)}(p) \neq T_n(R)$. Thus, we obtain $a_1 \neq 0$. In this case, we obtain:

$$N_{T_n(R)}(p) = \left\{ \begin{pmatrix} 0 & u_2 & u_3 & \cdots & u_n \\ 0 & 0 & u_2 & \cdots & u_{n-1} \\ 0 & 0 & 0 & \cdots & u_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \mid u_i \in R \right\}$$

$$= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \cdot T_n(R),$$

where, $\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$ is a nilpotent element of $T_n(R)$.
Therefore, $T_n(R)$ is a nilpotent p.p. ring. □

From Proposition 2.6, one may suspect that the $n\times n$ upper triangular matrix ring over a domain is a nilpotent *p.p.* ring. But, the following example erases the possibility.

Example 2.7. Let R be a domain and let $T_3(R)$ be the 3×3 upper triangular matrix ring over R. Let $p = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in T_3(R)$. By a routine computation, we have $N_{T_3(R)}(p) = \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & 0 \end{pmatrix} \mid x_{ij} \in R \right\} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot T_3(R)$, where $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is not a nilpotent element. Therefore, $T_3(R)$ is not a nilpotent p.p. ring.

For the proofs of the next two Lemmas, see [12].

Lemma 2.8. *Let R be a semicommutative ring. Then, $nil(R)$ is an ideal of R.*

Lemma 2.9. *Let R be a semicommutative ring. Then, $f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x]$ is a nilpotent element of $R[x]$ if and only if $a_i \in nil(R)$, for all $0 \le i \le n$.*

Lemma 2.10. *Let R be a semicommutative ring. If $ab \in nil(R)$, for $a, b \in R$, then $aRbR \subseteq nil(R)$.*

Proof. Suppose $ab \in nil(R)$. Then, $abs \in nil(R)$ for any $s \in R$, since $nil(R)$ is an ideal of R. Thus, there exists a positive integer n such that $(abs)^n = absabs \cdots abs = 0$, and so $arbsarbs \cdots arbs = 0$, for any $r \in R$, because R is a semicommutative ring. Hence, $arbs \in nil(R)$, for each $r \in R$ and $s \in R$. Therefore $aRbR \subseteq nil(R)$. □

Proposition 2.11. *Let R be a semicommutative ring. Then, R is a nilpotent p.p. ring if and only if $R[x]$ is a nilpotent p.p. ring.*

Proof. Suppose that R is a nilpotent *p.p.* ring. Let $f(x) = a_0 + a_1x + \cdots + a_mx^m \in R[x]$, with $N_{R[x]}(f(x)) \neq R[x]$. We show that $N_{R[x]}(f(x))$ is generated by a nilpotent element. If $g(x) = b_0 + b_1x + \cdots + b_nx^n \in N_{R[x]}(f(x))$, then we have

$$f(x)g(x) = \left(\sum_{i=0}^{m} a_ix^i\right)\left(\sum_{j=0}^{n} b_jx^j\right) = \sum_{s=0}^{m+n}\left(\sum_{i+j=s} a_ib_j\right)x^s \in nil(R[x]).$$

We have the following system of equations by Lemma 2.9:

$$\Delta_s = \sum_{i+j=s} a_ib_j \in nil(R), \quad s = 0, 1, \cdots, m+n.$$

We will show that $a_ib_j \in nil(R)$ by induction on $i + j$.

If $i + j = 0$, then $a_0b_0 \in nil(R), b_0a_0 \in nil(R)$.

Now, suppose that s is a positive integer such that $a_ib_j \in nil(R)$, when $i + j < s$. We will show that $a_ib_j \in nil(R)$, when $i + j = s$. Consider the following equation:

$$(*) : \ \Delta_s = a_0b_s + a_1b_{s-1} + \cdots + a_sb_0 \in nil(R).$$

Multiplying Eq.$(*)$ by b_0 from left, we have $b_0a_sb_0 = b_0\Delta_s - (b_0a_0)b_s - (b_0a_1)b_{s-1} - \cdots - (b_0a_{s-1})b_1$. By the induction hypothesis, $a_ib_0 \in nil(R)$, for each i, $0 \leq i < s$, and so $b_0a_i \in nil(R)$, for each i, $0 \leq i < s$. Thus, $b_0a_sb_0 \in nil(R)$ and so $b_0a_s \in nil(R), a_sb_0 \in nil(R)$. Multiplying Eq.$(*)$ by $b_1, b_2, \cdots, b_{s-1}$ from the left side, respectively, yields $a_{s-1}b_1 \in nil(R), a_{s-2}b_2 \in nil(R), \cdots, a_0b_s \in nil(R)$, in turn. This means that $a_ib_j \in nil(R)$, when $i + j = s$. Therefore, by induction we obtain $a_ib_j \in nil(R)$, for each i, j, and so $b_j \in N_R(a_i)$, for for each i, $0 \leq i \leq m$ and j, $0 \leq j \leq n$. If $N_R(a_i) = R$, for each i, $0 \leq i \leq m$, then $a_ir \in nil(R)$ for each i, $0 \leq i \leq m$ and each $r \in R$. So, for any $u(x) = u_0 + u_1x + \cdots + u_tx^t \in R[x]$, we have $a_iu_j \in nil(R)$ for each i,

$0 \leq i \leq m$ and each j, $0 \leq j \leq t$. Thus,

$$f(x)u(x) = \sum_{s=0}^{m+t} \left(\sum_{i+j=s} a_i u_j \right) x^s \in nil(R[x]),$$

by Lemma 2.9, and so $u(x) \in N_{R[x]}(f(x))$. Thus, we obtain $N_{R[x]}(f(x)) = R[x]$. This is contrary to the fact that $N_{R[x]}(f(x)) \neq R[x]$. Thus, there exists an i, $0 \leq i \leq m$, such that $N_R(a_i) \neq R$. Since R is a nilpotent *p.p.* ring, there exists some $c \in nil(R)$ with $N_R(a_i) = cR$. Now, we show that $N_{R[x]}(f(x)) = c \cdot R[x]$. Since $b_j \in N_R(a_i) = cR$ for each j, $0 \leq j \leq n$, there exists $r_j \in R$ such that $b_j = cr_j$, and so $g(x) = c(r_0 + r_1 x + \cdots + r_n x^n) \in c \cdot R[x]$. Hence, $N_{R[x]}(f(x)) \subseteq c \cdot R[x]$. On the other hand, for $h(x) = h_0 + h_1 x + \cdots + h_p x^p \in R[x]$, we have

$$f(x) \cdot ch(x) = \left(\sum_{i=0}^{m} a_i x^i \right) \left(\sum_{j=0}^{p} ch_j x^j \right) = \sum_{s=0}^{m+p} \left(\sum_{i+j=s} a_i ch_j \right) x^s.$$

Since $nil(R)$ is an ideal of R and $c \in nil(R)$, we obtain $a_i ch_j \in nil(R)$ and so $f(x) \cdot ch(x) \in nil(R[x])$, by Lemma 2.9. Hence, $N_{R[x]}(f(x)) \supseteq c \cdot R[x]$, and so $N_{R[x]}(f(x)) = c \cdot R[x]$, where $c \in nil(R[x])$. Therefore, $R[x]$ is a nilpotent *p.p.* ring.

Conversely, assume that $R[x]$ is a nilpotent *p.p.* ring. Let $p \in R$, with $N_R(p) \neq R$. If $N_{R[x]}(p) = R[x]$, then we have $N_R(p) = N_{R[x]}(p) \cap R = R$, by Lemma 2.4, which is a contradiction. Thus, we obtain $N_{R[x]}(p) \neq R[x]$. Since $R[x]$ is a nilpotent *p.p.* ring, there exists $u(x) = u_0 + u_1 x + \cdots + u_s x^s \in nil(R[x])$ such that $N_{R[x]}(p) = u(x) \cdot R[x]$. Since $u(x) = u_0 + u_1 x + \cdots + u_s x^s \in nil(R[x])$, we obtain $u_i \in nil(R)$ for each i, $0 \leq i \leq s$, by Lemma 2.9. Now, we show that $N_R(p) = u_0 \cdot R$. Since $u_0 \in nil(R)$ and $nil(R)$ is an ideal of R, we have $pu_0 r \in nil(R)$ for each $r \in R$. Thus, $u_0 r \in N_R(p)$, for each $r \in R$, and so $N_R(p) \supseteq u_0 \cdot R$. Suppose that $m \in N_R(p)$. Then, $m \in N_{R[x]}(p)$, and so there exists $p(x) = p_0 + p_1 x + \cdots + p_q x^q \in R[x]$ such that $m = u(x)p(x)$. Hence, $m = u_0 p_0 \in u_0 \cdot R$, and so $N_R(p) \subseteq u_0 \cdot R$. Therefore, $N_R(p) = u_0 \cdot R$, and so R is a nilpotent *p.p.* ring. □

The ring of Laurent polynomial in x, with coefficient in R, consists of all formal sums $\sum_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integers. We denote this ring by $R[x; x^{-1}]$. If $f(x)$ is a nilpotent element of $R[x; x^{-1}]$, then we say that $f(x) \in nil(R[x; x^{-1}])$.

Lemma 2.12. *Let R be a semicommutative ring. Then, $f(x) = \sum_{i=k}^{n} a_i x^i \in R[x;x^{-1}]$ is a nilpotent element of $R[x;x^{-1}]$ if and only if $a_i \in nil(R)$, for each i, $k \leq i \leq n$.*

Proof. There exists a positive integer t such that $f(x) \cdot x^t \in R[x]$. Note that $(f(x))^k = 0$ if and only if $(f(x) \cdot x^t)^k = 0$, where k is a positive integer. Then, we complete the proof by Lemma 2.9. □

Lemma 2.13. *Let R be a semicommutative ring, $f(x) = \sum_{i=k}^{m} a_i x^i \in R[x;x^{-1}]$ and $g(x) = \sum_{j=l}^{n} b_j x^j \in R[x;x^{-1}]$. Then, we have $f(x)g(x) \in nil(R[x;x^{-1}])$ if and only if $a_i b_j \in nil(R)$, for each i, $k \leq i \leq m$ and for each j, $l \leq j \leq n$.*

Proof. Suppose that $a_i b_j \in nil(R)$, for each i, $k \leq i \leq m$ and each j, $l \leq j \leq n$. Then,

$$f(x)g(x) = \sum_{s=k+l}^{m+n} \left(\sum_{i+j=s} a_i b_j \right) x^s \in nil(R[x;x^{-1}]),$$

by Lemma 2.12. So it suffices to show that $a_i b_j \in nil(R)$ for each i, j, when $f(x)g(x) \in nil(R[x;x^{-1}]$. There exist positive integers u and v such that $f(x)x^u \in R[x]$ and $g(x)x^v \in R[x]$. Since $(f(x)g(x))^k = 0$ if and only if $(f(x)x^u g(x)x^v)^k = 0$, where k is a positive integer, same as the proof of Proposition 2.11, we obtain that $a_i b_j \in nil(R)$, for each i, j. □

Proposition 2.14. *Let R be a semicommutative ring. If R is a nilpotent p.p. ring, then so is $R[x;x^{-1}]$.*

Proof. Let $f(x) = \sum_{i=k}^{m} a_i x^i \in R[x;x^{-1}]$, with $N_{R[x;x^{-1}]}(f(x)) \neq R[x;x^{-1}]$. We show that $N_{R[x;x^{-1}]}(f(x))$ is generated by a nilpotent element. If $g(x) = \sum_{j=l}^{n} b_j x^j \in N_{R[x;x^{-1}]}(f(x))$, then $f(x)g(x) \in nil(R[x;x^{-1}])$. Then, we obtain $a_i b_j \in nil(R)$, for each i, j, by lemma 2.13, and so $b_j \in N_R(a_i)$ for each j, $l \leq j \leq n$ and each i, $k \leq i \leq m$. If $N_R(a_i) = R$, for each i, $k \leq i \leq m$, then for each $h(x) = \sum_{j=s}^{t} h_j x^j \in R[x;x^{-1}]$, we have $a_i h_j \in nil(R)$, for each i, $k \leq i \leq m$ and $s \leq j \leq t$. Thus, $f(x)h(x) \in nil(R[x;x^{-1}])$, by Lemma 2.13, and so $h(x) \in N_{R[x;x^{-1}]}(f(x))$. Hence, we obtain $N_{R[x;x^{-1}]}(f(x)) = R[x;x^{-1}]$, which is a contradiction. Thus, there exists an i, $k \le i \le m$, such that $N_R(a_i) \neq R$. Since R is a nilpotent *p.p.* ring, there exists some $c \in nil(R)$, with $N_R(a_i) = cR$. Now, we show that $N_{R[x;x^{-1}]}(f(x)) = c \cdot R[x;x^{-1}]$. Since $b_j \in N_R(a_i)$, for each j, $l \le j \le n$, there exists $r_j \in R$ such that $b_j = c \cdot r_j$. Thus, $g(x) = \sum_{j=l}^{n} b_j x^j = c(\sum_{j=l}^{n} r_j x^j) \in c \cdot R[x;x^{-1}]$. Hence, $N_{R[x;x^{-1}]}(f(x)) \subseteq c \cdot R[x;x^{-1}]$. Let $q(x) = \sum_{j=v}^{t} q_j x^j \in R[x;x^{-1}]$. Since $c \in nil(R)$ and $nil(R)$ is an ideal of R, we obtain $a_i c q_j \in nil(R)$, for each i,j, and so $f(x) \cdot cq(x) \in nil(R[x;x^{-1}])$, by Lemma 2.13. Thus, $N_{R[x;x^{-1}]}(f(x)) \supseteq c \cdot R[x;x^{-1}]$. Hence, $N_{R[x;x^{-1}]}(f(x)) = c \cdot R[x;x^{-1}]$, where $c \in nil(R[x;x^{-1}])$. Therefore, $R[x;x^{-1}]$ is a nilpotent *p.p.* ring. □

3. The Ore extensions over nilpotent *p.p.* rings

Let α be an endomorphism of R and $\delta : R \longrightarrow R$ an additive map of R. The application δ is said to be an $\alpha-$derivation if $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$. The Ore extension $S = R[x;\alpha,\delta]$ is the set of polynomials $\sum_{i=0}^{m} a_i x^i$ with the usual sum, and the multiplication rule as $xa = \alpha(a)x + \delta(a)$. Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x;\alpha,\delta]$. We say that $f(x) \in nil(R)[x;\alpha,\delta]$ if and only if $a_i \in nil(R)$, for each i, $0 \le i \le n$. If $f(x) \in R[x;\alpha,\delta]$ is a nilpotent element of $R[x;\alpha,\delta]$, then we say $f(x) \in nil(R[x;\alpha,\delta])$. For $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x;\alpha,\delta]$, we denote by $\{a_0, a_1, \cdots, a_n\}$ the set of coefficients of $f(x)$. Let $a_i \in R$, $1 \le i \le n$, and denote by $a_1 a_2 \cdots a_n$ the product of all $a_i, 1 \le i \le n$.

Let δ be an α-derivation of R. For integers i,j, with $0 \le i \le j$, $f_i^j \in End(R,+)$ will denote the map which is the sum of all possible words in α,δ built with i letters α and $j-i$ letters δ. For instance, $f_0^0 = 1, f_j^j = \alpha^j, f_0^j = \delta^j$ and $f_{j-1}^j = \alpha^{j-1}\delta + \alpha^{j-2}\delta\alpha + \cdots + \delta\alpha^{j-1}$. The next two lemmas appear in [11] and [6], respectively.

Lemma 3.1. *For any positive integer n and $r \in R$, we have $x^n r = \sum_{i=0}^{n} f_i^n(r) x^i$ in the ring $R[x;\alpha,\delta]$.*

Lemma 3.2. *Let R be an (α,δ)-compatible ring. Then, we have the followings:*

(1) If $ab = 0$, then $a\alpha^n(b) = \alpha^n(a)b = 0$, for every positive integer n.
(2) If $\alpha^k(a)b = 0$, for a positive integer k, then $ab = 0$.

(3) If $ab=0$, then $\alpha^n(a)\delta^m(b)=0=\delta^m(a)\alpha^n(b)$, for every positive integers m,n.

Lemma 3.3. *Let δ be an α-derivation of R. If R is an (α,δ)-compatible ring, then $ab=0$ implies $af_i^j(b)=0$, for each i,j, $j\geq i\geq 0$ and $a,b\in R$.*

Proof. If $ab=0$, then $a\alpha^i(b)=a\delta^j(b)=0$, for each $i\geq 0$ and each $j\geq 0$, because R is (α,δ)-compatible. Then, $af_i^j(b)=0$ for each i,j. □

Lemma 3.4. *Let δ be an α-derivation of R. If R is (α,δ)-compatible and reversible, then $ab\in nil(R)$ implies $af_i^j(b)\in nil(R)$, for each i,j, $j\geq i\geq 0$ and $a,b\in R$.*

Proof. Since $ab\in nil(R)$, there exists a positive integer k such that $(ab)^k=0$. $0=(ab)^k=abab\cdots ab\Rightarrow abab\cdots abaf_i^j(b)=0\Rightarrow af_i^j(b)ab\cdots ab=0\Rightarrow af_i^j(b)ab\cdots abaf_i^j(b)=0\Rightarrow af_i^j(b)af_i^j(b)ab\cdots ab=0\Rightarrow\cdots\Rightarrow af_i^j(b)\in nil(R)$. □

Lemma 3.5. *Let R be an $(\alpha,\delta)-$compatible ring. If $a\alpha^m(b)\in nil(R)$ for $a,b\in R$, and m is a positive integer, then $ab\in nil(R)$.*

Proof. Since $a\alpha^m(b)\in nil(R)$, there exists some positive integer n such that $(a\alpha^m(b))^n=0$. In the following computations, we use freely the condition that R is $(\alpha,\delta)-$compatible:

$$\begin{aligned}
&(a\alpha^m(b))^n=\underbrace{a\alpha^m(b)a\alpha^m(b)\cdots a\alpha^m(b)}_{n}=0\\
&\Rightarrow a\alpha^m(b)a\alpha^m(b)\cdots a\alpha^m(b)ab=0\\
&\Rightarrow a\alpha^m(b)a\alpha^m(b)\cdots a\alpha^m(b)\alpha^m(ab)=0\\
&\Rightarrow a\alpha^m(b)a\alpha^m(b)\cdots a\alpha^m(b)a\alpha^m(bab)=0\\
&\Rightarrow a\alpha^m(b)a\alpha^m(b)\cdots a\alpha^m(b)abab=0\\
&\Rightarrow\cdots\Rightarrow ab\in nil(R).
\end{aligned}$$

□

Proposition 3.6. *Let R be a reversible and $(\alpha,\delta)-$compatible ring and $f(x)=a_0+a_1x+\cdots+a_nx^n\in R[x;\alpha,\delta]$. Then, $f(x)\in nil(R[x;\alpha,\delta])$ if and only if $a_i\in nil(R)$ for each i, $0\leq i\leq n$.*

Proof.($\Longrightarrow$) Suppose $f(x) \in nil(R[x;\alpha,\delta])$. There exists a positive integer k such that $f(x)^k = (a_0 + a_1x + \cdots + a_nx^n)^k = 0$. Then,

$$f(x)^k = \text{“}\textit{lower terms}\text{”} + a_n\alpha^n(a_n)\alpha^{2n}(a_n)\cdots\alpha^{(k-1)n}(a_n)x^{nk}.$$

Hence, $a_n\alpha^n(a_n)\alpha^{2n}(a_n)\cdots\alpha^{(k-1)n}(a_n) = 0$, and $\alpha-$compatibility and reversibility of R gives $a_n \in nil(R)$. So by Lemma 3.4, $a_n = 1 \cdot a_n \in nil(R)$ implies $1 \cdot f_s^t(a_n) = f_s^t(a_n) \in nil(R)$, for each s, $0 \le s \le t$. Let $Q = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$. Then, we have

$$\begin{aligned} 0 &= (Q + a_nx^n)^k \\ &= (Q + a_nx^n)(Q + a_nx^n)\cdots(Q + a_nx^n) \\ &= (Q^2 + Q\cdot a_nx^n + a_nx^n\cdot Q + a_nx^n\cdot a_nx^n) \\ &\quad \cdot(Q + a_nx^n)\cdots(Q + a_nx^n) = \cdots = Q^k + \Delta, \end{aligned}$$

where, $\Delta \in R[x;\alpha,\delta]$. Note that the coefficients of Δ can be written as sums of monomials in a_i and $f_u^v(a_j)$, where $a_i, a_j \in \{a_0, a_1, \cdots, a_n\}$ and $v \ge u \ge 0$ are positive integers, and each monomial has a_n or $f_s^t(a_n)$. Since $nil(R)$ of a reversible ring R is an ideal, we obtain that each monomial is in $nil(R)$, and so $\Delta \in nil(R)[x;\alpha,\delta]$. Thus, we obtain:

$$\begin{aligned} &(a_0 + a_1x + \cdots + a_{n-1}x^{n-1})^k \\ &= \text{“}\textit{lower terms}\text{”} + a_{n-1}\alpha^{n-1}(a_{n-1})\cdots\alpha^{(n-1)(k-1)}(a_{n-1})x^{(n-1)k} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \in nil(R)[x;\alpha,\delta]. \end{aligned}$$

Hence, $a_{n-1}\alpha^{n-1}(a_{n-1})\cdots\alpha^{(k-1)(n-1)}(a_{n-1}) \in nil(R)$, and so $a_{n-1} \in nil(R)$, by Lemma 3.5. Using induction on n, we obtain $a_i \in nil(R)$, for each i, $0 \le i \le n$.

($\Longleftarrow$) Let $k > 1$ such that $a_i^k = 0$, for each i, $0 \le i \le n$. We claim that $f(x)^{(n+1)k+1} = (a_0 + a_1x + \cdots + a_nx^n)^{(n+1)k+1} = 0$. From

$$\begin{aligned} &(\sum_{i=0}^n a_ix^i)^2 = (\sum_{i=0}^n a_ix^i)\ (\sum_{i=0}^n a_ix^i) \\ = \ &(\sum_{i=0}^n a_ix^i)a_0 + (\sum_{i=0}^n a_ix^i)a_1x + \cdots \\ &+ (\sum_{i=0}^n a_ix^i)a_sx^s + \cdots + (\sum_{i=0}^n a_ix^i)a_nx^n \\ = \ &\sum_{i=0}^n a_if_0^i(a_0) + \left(\sum_{i=1}^n a_if_1^i(a_0) + \sum_{i=0}^n a_if_0^i(a_1)\right)x \\ &+ \left(\sum_{i=2}^n a_if_2^i(a_0) + \sum_{i=1}^n a_if_1^i(a_1) + \sum_{i=0}^n a_if_0^i(a_2)\right)x^2 + \cdots \\ &+ \left(\sum_{s+t=k} (\sum_{i=s}^n a_if_s^i(a_t))\right)x^k + \cdots + a_n\alpha^n(a_n)x^{2n}, \end{aligned}$$

it is easy to check that the coefficients of $(\sum_{i=0}^{n} a_i x^i)^{(n+1)k+1}$ can be written as sums of monomials of length $(n+1)k+1$ in a_i and $f_u^v(a_j)$, where $a_i, a_j \in \{a_0, a_1, \cdots, a_n\}$ and $v \geq u \geq 0$ are positive integers. Consider each monomial $\underbrace{a_{i1} f_{s_2}^{t_2}(a_{i_2}) \cdots f_{s_p}^{t_p}(a_{i_p})}_{(n+1)k+1}$ where, $a_{i_1}, a_{i_2}, \cdots a_{i_p} \in \{a_0, a_1, \cdots, a_n\}$, and $t_j, s_j (t_j \geq s_j, 2 \leq j \leq p)$ are nonnegative integers. We will show that $a_{i1} f_{s_2}^{t_2}(a_{i_2}) \cdots f_{s_p}^{t_p}(a_{i_p}) = 0$. If the number of a_0 in $a_{i1} f_{s_2}^{t_2}(a_{i_2}) \cdots f_{s_p}^{t_p}(a_{i_p})$ is greater than k, then we write $a_{i1} f_{s_2}^{t_2}(a_{i_2}) \cdots f_{s_p}^{t_p}(a_{i_p})$ as:

$$b_1(f_{s_{01}}^{t_{01}}(a_0))^{j_1} b_2(f_{s_{02}}^{t_{02}}(a_0))^{j_2} \cdots b_v(f_{s_{0v}}^{t_{0v}}(a_0))^{j_v} b_{v+1},$$

where, $j_1 + j_2 + \cdots j_v > k, 1 \leq j_1, j_2 \cdots, j_v$ and $b_q (q = 1, 2, \cdots, v+1)$ is a product of some elements chosen from $\{a_{i1}, f_{s_2}^{t_2}(a_{i_2}), \cdots f_{s_p}^{t_p}(a_{i_p})\}$ or is equal to 1. Since $a_0^{j_1+j_2+\cdots j_v} = 0$ and R is reversible and $(\alpha, \delta)-$compati– ble, we have $0 = a_0^{j_1+j_2+\cdots+j_v} = \underbrace{a_0 a_0 \cdots a_0}_{j_1+j_2+\cdots+j_v} \Longrightarrow a_0 a_0 \cdots (f_{s_{01}}^{t_{01}}(a_0)) = 0 \Longrightarrow (f_{s_{01}}^{t_{01}}(a_0)) a_0 \cdots a_0 = 0 \Longrightarrow (f_{s_{01}}^{t_{01}}(a_0))^{j_1} a_0 \cdots a_0 = 0 \Rightarrow \cdots \Rightarrow (f_{s_{01}}^{t_{01}}(a_0))^{j_1} (f_{s_{02}}^{t_{02}}(a_0))^{j_2} \cdots (f_{s_{0v}}^{t_{0v}}(a_0))^{j_v} = 0 \Longrightarrow b_1(f_{s_{01}}^{t_{01}}(a_0))^{j_1} b_2(f_{s_{02}}^{t_{02}}(a_0))^{j_2} \cdots b_v(f_{s_{0v}}^{t_{0v}}(a_0))^{j_v} b_{v+1} = 0$. Thus, $a_{i1} f_{s_2}^{t_2}(a_{i_2}) \cdots f_{s_p}^{t_p}(a_{i_p}) = 0$. If the number of a_i in $a_{i1} f_{s_2}^{t_2}(a_{i_2}) \cdots f_{s_p}^{t_p}(a_{i_p})$ is greater than k, then similar discussion yields $a_{i1} f_{s_2}^{t_2}(a_{i_2}) \cdots f_{s_p}^{t_p}(a_{i_p}) = 0$. Thus, each term appearing in $(\sum_{i=0}^{n} a_i x^i)^{(n+1)k+1}$ equals 0. Therefore, $\sum_{i=0}^{n} a_i x^i \in R[x; \alpha, \delta]$ is a nilpotent element. □

Corollary 3.7. *Let R be a reversible and $\alpha-$compatible ring, and $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x; \alpha]$. Then, $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in nil(R[x; \alpha])$ if and only if $a_i \in nil(R)$, for each i, $0 \leq i \leq n$.*

Proposition 3.8. *Let R be a reversible and $(\alpha, \delta)-$compatible ring. Then for $f = \sum_{i=0}^{m} a_i x^i, g = \sum_{j=0}^{n} b_j x^j \in R[x; \alpha, \delta]$, $fg \in nil(R[x; \alpha, \delta])$ if and only if $a_i b_j \in nil(R)$, for each i, j, $0 \leq i \leq m, 0 \leq j \leq n$.*

Proof. (⇒) Let $f = \sum_{i=0}^{m} a_i x^i, g = \sum_{j=0}^{n} b_j x^j \in R[x; \alpha, \delta]$ be such that $fg \in nil(R[x; \alpha, \delta])$. Then,

$$
\begin{aligned}
fg &= (\sum_{i=0}^{m} a_i x^i)(\sum_{j=0}^{n} b_j x^j) \\
&= (\sum_{i=0}^{m} a_i x^i) b_0 + (\sum_{i=0}^{m} a_i x^i) b_1 x + \cdots + (\sum_{i=0}^{m} a_i x^i) b_n x^n \\
&= \sum_{i=0}^{m} a_i f_0^i(b_0) + \left(\sum_{i=1}^{m} a_i f_1^i(b_0) + \sum_{i=0}^{m} a_i f_0^i(b_1) \right) x + \cdots \\
&+ \left(\sum_{s+t=k} (\sum_{i=s}^{m} a_i f_s^i(b_t)) \right) x^k + \cdots + a_m \alpha^m(b_n) x^{m+n} \in nil(R[x;\alpha,\delta]).
\end{aligned}
$$

Then, we have the following system of equations, by Proposition 3.6:

$$
\begin{aligned}
&(1) \quad \Delta_{m+n} = a_m \alpha^m(b_n) \in nil(R), \\
&(2) \quad \Delta_{m+n-1} = a_m \alpha^m(b_{n-1}) + a_{m-1}\alpha^{m-1}(b_n) + a_m f_{m-1}^m(b_n) \in nil(R), \\
&(3) \quad \Delta_{m+n-2} = a_m \alpha^m(b_{n-2}) + \sum_{i=m-1}^{m} a_i f_{m-1}^i(b_{n-1}) \\
&\qquad\qquad + \sum_{i=m-2}^{m} a_i f_{m-2}^i(b_n) \in nil(R), \\
&\qquad\qquad \vdots \\
&(4) \quad \Delta_k = \sum_{s+t=k} (\sum_{i=s}^{m} a_i f_s^i(b_t)) \in nil(R).
\end{aligned}
$$

From Eq. (1), $a_m b_n \in nil(R)$. Now, we show that $a_i b_n \in nil(R)$, for each i, $0 \leq i \leq m$. If we multiply Eq. (2) on the left side by b_n, then $b_n a_{m-1}\alpha^{m-1}(b_n) = b_n \Delta_{m+n-1} - (b_n a_m \alpha^m(b_{n-1}) + b_n a_m f_{m-1}^m(b_n)) \in nil(R)$, since $nil(R)$ of a semicommutative ring is an ideal. Thus, by Lemma 3.5, we obtain $b_n a_{m-1} b_n \in nil(R)$, and so we have $b_n a_{m-1} \in nil(R), a_{m-1} b_n \in nil(R)$. If we multiply Eq. (3) on the left side by b_n, then we obtain $b_n a_{m-2} f_{m-2}^{m-2}(b_n) = b_n a_{m-2}\alpha^{m-2}(b_n) = b_n \Delta_{m+n-2} - b_n a_m \alpha^m(b_{n-2}) - b_n a_{m-1} f_{m-1}^{m-1}(b_{n-1}) - b_n a_m f_{m-1}^m(b_{n-1}) - b_n a_{m-1} f_{m-2}^{m-1}(b_n) - b_n a_m f_{m-2}^m(b_n) = b_n \Delta_{m+n-2} - (b_n a_m)\alpha^m(b_{n-2}) - (b_n a_{m-1}) f_{m-1}^{m-1}(b_{n-1}) - (b_n a_m) f_{m-1}^m(b_{n-1}) - (b_n a_{m-1})\ f_{m-2}^{m-1}(b_n) - (b_n a_m)\ f_{m-2}^m(b_n) \in nil(R)$, since $nil(R)$ is an ideal of R. Thus, we obtain $a_{m-2} b_n \in nil(R)$ and $b_n a_{m-2} \in nil(R)$. Continuing this procedure yields that $a_i b_n \in nil(R)$, for each i, $0 \leq i \leq m$, and so $a_i f_s^t(b_n) \in nil(R)$, for any $t \geq s \geq 0$ and any i, $0 \leq i \leq m$, by Lemma 3.4. Thus, it is easy to verify that $(\sum_{i=0}^{m} a_i x^i)(\sum_{j=0}^{n-1} b_j x^j) \in nil(R)[x;\alpha,\delta]$. Applying the preceding argument repeatedly, we obtain that $a_i b_j \in nil(R)$, for each i, $0 \leq i \leq m, 0 \leq j \leq n$. □

($\Leftarrow$) Suppose that $a_ib_j \in nil(R)$, for each i,j. Then, $a_if_s^i(b_j) \in nil(R)$, for each i,j and each positive integers, $i \geq s \geq 0$, by Lemma 3.4. Thus,

$$\sum_{s+t=k}(\sum_{i=s}^{m} a_if_s^i(b_t)) \in nil(R), \quad k = 0,1,2,\cdots m+n.$$

Hence, $fg = \sum_{k=0}^{m+n}(\sum_{s+t=k}(\sum_{i=s}^{m} a_if_s^i(b_t)))x^k \in nil(R[x;\alpha,\delta])$, by Proposition 3.6.

Proposition 3.9. *Let R be an $(\alpha,\delta)-$compatible and reversible ring. If R is a nilpotent p.p. ring, then so is $S = R[x;\alpha,\delta]$.*

Proof. Let $f(x) = a_0 + a_1x + \cdots + a_mx^m \in S = R[x;\alpha,\delta]$, with $N_S(f(x)) \neq S$. If $g(x) = b_0 + b_1x + \cdots + b_nx^n \in N_S(f(x))$, then $f(x)g(x) \in nil(S)$. Thus, we have $a_ib_j \in nil(R)$ for each i,j, by Proposition 3.8, and so $b_j \in N_R(a_i)$, for each i,j, $0 \leq i \leq m$ and $0 \leq j \leq n$. If $N_R(a_i) = R$, for each i, $0 \leq i \leq m$, then for any $h(x) = h_0 + h_1x + \cdots + h_lx^l \in S = R[x;\alpha,\delta]$, we have $a_ih_j \in nil(R)$, for each i,j, and so $f(x)h(x) \in nil(S)$, by Proposition 3.8. Thus, $N_S(f(x)) = S$, which is a contradiction. So, there exists an i, $0 \leq i \leq m$ such that $N_R(a_i) \neq R$. Since R is a nilpotent *p.p.* ring, there exists a $c \in nil(R)$, with $N_R(a_i) = cR$. Now, we show that $N_S(f(x)) = c \cdot S$. Since $b_j \in N_R(a_i) = cR$, for each j, $0 \leq j \leq n$, there exists $r_j \in R$ such that $b_j = cr_j$ for each j, $0 \leq j \leq n$. Hence $g(x) = b_0 + b_1x + \cdots + b_nx^n = c(r_0 + r_1x + \cdots + r_nx^n) \in c \cdot S$. Thus, $N_S(f(x)) \subseteq c \cdot S$. On the other hand, any $u(x) = u_0 + u_1x + \cdots + u_qx^q \in S = R[x;\alpha,\delta]$. Since $c \in nil(R)$ and $nil(R)$ is an ideal of R, we obtain $a_icu_j \in nil(R)$, for each i,j, and so $f(x) \cdot cu(x) \in nil(S)$, by Proposition 3.8. Thus, we obtain $N_S(f(x)) \supseteq c \cdot S$. Hence, $N_S(f(x)) = c \cdot S$, where $c \in nil(S)$. Therefore, $S = R[x;\alpha,\delta]$ is a nilpotent *p.p.* ring. □

Corollary 3.10. *Let R be an $\alpha-$compatible and reversible ring. If R is a nilpotent p.p.ring, then so is $R[x;\alpha]$.*

Proposition 3.11. *Let R be an $\alpha-$compatible and reversible ring. Then, R is a nilpotent p.p. ring if and only if $R[x;\alpha]$ is a nilpotent p.p. ring.*

Proof. Suppose that R is a nilpotent *p.p.* ring. Then, so is $R[x;\alpha]$, by Corollary 3.10. So it suffices to show that R is a nilpotent *p.p.* ring, when $R[x;\alpha]$ is a nilpotent *p.p.* ring. Let $p \in R$, with $N_R(p) \neq R$. If $N_{R[x;\alpha]}(p) = R[x;\alpha]$, then $N_R(p) = N_{R[x;\alpha]}(p) \cap R = R$, by Lemma 2.4, which is a contradiction. Thus, we have $N_{R[x;\alpha]}(p) \neq R[x;\alpha]$. Since $R[x;\alpha]$ is a nilpotent *p.p.* ring, there exists $f(x) = a_0 + a_1x + \cdots + a_mx^m \in nil(R[x;\alpha])$ such that $N_{R[x;\alpha]}(p) = f(x) \cdot R[x;\alpha]$. Since $f(x) = a_0 + a_1x + \cdots + a_mx^m \in nil(R[x;\alpha])$, we have $a_i \in nil(R)$, for each i, $0 \leq i \leq m$, by Corollary 3.7. Now, we show that $N_R(p) = a_0R$. Since $a_0 \in nil(R)$ and $nil(R)$ is an ideal of R, we obtain $p \cdot a_0R \subseteq nil(R)$, and so $N_R(p) \supseteq a_0R$. If $m \in N_R(p)$, then $m \in N_{R[x;\alpha]}(p)$. Thus, there exists $h(x) = h_0 + h_1x + \cdots + h_qx^q \in R[x;\alpha]$ such that

$$m = f(x)h(x) = \sum_{s=0}^{m+q} \left(\sum_{i+j=s} a_i\alpha^i(h_j) \right) x^s.$$

Thus, we have $m = a_0h_0 \in a_0R$, and so $N_R(p) \subseteq a_0R$. Hence, $N_R(p) = a_0R$, where $a_0 \in nil(R)$. Therefore, R is a nilpotent *p.p.* ring. □

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Lunqun Ouyang
Department of Mathematics, Huan University of Science and Technology, Xiangtan, Hunan 411201, P.R. China.
Email: ouyanglqtxy@163.com